Lecture 26
Trig Integrals

11/12

Test Review 3 - failed 9/33

1. Area between curves ✓
2. Volume of revolution
   y = 1 = horiz line
3. Pair of tricky proplems

$$\lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{\left(1 + i\frac{3}{n}\right)^2}_{f(x)} \cdot \underbrace{\frac{3}{n}}_{\Delta x}$$

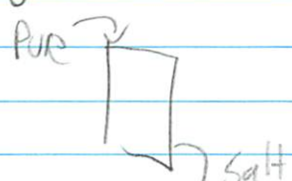
$$\int_1^4 x^2\,dx$$ ← will see again

knowing Riemann sums rule

4. Finding COM
   - dh
5. Proof $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$
   - I thought I got — Disaster
6. Differential equation
   - He was not here that lecture
   - Low on time — Disaster
   - kinda bonus

   Brine leaving a tank

   salt → 0



Differential eq - rate of salt water leaves tank
- name things in problem
- weight ↓ (t)

$S(t)$ want $\frac{ds}{dt}$ ← did ← units kg/min differential - amount leaving charging over time

$\frac{ds}{dt} \neq -10\,L/min \ \frac{15}{1000\,L}$

↑ but amt salt leaving **not** constant

$-10\,L/min \ \frac{S(t)}{1000\,L}$ ← just write!

$\frac{ds}{dt} = \frac{-s}{100}$ seperate variables

ln
exponentiate
solve for

<u>Method of Substitution</u>

$\int \sqrt{x^3+1}\ x^5\,dx$

$u(x) = x^3+1$
$du = 3x^2\,dx$

split so can find du hiding in here

$\int \sqrt{u(x)} \cdot \underset{u-1}{x^3} \cdot \frac{1}{3} \cdot \underbrace{3x^2\,dx}_{du}$

$= \frac{1}{3}\int u^{1/2}(u-1)\,du$

$= \frac{1}{3}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$

$= \frac{1}{3}\left(\frac{2}{5}(x^3+1)^{5/2} - \frac{2}{3}(x^3+1)^{3/2}\right) + C$

Today Integrals of form

$$\int \sin^m x \cos^n x \, dx$$

Plan: use substitution + trig identities

Identies $\sin^2 x + \cos^2 x = 1$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\sin x \cos x = \frac{1}{2} \sin 2x$

- if forget → guess identity + plug in 0
  check it make sense

$$\int \cos^3 x \, dx$$

$\boxed{\cancel{u(x) = \cos x \quad du = -\sin x \, dx}}$ Bad idea for now

$$\int \cos^2 x \cos x \, dx$$

use trig identity $(1 - \sin^2 x)$

$$\int \underbrace{(1 - \sin^2 x)}_{\text{function } \sin x} \cos x \, dx$$

look !!

$u(x) = \sin x$

$du = \cos x \, dx$

$$= \int (1-u^2)\, du$$

actually integrate

$$= u - \tfrac{1}{3}u^3 + C \quad \text{where } u(x) = \sin x$$

$$= \boxed{\sin x - \tfrac{1}{3}(\sin^3 x) + C}$$

ex2

$$\int \sin^5 x \cos^2 x \, dx$$

split

$$\int (\sin^2 x)^2 \cos^2 x \cdot \sin x \, dx$$

$$\int (1-\cos^2 x)^2 \cdot \cos^2 x \cdot \sin x \, dx$$

$$u(x) = \cos x$$
$$du = -\sin x \, dx$$

$$-\int (1-u^2)^2 u^2 \, du$$

$$-\int (1-2u^2+u^4) u^2 \, du$$

$$-\int (u^2 - 2u^4 + u^6) \, du$$

...

Strategy

If either m or n is odd – we have a strategy

say $m = 2k+1$

chop away all powers but 1

$$\int (\sin^{2k} x \cos^n x) \sin x \, dx$$

trig identity

$$\int (1-\cos^2 x)^k \cos^n x \sin x \, dx$$

What if m and n even

$\int \sin^4 x \, dx$

$\int (\sin^2 x)^2 \, dx$ ← half angle identity

$\int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$

$\int \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) \, dx$

ok ok still annoying

$\int \frac{1}{4} dx - \frac{1}{4}\int 2\cos 2x \, dx + \int \frac{1}{4}\cos^2 2x \, dx$

each stage lose half of powers as before

26 → 13 → now odd

24 → 12 → 6 → 3 → now odd

if mix sin + cos use the 2x angle identities

If m, n are both even
- use half angle identities to express as a polynomial in cos 2x
  Reassess at end

$\int \sin^2 x \cos^{10} x \, dx$

↑ apply half angle identies to both

$\int \frac{1}{2}(1-\cos 2x)\left(\frac{1}{2}(1-\cos 2x)\right)^5 dx$

... keep going

both cant
both ugly

✓ or

$\int \left(\frac{1}{2}\sin 2x\right)^2 \left(\frac{1}{2}(1+\cos 2x)\right)^4 dx$

Similar tricks for

$$\int \tan^m x \sec^n x \, dx$$

- same sort of game

$$\frac{d}{dx} \tan x = \sec^2 x$$

a bit harder

$$\frac{d}{dx} \sec x = \sec x \tan x$$

Identity

$$\sec^2 x = 1 + \tan^2 x$$

$$\int \tan^6 x \sec^4 x \, dx$$

strategy - sub in tan and leave 2 sec
or sub in sec and leave sec tan

$$\int \tan^6 x \left(1 + \tan^2 x\right) \sec^2 x \, dx$$

$$u = \tan x$$
$$du = \sec^2 x \, dx$$

$$\int u^6 \left(1 + u^2\right) du$$

If n = even → good
If m = odd → good (need n ⊕)
other cases → use cleverness :)

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \ln|\sec x| + C$$

$= \ln|\sec x + \tan x| + C$ → can do same $\int \cot^m x \csc^n x \, dx$

$$\int \sec x \, dx = \int \frac{1}{\cos x} dx = \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} = \int \frac{\sec^2 + \sec x \tan x}{\sec x + \tan x}$$

$u = \sec x + \tan x$
$du = \sec^2 x + \sec x \tan x$

Lecture 26: Trigonometric Integrals and Substitution

Trigonometric Integrals

How do you integrate an expression like $\int \sin^n x \cos^m x \, dx$? ($n = 0, 1, 2...$ and $m = 0, 1, 2, \ldots$)

We already know that:

$$\int \sin x \, dx = -\cos x + c \quad \text{and} \quad \int \cos x \, dx = \sin x + c$$

Method A

Suppose either n or m is odd.

Example 1. $\int \sin^3 x \cos^2 x \, dx$.
Our strategy is to use $\sin^2 x + \cos^2 x = 1$ to rewrite our integral in the form:

$$\int \sin^3 x \cos^2 x \, dx = \int f(\cos x) \sin x \, dx$$

Indeed,

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$$

Next, use the substitution

$$u = \cos x \quad \text{and} \quad du = -\sin x \, dx$$

Then,

$$\int (1 - \cos^2 x) \cos^2 x \sin x \, dx = \int (1 - u^2) u^2 (-du)$$

$$= \int (-u^2 + u^4) du = -\frac{1}{3}u^3 + \frac{1}{5}u^5 + c = -\frac{1}{3}\cos^3 u + \frac{1}{5}\cos^5 x + c$$

Example 2.

$$\int \cos^3 x \, dx = \int f(\sin x) \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx$$

Again, use a substitution, namely

$$u = \sin x \quad \text{and} \quad du = \cos x \, dx$$

$$\int \cos^3 x \, dx = \int (1 - u^2) du = u - \frac{u^3}{3} + c = \sin x - \frac{\sin^3 x}{3} + c$$

Method B

This method requires *both* m and n to be even. It requires double-angle formulae such as

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

(Recall that $\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1-\sin^2 x) = 2\cos^2 x - 1$)
Integrating gets us

$$\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2} dx = \frac{x}{2} + \frac{\sin(2x)}{4} + c$$

We follow a similar process for integrating $\sin^2 x$.

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\int \sin^2 x\, dx = \int \frac{1-\cos(2x)}{2} dx = \frac{x}{2} - \frac{\sin(2x)}{4} + c$$

The full strategy for these types of problems is to keep applying Method B until you can apply Method A (when one of m or n is odd).

Example 3. $\int \sin^2 x \cos^2 x\, dx$.
Applying Method B twice yields

$$\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx = \int \left(\frac{1}{4} - \frac{1}{4}\cos^2 2x\right) dx$$

$$= \int \left(\frac{1}{4} - \frac{1}{8}(1+\cos 4x)\right) dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + c$$

There is a shortcut for Example 3. Because $\sin 2x = 2\sin x \cos x$,

$$\int \sin^2 x \cos^2 x\, dx = \int \left(\frac{1}{2}\sin 2x\right)^2 dx = \frac{1}{4}\int \frac{1-\cos 4x}{2} dx = \text{ same as above}$$

The next family of trig integrals, which we'll start today, but will not finish is:

$$\int \sec^n x \tan^m x\, dx \quad \text{where } n = 0, 1, 2, \ldots \text{ and } m = 0, 1, 2, \ldots$$

Remember that

$$\sec^2 x = 1 + \tan^2 x$$

which we double check by writing

$$\frac{1}{\cos^2 x} = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^3 x}$$

$$\boxed{\int \sec^2 x\, dx = \tan x + c \qquad \int \sec x \tan x\, dx = \sec x + c}$$

To calculate the integral of $\tan x$, write

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx$$

Let $u = \cos x$ and $du = -\sin x\,dx$, then

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int -\frac{du}{u} = -\ln(u) + c$$

$$\boxed{\int \tan x\,dx = -\ln(\cos x) + c}$$

(We'll figure out what $\int \sec x\,dx$ is later.)

Now, let's see what happens when you have an even power of secant. (The case n even.)

$$\int \sec^4 x\,dx = \int f(\tan x)\,\sec^2 x\,dx = \int (1+\tan^2 x)\,\sec^2 x\,dx$$

Make the following substitution:

$$u = \tan x$$

and

$$du = \sec^2 x\,dx$$

$$\int \sec^4 x\,dx = \int (1+u^2)du = u + \frac{u^3}{3} + c = \tan x + \frac{\tan^3 x}{3} + c$$

What happens when you have a odd power of tan? (The case m odd.)

$$\int \tan^3 x\,\sec x\,dx = \int f(\sec x)\,d(\sec x)$$

$$= \int (\sec^2 x - 1)\,\sec x\,\tan x\,dx$$

(Remember that $\sec^2 x - 1 = \tan^2 x$.)

Use substitution:

$$u = \sec x$$

and

$$du = \sec x\,\tan x\,dx$$

Then,

$$\int \tan^3 x\,\sec x\,dx = \int (u^2-1)du = \frac{u^3}{3} - u + c = \frac{\sec^3 x}{3} - \sec x + c$$

We carry out one final case: $n = 1, m = 0$

$$\int \sec x\,dx = \ln(\tan x + \sec x) + c$$

We get the answer by "advanced guessing," i.e., "knowing the answer ahead of time."

$$\int \sec x\,dx = \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx = \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}dx$$

Make the following substitutions:

$$u = \tan x + \sec x$$

and

$$du = (\sec^2 x + \sec x \tan x)\,dx$$

This gives

$$\int \sec x\,dx = \int \frac{du}{u} = \ln(u) + c = \ln(\tan x + \sec x) + c$$

Cases like $n = 3, m = 0$ or more generally n odd and m even are more complicated and will be discussed later.

Trigonometric Substitution

Knowing how to evaluate all of these trigonometric integrals turns out to be useful for evaluating integrals involving square roots.

Example 4. $y = \sqrt{a^2 - x^2}$

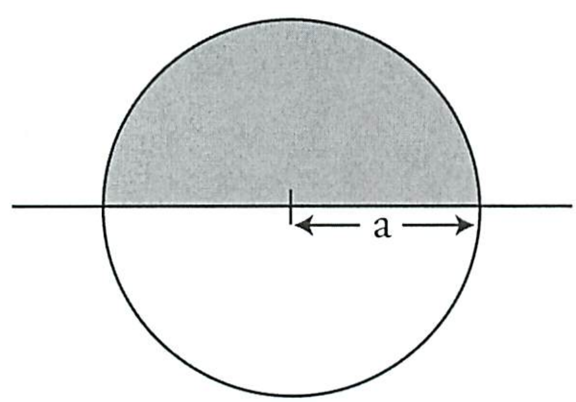


Figure 1: Graph of the circle $x^2 + y^2 = a^2$.

We already know that the area of the top half of the disk is

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = \frac{\pi a^2}{2}$$

What if we want to find this area?

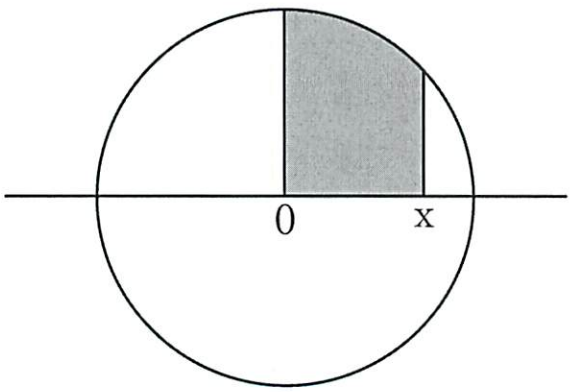


Figure 2: Area to be evaluated is shaded.

To do so, you need to evaluate this integral:

$$\int_{t=0}^{t=x} \sqrt{a^2-t^2}\,dt$$

Let $t = a\sin u$ and $dt = a\cos u\,du$. (Remember to change the limits of integration when you do a change of variables.)
Then,

$$a^2 - t^2 = a^2 - a^2\sin^2 u = a^2\cos^2 u; \quad \sqrt{a^2-t^2} = a\cos u$$

Plugging this into the integral gives us

$$\int_0^x \sqrt{a^2-t^2}\,dt = \int (a\cos u)\,a\cos u\,du = a^2\int_{u=0}^{u=\sin^{-1}(x/a)} \cos^2 u\,du$$

Here's how we calculated the new limits of integration:

$$\begin{aligned} t &= 0 \implies a\sin u = 0 \implies u = 0 \\ t &= x \implies a\sin u = x \implies u = \sin^{-1}(x/a) \end{aligned}$$

$$\begin{aligned} \int_0^x \sqrt{a^2-t^2}\,dt = a^2\int_0^{\sin^{-1}(x/a)} \cos^2 u\,du &= a^2\left(\frac{u}{2}+\frac{\sin 2u}{4}\right)\Big|_0^{\sin^{-1}(x/a)} \\ &= \frac{a^2\sin^{-1}(x/a)}{2} + \left(\frac{a^2}{4}\right)\left(2\sin(\sin^{-1}(x/a))\cos(\sin^{-1}(x/a))\right) \end{aligned}$$

(Remember, $\sin 2u = 2\sin u\,\cos u$.)

We'll pick up from here next lecture (Lecture 28 since Lecture 27 is Exam 3).

Skipped Lecture 27 in numbering

Lecture 28

11/13

PSet 7b posted

Techniques of integration

- Products of trig functions ← yesterday
- Inverse substitution to handle square roots of quadratics → today
- Rational Functions → next lecture
- everything else → integration by parts

$\int \sin^n x \cos^m x \, dx$

$\int \tan^n x \cot^m x \, dx$

$\int \cot^n x \csc^m x \, dx$

have nice identies

$\cos^2 x + \sin^2 x = 1$

can derive every ident from this by dividing

Hybrid product

$\int \tan^5 x \sin^2 x \, dx$

- can express in terms of sin + cos

$\int \frac{\sin^7 x}{\cos^5 x} dx = \int$?? prof does not know how to solve quickly

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Inverse substitution

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx$$

$\cancel{u = 9-x^2}$

$\cancel{du = -2x}\,dx$ ← not sitting around

crappy idea

Need clever idea!

- use u substitution in reverse
- let x be function θ

$x = 3\sin\theta$ ← make new function

$dx = 3\cos\theta\,d\theta$ seems like making worse

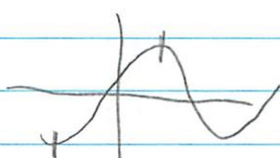
$$\int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\cdot 3\cos\theta\,d\theta$$

$$\int \frac{3\sqrt{1-\sin^2\theta}}{9\sin^2\theta}\cdot 3\cos\theta\,d\theta$$

← make sure abs value

$$\int \frac{3\,|\cos\theta|}{9\sin^2\theta}\cdot 3\cos\theta\,d\theta$$

- Limit range of θ so that $x = 3\sin\theta$ is 1 to 1 (has inverse)
- monotonic (horiz line test)



$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\cos\theta \in (0, 1)$ +

so can get rid abs value since never negitive

look at original → $x \in (-3, 3)$

value of θ needs to obtain these values

$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ sanity ✓

$$\int \frac{3\cos\theta}{9\sin^2\theta} \cdot 3\cos\theta \, d\theta$$

$$\int \frac{\cos^2\theta}{\sin^2\theta} \, d\theta$$

$$\frac{\sin^2 x + \cos^2 x = 1}{\sin^2 x \quad \sin^2 x \quad \sin^2 x}$$

$$\int \cot^2\theta \, d\theta$$

trig identity

$$\cot^2\theta + 1 = \csc^2\theta$$

$$\int -(1 - \csc^2\theta)\, d\theta$$

$$-\theta + \cot(\theta) + C$$

$$\cot(\theta) - \theta + C$$

← go back to $x = 3\sin\theta$

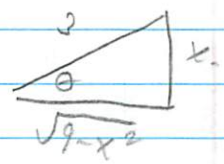
$$\cot\left(\sin^{-1}\left(\frac{x}{3}\right)\right) - \sin^{-1}\left(\frac{x}{3}\right) + C$$

$$\theta = \sin^{-1}\left(\frac{x}{3}\right)$$

← trig on trig inverse bad

$\text{trig}(\text{trig}^{-1}(x)) \rightarrow$ simplify

$$\cot\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$$



$$\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

$$\sin(\theta) = \frac{x}{3}$$

$$\theta = \sin^{-1}\left(\frac{x}{3}\right)$$

? full credit :)

$$\cot(\theta) = \frac{\sqrt{9-x^2}}{x}$$

General Rules

$\sqrt{a^2-x^2} \rightarrow x = a\sin\theta$

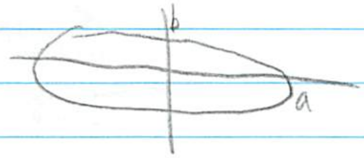
use $\sin^2\theta+\cos^2\theta = 1$

$\sqrt{a^2+x^2} \rightarrow x = a\tan\theta$

$\tan^2\theta + 1 = \sec^2\theta$

$\sqrt{x^2-a^2} \rightarrow x = a\sec\theta$

Can find area of ellipse



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sanity ✓ circle is elipse where $A = B$

If $a = b$ → then area elipse is just area of a circle $= \pi a^2$

In general we guess area $= \pi ab$

Check

$$4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

very hard + confusing - like everything is in begining

$$4\frac{b}{a}\int_0^{\frac{\pi}{2}} \sqrt{a^2-a^2\sin^2\theta}\; a\cos\theta\, d\theta$$

$$4ab\int_0^{\pi/2} \cos^2\theta\, d\theta$$

half angle identity

$$\boxed{\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)}$$

$$2ab \cdot \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$2ab\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_{\theta=0}^{\theta=\pi/2}$$

$$2ab\left(\frac{\pi}{2} + \frac{\sin 2\theta}{2}\right)$$

conic sections

$$2ab\left(\frac{\pi}{2}+0\right) = \pi ab$$

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example

$$\int \frac{x}{\sqrt{3-2x-x^2}}\, dx$$

complete the square

x^2+2x-3

$$x^2+2x-3 = \underbrace{(x+1)^2}_{x^2+2x+1} + k$$

$k+1=-3$

$k=-4$

$(x+1)^2-4$

$$\int \frac{x}{\sqrt{4-(x+1)^2}}\, dx$$

$u = x+1$

$du = dx$

$$\int \frac{u-1}{\sqrt{4-u^2}}\, du$$

$u = 2\sin\theta$

$$\int \frac{u}{\sqrt{4-u^2}}\, du - \int \frac{1}{\sqrt{4-u^2}}$$

← usual trig sub

$u = 2\sin\theta$

$du = 2\cos\theta\, d\theta$

↑ regular old substitution

$w = 4-u^2$

$dw = -2u\,du$

easy

$$\int \frac{x^3}{(4x^2+9)^{3/2}}\,dx$$

↑ cubing square root $\left(\sqrt{4x^2+9}\right)^3$

one way

$$2x = 3\tan\theta$$
$$4x^2 = 9\tan^2\theta$$

$$\int \frac{x^3}{\left(2\sqrt{x^2+9/4}\right)^3} \qquad x = \frac{3}{2}\tan\theta$$

Fashionable in 19th century to see what functions arose by intergrating $\sqrt{}$

$\sqrt{\text{cubics}}$ = really hard, eliptic curve

Lecture 28: Integration by Inverse Substitution; Completing the Square

Trigonometric Substitutions, continued

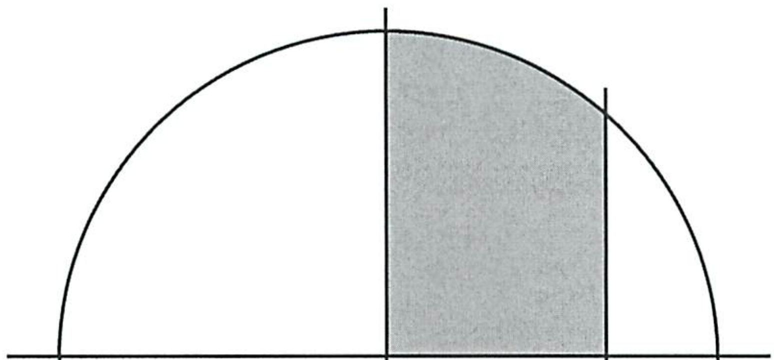


Figure 1: Find area of shaded portion of semicircle.

$$\int_0^x \sqrt{a^2 - t^2}\,dt$$

$$t = a\sin u; \quad dt = a\cos u\,du$$

$$a^2 - t^2 = a^2 - a^2\sin^2 u = a^2\cos^2 u \implies \sqrt{a^2 - t^2} = a\cos u \quad \text{(No more square root!)}$$

Start: $x = -a \Leftrightarrow u = -\pi/2$; Finish: $x = a \Leftrightarrow u = \pi/2$

$$\int \sqrt{a^2 - t^2}\,dt = \int a^2\cos^2 u\,du = a^2 \int \frac{1+\cos(2u)}{2}du = a^2\left[\frac{u}{2} + \frac{\sin(2u)}{4}\right] + c$$

$$\text{(Recall, } \cos^2 u = \frac{1+\cos(2u)}{2}\text{).}$$

We want to express this in terms of x, not u. When $t = 0$, $a\sin u = 0$, and therefore $u = 0$. When $t = x$, $a\sin u = x$, and therefore $u = \sin^{-1}(x/a)$.

$$\frac{\sin(2u)}{4} = \frac{2\sin u\cos u}{4} = \frac{1}{2}\sin u\cos u$$

$$\sin u = \sin\left(\sin^{-1}(x/a)\right) = \frac{x}{a}$$

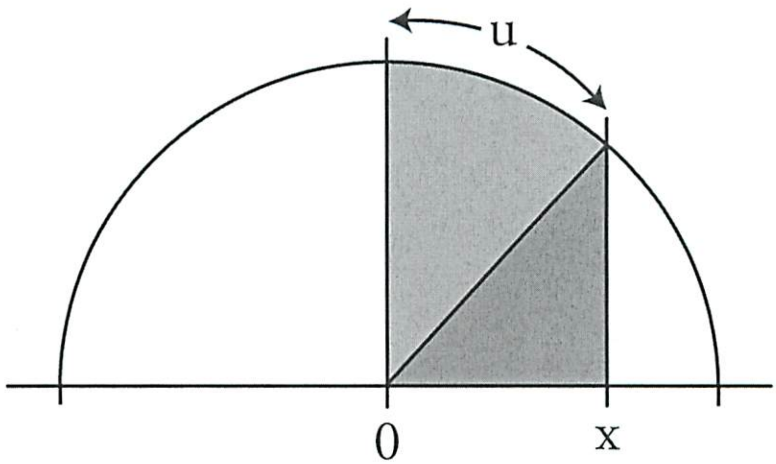


Figure 3: Area divided into a sector and a triangle.

Here is a list of integrals that can be computed using a trig substitution and a trig identity.

integral	substitution	trig identity
$\int \frac{dx}{\sqrt{x^2+1}}$	$x = \tan u$	$\tan^2 u + 1 = \sec^2 u$
$\int \frac{dx}{\sqrt{x^2-1}}$	$x = \sec u$	$\sec^2 u - 1 = \tan^2 u$
$\int \frac{dx}{\sqrt{1-x^2}}$	$x = \sin u$	$1 - \sin^2 u = \cos^2 u$

Let's extend this further. How can we evaluate an integral like this?

$$\int \frac{dx}{\sqrt{x^2+4x}}$$

When you have a linear and a quadratic term under the square root, complete the square.

$$x^2 + 4x = (\text{something})^2 \pm \text{constant}$$

In this case,

$$(x+2)^2 = x^2 + 4x + 4 \implies x^2 + 4x = (x+2)^2 - 4$$

Now, we make a substitution.

$$v = x + 2 \quad \text{and} \quad dv = dx$$

Plugging these in gives us

$$\int \frac{dx}{\sqrt{(x+2)^2 - 4}} = \int \frac{dv}{\sqrt{v^2-4}}$$

Now, let

$$v = 2\sec u \quad \text{and} \quad dv = 2\sec u \tan u$$

$$\int \frac{dv}{\sqrt{v^2-4}} = \int \frac{2\sec u \tan u \, du}{2\tan u} = \int \sec u \, du$$

How can we find $\cos u = \cos\left(\sin^{-1}(x/a)\right)$? Answer: use a right triangle (Figure 2).

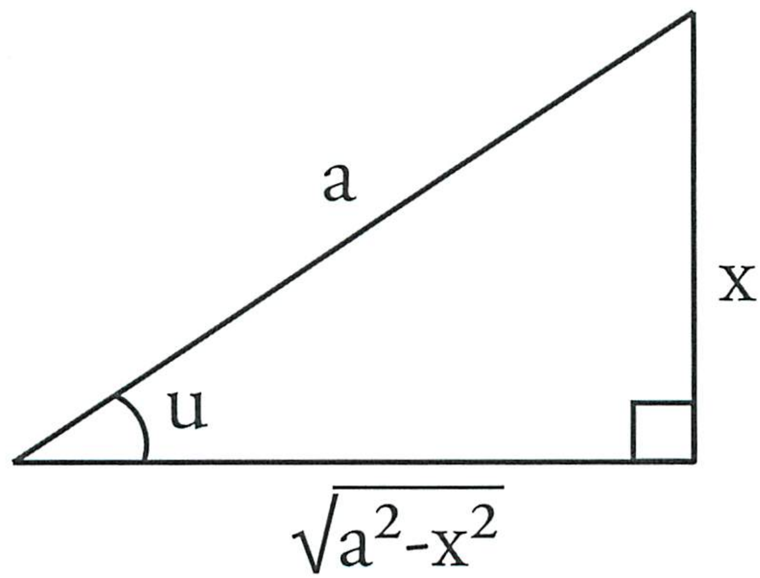


Figure 2: $\sin u = x/a$; $\cos u = \sqrt{a^2 - x^2}/a$.

From the diagram, we see

$$\cos u = \frac{\sqrt{a^2 - x^2}}{a}$$

And finally,

$$\int_0^x \sqrt{a^2 - t^2}\, dt = a^2\left[\frac{u}{4} + \frac{1}{2}\sin u \cos u\right] - 0 = a^2\left[\frac{\sin^{-1}(x/a)}{2} + \frac{1}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2 - x^2}}{a}\right]$$

$$\int_0^x \sqrt{a^2 - t^2}\, dt = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2 - x^2}$$

When the *answer* is this complicated, the route to getting there has to be rather complicated. There's no way to avoid the complexity.

Let's double-check this answer. The area of the upper shaded sector in Figure 3 is $\frac{1}{2}a^2 u$. The area of the lower shaded region, which is a triangle of height $\sqrt{a^2 - x^2}$ and base x, is $\frac{1}{2}x\sqrt{a^2 - x^2}$.

Remember that
$$\int \sec u \, du = \ln(\sec u + \tan u) + c$$
Finally, rewrite everything in terms of x.
$$v = 2\sec u \Leftrightarrow \cos u = \frac{2}{v}$$
Set up a right triangle as in Figure 4. Express $\tan u$ in terms of v.

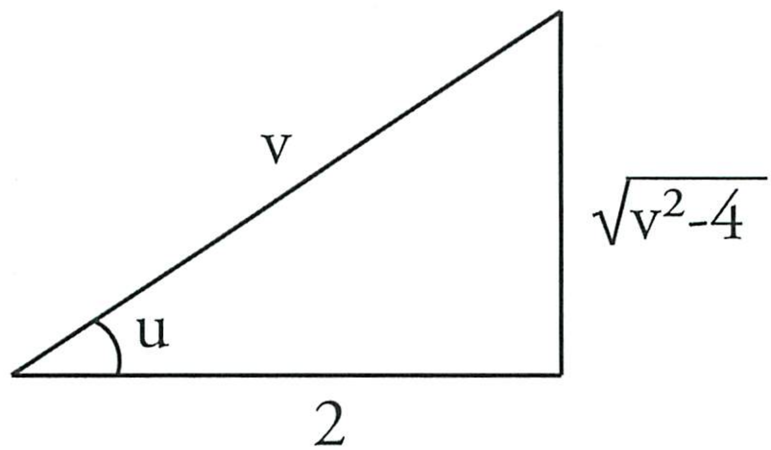


Figure 4: $\sec u = v/2$ or $\cos u = 2/v$.

Just from looking at the triangle, we can read off
$$\sec u = \frac{v}{2} \quad \text{and} \quad \tan u = \frac{\sqrt{v^2-4}}{2}$$
$$\int 2\sec u \, du = \ln\left(\frac{v}{2} + \frac{\sqrt{v^2-4}}{2}\right) + c$$
$$= \ln(v + \sqrt{v^2-4}) - \ln 2 + c$$
We can combine those last two terms into another constant, $\tilde{c}$.
$$\int \frac{dx}{\sqrt{x^2+4x}} = \ln(x + 2 + \sqrt{x^2+4x}) + \tilde{c}$$

Here's a teaser for next time. In the next lecture, we'll integrate all rational functions. By "rational functions," we mean functions that are the ratios of polynomials:
$$\frac{P(x)}{Q(x)}$$

It's easy to evaluate an expression like this:
$$\int \left(\frac{1}{x-1} + \frac{3}{x+2}\right) dx = \ln|x-1| + 3\ln|x+2| + c$$

If we write it a bit differently, however, it becomes much harder to integrate:

$$\frac{1}{x-1} + \frac{3}{x+2} = \frac{x+2+3(x-1)}{(x-1)(x+2)} = \frac{4x-1}{x^2+x-2}$$

$$\int \frac{4x-1}{x^2+x-2} = ???$$

How can we reorganize what to do starting from $(4x-1)/(x^2+x-2)$? Next time, we'll see how. It involves some algebra.

Recitation

11/16/09

1. $\int \frac{dx}{\sqrt{4+x^2}}$ get to $\int f(\theta)\, d\theta$

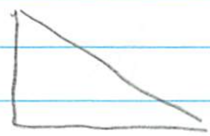
no x outside
- no u-sub

opposed →

$\tan\theta = \frac{x}{2}$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$



try sub in for x

denom $\frac{\sqrt{4+(2\tan\theta)^2}}{\sqrt{4+4\tan^2\theta}}$ or $\int \frac{2\sec^2\theta\, d\theta}{\sqrt{4+4\tan^2\theta}}$

need trig identity
$1+\tan^2\theta = \sec^2\theta$

$\int \frac{2\sec^2\theta\, d\theta}{\sqrt{4(1+\tan^2\theta)}}$

$\int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}}$

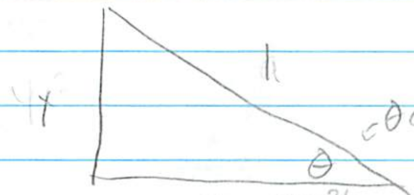
$\int \frac{\cancel{2}\sec^2\theta\, d\theta}{\cancel{2}\sec\theta}$

$\int \sec\theta\, d\theta$

↑ found in class + evaluate
at very end lecture 26

2. $\int \frac{4x^2}{(1-x^2)^{3/2}}\,dx$ get $g(x)$

← both ² would not work



$\sqrt{1-x^2}^{\,3/2}$ ← know $\sqrt{a+x}$ ← $\sqrt{a^2+b^2}$ $a^2+b^2=c^2$

↑ knew it had to be c so hyp

$x = \sin\theta$ ← $\sin\theta = \frac{x}{1}$

$dx = \cos\theta\,d\theta$

$\int \frac{4(\sin\theta)^2 \cos\theta\,d\theta}{(1-\sin^2\theta)^{3/2}}$

↑ trig identity

$\int \frac{4\sin^2\theta \cos\theta\,d\theta}{\cos^2\theta^{\,3/2}}$ $\left(\sqrt{\cos^2\theta}\right)^3$

↑ $\frac{2}{1}\cdot\frac{3}{2} = \cos^3\theta$ ↑ or

$\int \frac{4\sin^2\theta \cancel{\cos\theta}\,d\theta}{\cos^{\cancel{3}2}\theta}$

$\int \frac{4\sin^2\theta\,d\theta}{\cos^2\theta} = \int 4\tan^2\theta\,d\theta$

$\int 4(\sec^2\theta - 1)\,d\theta$

$4(\tan\theta - \theta)$

↑ plug in for θ - find tan θ in terms of x

- use triangle

$\tan\theta = \frac{x}{\sqrt{1-x^2}}$

$4\left(\frac{x}{\sqrt{1-x^2}} - \arcsin(x)\right)$ ← in terms of x

↑ know $x = \sin\theta$

so: $\arcsin x = \theta$

Sidebar

$\int x\sqrt{1-x}\,dx$

- not trig sub
- no $\sqrt{1-x^2}$
- could do u sub $u = 1-x$

$du = -dx$

$x = 1-u$

$-\int (1-u)\,u^{1/2}\,du$

$-\int u^{1/2} - u^{3/2}\,du$

Lecture 29

Integrate any Rational Function 11/17

rational function $\frac{P(x)}{Q(x)}$ w/ P,Q polynomials

ex $\int \frac{x^3+x}{x-1}\,dx$

- is a division algorithm for polynomials with the remainder having a smaller degree than denominator

Step 0

Long division

$$\begin{array}{r} x^2+x+2 \\ x-1 \overline{)\, x^3+x } \\ -(x^3+x) \\ x^2+x \\ -(x^2+x) \\ 2x \\ (2x-2) \\ 2 \end{array}$$

just looking at x

← multiply top by side (divisor)

what you get on for subtraction

$x^2+x+2+\frac{2}{x-1}$

Back to example

$$\int (x^2+x+2)+\left(\frac{2}{x-1}\right)dx$$

$$\frac{1}{3}x^3+\frac{1}{2}x^2+2x+2\ln|x-1|+C$$

(Division $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$

degree $R(x) <$ degree $Q(x)$

Example 2 $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

Choose $u(x) = 2x^3+3x^2-2x$

$du = 6x^2+6x-2$ ← 1 in a million chance this will work

<u>Step 1</u> Factor denominator as much as possible

- Step 2 depends on how nicely denom factors

Factor $2x^3+3x^2-2x$

- can't use cubic formula

$x(2x^2+3x-2)$

$x(2x-1)(x+2)$

<u>Step 2 Case 1</u>

If denominator clearly factors w/ no repeated linear factors

↑ bad $\boxed{x\underbrace{(2x-1)^2}_{\text{no good}}(x+2)}$

clever idea

Write $\dfrac{x^2+2x-1}{x(2x-1)(x+2)} = \dfrac{A}{x} + \dfrac{B}{2x-1} + \dfrac{C}{x+2}$

How to solve for A, B, C?

- make a common denominator
- set resulting numerators equal

$$x^2+2x-1 = A(2x-1)(x+2) + B(x)(x+2) + C(x)(2x-1)$$

equality of 2 polynomials

Not from Lecture

~~test doing algebra~~ $\boxed{\text{on own } 11/18}$ 11/18

$$x^2 + 2x - 1 = (2x^2 + 4x - x - 2)A + (x^2 + 2x)B + (2x^2 - x)C$$

$x^2)$ $1 = 2A + B + 2C$

$x)$ $2 = 3A + 2B - C$ ✓ ~~Worked out~~

$1)$ $-1 = -2A$

$A = \frac{1}{2}$

$1 = 2(\frac{1}{2}) + B + 2C$

$0 = B + 2C$

$2 = 3(\frac{1}{2}) + 2B - C$

$\frac{1}{2} = 2B - C$

$1 = 4B - 2C$

$B + 2C = 4B - 2C$

$-B \quad +2C \quad -B \quad +2C$

$4C = 3B$

$2 = 4A + 2B + 4C$

$2 = 4(\frac{1}{2}) + 2B + 3B$

$0 = 5B$

$B = 0$

$0 = 2B + 4C$

$0 = 2B + 3B$

$0 = 5B$

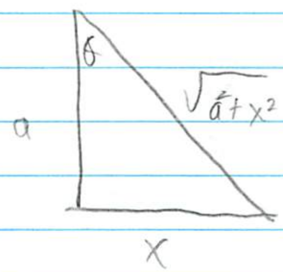
what now

$C = 0$

i guess that right

can do something on computer

3. $\int x^2 \sqrt{a^2+x^2}\,dx$ to $\int f(\theta)\,d\theta$



think $x = a\tan\theta$

$\theta = \tan\frac{x}{a}$

$x = a\tan\theta$

$dx = a\sec^2\theta\,d\theta$

Do some expanding
(skipping algebra)

$$(2A+B+2C)x^2+(3A+2B-C)x-2A$$
$$=x^2+2x-1$$

2 polynomials are = only if all coefficents are =

$2A+B+2C=1$
$3A+2B-C=2$) system of equations to solve
$-2A=-1$ 3 equations w/ 3 unknowns

↑ 3 linearly independent algebraic equations w/ unique solutions (thats why no repeated linear factors)

Short cut (Heaviside's Cover up Method) For case 1

$$\frac{x^2+2x-1}{x(2x-1)(x+2)}=\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}$$

Multiply both sides by $x+2$

$$\frac{(x^2+2x-1)}{x(2x-1)}=\frac{A(x+2)}{x}+\frac{B(x+2)}{2x-1}+C$$

Plug in $x=-2$

$$C=\frac{(-2)^2+2(-2)-1}{(-2)(-4-1)}=\boxed{-\frac{1}{10}}$$

Repeat for $(2x-1)$
Plug in $x=\frac{1}{2}$
...

Case 2 Denominator factors into linear roots (with some repeating)

$$\int \frac{x^3 - x + 1}{x^2(x-1)^3}\,dx$$

New clever idea

$$\frac{x^3 - x + 1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

each power of repeated root up to max
would be 4 terms if
$x^2(x-1)^7$

Can get some things from Heavysides'
- in this case B and E

then must common denom
3 eq w/ 3 unknowns

was 5 originally
but 2 w/ Heavysides

$$\frac{x^2+2x}{(x-1)(x+3)^2}$$ - don't need step 0

$$\frac{A}{(x-1)} + \frac{B}{(x+3)} + \frac{C}{(x+3)^2}$$

Case 3 - Can't factor denominator into linear factors

How bad can it be?

Fundemental Theorm of Algebra

- over complex number $(a+b\sqrt{-1})$ a and b real
- every polynomial factors completely into linear factors

If $a+bi$ is a root of $Q(x)$. What else is a root?
- So is $a-bi$ (complex conjugate - comes in pairs)
- $Q(x)$ has real coefficents
  - nothing imaginory left since $\sqrt{-1}\sqrt{-1}=-1$
- can always factor into linear + quadratic pieces

$$(x-(a+bi))(x-(a-bi))$$

Example

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx$$

$$\frac{\text{top}}{x(x^2+4)}$$ ← real factors only

$$\boxed{\frac{A}{x}+\frac{B}{x^2}=\frac{Ax+B}{x^2}}$$

Clever idea #3 $\frac{2x^2-x+4}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$

coverup
or system linear equations

Case 4 Repeated Quadratics

- what you think it is

If can find quadratics (hard) then you can solve anything

$$\frac{A}{x}+\frac{Bx+C}{x^2+4}+\frac{Dx+E}{(x^2+4)^2}$$

Can we integrate complex things?

$$\int \frac{1-x+2x^2-x^3}{x\,(x^2+1)^2}\,dx$$

$$\frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2} \right) dx$$

split in 2 integrals

$$\int \frac{1}{x}\,dx - \int \frac{x}{x^2+1} - \int \frac{1}{x^2+1} + \int \frac{x}{(x^2+1)^2}\,dx$$

$\ln x$	u sub	$\tan^{-1}(x)$	u-sub
	$u(x) = x^2+1$	use tan	$w = x^2+1$
	$du = 2x\,dx$	substitution	$dw = 2x\,dx$

Can now integrate every Rational function

Lecture 29: Partial Fractions

We continue the discussion we started last lecture about integrating rational functions. We defined a rational function as the ratio of two polynomials:

$$\frac{P(x)}{Q(x)}$$

We looked at the example

$$\int \left[\frac{1}{x-1} + \frac{3}{x+2}\right] dx = \ln|x-1| + 3\ln|x+2| + c$$

That same problem can be disguised:

$$\frac{1}{x-1} + \frac{3}{x+2} = \frac{(x+2)+3(x-1)}{(x-1)(x+2)} = \frac{4x-1}{x^2+x-2}$$

which leaves us to integrate this:

$$\int \frac{4x-1}{x^2+x-2}\, dx = ???$$

Goal: we want to figure out a systematic way to split $\frac{P(x)}{Q(x)}$ into simpler pieces.

First, we factor the denominator $Q(x)$.

$$\frac{4x-1}{x^2+x-2} = \frac{4x-1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

There's a slow way to find A and B. You can clear the denominator by multiplying through by $(x-1)(x+2)$:

$$(4x-1) = A(x+2) + B(x-1)$$

From this, you find

$$4 = A + B \quad \text{and} \quad -1 = 2A - B$$

You can then solve these simultaneous linear equations for A and B. This approach can take a very long time if you're working with 3, 4, or more variables.

There's a faster way, which we call the "cover-up method". Multiply both sides by $(x-1)$:

$$\frac{4x-1}{x+2} = A + \frac{B}{x+2}(x-1)$$

Set $x = 1$ to make the B term drop out:

$$\frac{4-1}{1+2} = A$$

$$A = 1$$

The fastest way is to do this in your head or physically *cover up* the struck-through terms. For instance, to evaluate B:

$$\frac{4x-1}{(x-1)\cancel{(x+2)}} = \frac{\cancel{A}}{\cancel{x}-1} + \frac{B}{\cancel{(x+2)}}$$

Implicitly, we are multiplying by $(x+2)$ and setting $x=-2$. This gives us

$$\frac{4(-2)-1}{-2-1} = B \quad \Longrightarrow \quad B = 3$$

What we've described so far works when $Q(x)$ factors completely into *distinct* factors and the degree of P is less than the degree of Q.

If the factors of Q repeat, we use a slightly different approach. For example:

$$\frac{x^2+2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

Use the cover-up method on the highest degree term in $(x-1)$.

$$\frac{x^2+1}{x+2} = B + [\text{stuff}](x-1)^2 \quad \Longrightarrow \quad \frac{1^2+2}{1+2} = B \quad \Longrightarrow \quad B = 1$$

Implicitly, we multiplied by $(x-1)^2$, then took the limit as $x \to 1$.

C can also be evaluated by the cover-up method. Set $x=-2$ to get

$$\frac{x^2+2\ ^2}{(x-1)} = C + [\text{stuff}](x+2) \quad \Longrightarrow \quad \frac{(-2)^2+2}{(-2-1)^2} = C \quad \Longrightarrow \quad C = \frac{2}{3}$$

This yields

$$\frac{x^2+2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{1}{(x-1)^2} + \frac{2/3}{x+2}$$

Cover-up can't be used to evaluate A. Instead, plug in an easy value of x: $x=0$.

$$\frac{2}{(-1)^2(2)} = \frac{A}{-1} + 1 + \frac{1}{3} \Longrightarrow 1 = 1 + \frac{1}{3} - A \Longrightarrow A = \frac{1}{3}$$

Now we have a complete answer:

$$\frac{x^2+2}{(x-1)^2(x+2)} = \frac{1}{3(x-1)} + \frac{1}{(x-1)^2} + \frac{2}{3(x+2)}$$

Not all polynomials factor completely (without resorting to using complex numbers). For example:

$$\frac{1}{(x^2+1)(x-1)} = \frac{A_1}{x-1} + \frac{B_1x+C_1}{x^2+1}$$

We find A_1, as usual, by the cover-up method.

$$\frac{1}{1^2+1} = A_1 \quad \Longrightarrow \quad A_1 = \frac{1}{2}$$

Now, we have

$$\frac{1}{(x^2+1)(x-1)} = \frac{1/2}{x-1} + \frac{B_1x+C_1}{x^2+1}$$

Plug in $x=0$.

$$\frac{1}{1(-1)} = -\frac{1}{2} + \frac{C_1}{1} \quad \Longrightarrow \quad C_1 = -\frac{1}{2}$$

Now, plug in any value other than $x = 0, 1$. For example, let's use $x = -1$.

$$\frac{1}{2(-2)} = \frac{1/2}{-2} + \frac{B_1(-1)-1/2}{2} \Longrightarrow 0 = -\frac{B_1 - 1/2}{2} \Longrightarrow B_1 = -\frac{1}{2}$$

Alternatively, you can multiply out to clear the denominators (not done here).

Let's try to integrate this function, now.

$$\int \frac{dx}{(x^2+1)(x-1)} = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{x\,dx}{x^2+1} - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln|\,x^2+1\,| - \frac{1}{2}\tan^{-1}x + c$$

What if we're faced with something that looks like this?

$$\int \frac{dx}{(x-1)^{10}}$$

This is actually quite simple to integrate:

$$\int \frac{dx}{(x-1)^{10}} = -\frac{1}{9}(x-1)^{-9} + c$$

What about this?

$$\int \frac{dx}{(x^2+1)^{10}}$$

Here, we would use trig substitution:

$$x = \tan u \quad \text{and} \quad dx = \sec^2 u du$$

and the trig identity

$$\tan^2 u + 1 = \sec^2 u$$

to get

$$\int \frac{\sec^2 u\,du}{(\sec^2 u)^{10}} = \int \cos^{18} u\,du$$

From here, we can evaluate this integral using the methods we introduced two lectures ago.

Recitation

11/18

11 min late

$$\int \frac{x}{(x+1)^2+1}\,dx = \int \frac{u-1}{u^2+1}\,du = \int \frac{u}{u^2+1}\,du - \frac{1}{u^2+1}\,du$$

$$\frac{\sim}{(x+1)^2(x^2+2)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{(x^2+2)^2} + \frac{Gx+H}{(x^2+2)^3}$$

Heavysides

$$\frac{6}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

multiply through by denominator
kill off a numerator (set = 0)

A and B must be true for every x

(x^2+2) never 0

Multiply both sides by $x-1$

$$\frac{6}{x+1} = A + \frac{B(x-1)}{x+1}$$

Pick any x to solve for A + B
Best is to make other term disappear
Set $x=1$, solve for A

$$6 = A(x+1) + B(x-1)$$

simplify right hand side
sum of numerators

make $x=1$ $A=3$
$x=-1$ $B=-3$

$$\int \frac{6}{x^2-1} = \int \frac{3}{x-1}\,dx - \int \frac{3}{x+1}\,dx$$

now can integrate w/ ln

Sometimes cover up does not work
So solve w/ system of equations

$$6 = A(x+1) + B(x-1)$$
$$6 = Ax + A + Bx - B$$

polynomial

$$6 = (A+B)x + (A-B)$$

$0 = A + B$ (x - 1st order on each side) → solve $A = 3$
$6 = A - B$ (0 order on both sides) $B = -3$

$$x^2 + 6x + 3 = Ax(x+1) + B(x^2+2)$$
$$= Ax^2 + Ax + Bx^2 + 2B$$

won't work out

Sidebar when does not work

3 types when have denominator of linear + quadratics
1. ln problems
2. u sub
3. trig sub

---

1 $\int \frac{2x-3}{x^3+x}\,dx$ no nice u, no ln

$$\frac{2x-3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\frac{2x-3}{x^2+1} = A + \frac{(Bx+C)x}{x^2+1}$$ set $x = 0$, so that disappears

$A = -3$

$$2x-3 = A(x^2+1) + (Bx+c)x$$

$$2x-3 = Ax^2 + A + Bx^2 + Cx$$

$\uparrow -3 \quad \uparrow -3$

$$2x-3 = -3x^2 - 3 + Bx^2 + cx$$

$$3x^2 + 2x = Bx^2 + cx$$

$B = 3$ equations are coefficents

$C = 2$

$$\cancel{2x-3 = -3(x^2+1) + (3x+2)x}$$

$$\cancel{2x-3}$$

$$\int \frac{-3}{x} + \int \frac{3x+2}{x^2+1}$$

$$-3\ln x + \int \frac{3x}{x^2+1}dx + \int \frac{2}{x^2+1}dx$$

$$-3\ln x + \quad u = x^2+1 \quad + 2\int \frac{1}{x^2+1}$$

$$du = 2x\,dx$$

$$\int \frac{\frac{3}{2}}{u} du \quad + \text{ trig!, should have memorized}$$

$$-3\ln x + \frac{3}{2u}du + 2\tan^{-1}(x)$$

$$-3\ln x + \frac{3}{2}\ln u + 2\tan^{-1}(x)$$

$$-3\ln x + \frac{3}{2}\ln(x^2+1) + 2\tan^{-1}(x)$$

Lecture 30
Integration by Parts

11/18

P-set 7 due tomorrow

<u>Today</u> Last day of integration
so test soon → 4 techniques + 1 or 2
on completly diff way of representing functions
– parametric equations

Pset 8 is last one

$\int x \sin x \, dx$

– use integration by parts – new method
– uses product rule

$$\frac{d}{dx} f(x) g(x) = f'(x) + g(x) = f(x) g'(x)$$

$\int dx \qquad \int dx$

$$\int \frac{d}{dx} f(x) g(x) dx = \int f'(x) g(x) \, dx + \int f(x) g'(x) \, dx$$

$f(x) g(x)$

these 2 integrals are related
if can solve one, can solve the other
– just must solve one

$$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) \, dx$$

or $f(x) = u \qquad g(x) = v$
$f'(x) dx = du \qquad g'(x) = dv$

$$\boxed{\int u \, dv = u \cdot v - \int v \, du}$$

In our example: $x \sin x$

choose

$f(x) = \sin x$ $\quad$ $g'(x) = x$

differentiate $\quad$ integrate

$f'(x) = \cos x$ $\quad$ $g(x) = \frac{x^2}{2}$

$$\int x \sin x \, dx = \frac{1}{2} x^2 \sin x - \int \frac{1}{2} x^2 \cos x \, dx$$

if want to solve $\quad$ solve this

but this is worse (since $\frac{x^2}{2}$)

so try other

$f(x) = x$ $\quad$ $g'(x) = \sin x$

$f'(x) = 1$ $\quad$ $g(x) = -\cos x$

$$\int x \sin x \, dx = -x\cos x - \int (-\cos x) \, dx$$

$$= \boxed{-x\cos x + \sin x + C}$$

If definite integral

$$f(b)g(b) - f(a)g(a) = \int_a^b f'(x) g(x) \, dx + \int_a^b f(x) g'(x) \, dx$$

$$\int_a^b u \, dv = uv\Big|_a^b - \int_a^b v \, du$$

example 2

$$\int \ln x \, dx = \int \ln(x) \cdot 1 \, dx$$

does have 2 parts

$u = \ln x$ $\quad$ $dv = 1\,dx$

$du = \frac{1}{x}$ $\quad$ $v = x$

$$\ln x \cdot x - \int x \tfrac{1}{x} \, dx$$

$$x \ln x - x + c$$

$$\int e^t \cdot t^2 \, dt$$

should get simpler $\quad$ must be able to integrate

$u = t^2$ $\quad$ $dv = e^t \, dt$

$du = 2t\,dt$ $\quad$ $v = e^t$

$$t^2 e^t - \int e^t \cdot 2t \, dt$$

can't integrate immedity

do it again (integrate by parts)

$u = 2t$ $\quad$ $dv = e^t dt$

$du = 2\,dt$ $\quad$ $v = e^t$

differentials

$$2t e^t - \int 2 e^t \, dt$$

$$2te^t - 2e^t + c$$

Finish ...

Bad $\int \tan^n x \sec^m x \, dx$ n even, m odd

Convert to a sum of products of pure $\sec^m x$

$\tan^n x = (\sec^2 x - 1)^{n/2}$

Remains to integrate $\int \sec^m x \, dx$ m odd

$\int \sec x \, dx \to$ sneaky trick $\to \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx$

$= \ln|\sec x + \tan x| + C$

$\int \sec^3 x \, dx \to \int \sec^2 x \sec x \, dx$

~~$u = \sec x \quad dv = \sec^2 x \, dx$~~

~~$du = \sec x \tan x \quad v = \tan x$~~

~~$\int \sec^3 x \, dx = \tan x \sec x - \int \sec x \tan^2 x \, dx$~~

~~dead end~~ $\longrightarrow$ actually not, next page

~~$u = \sec^3 x \quad dv = dx$~~

~~$du = 3\sec^2 x (\sec x \tan x) \quad v = x$~~

~~chain rule~~

way worse

~~$u = \sec^2 x \quad dv = \sec x \, dx$~~

~~$du = 2\sec^2 x \tan x \, dx$~~

nope

$$\cancel{\int \sec^2 x \sec x \, dx}$$

$$\cancel{\int (\tan^2 x + 1) \sec x \, dx}$$

$$\cancel{\ln|\sec x + \tan x| + \int \tan^2 x \sec x \, dx}$$

$$\int \sec^3 x \, dx = \tan x \sec x - \int \tan^2 x \sec x \, dx$$

$$\int (\sec^2 x - 1) \sec x \, dx$$

- look like circle
- but opposet sign

$$\int \sec^3 x \, dx = \tan x \sec x - \left(\int \sec^3 x \, dx - \int \sec x \, dx \right)$$

$$2 \int \sec^3 x \, dx = \tan x \sec x + \int \sec x \, dx$$

↓ memorized

$$\int \sec^3 x \, dx = \frac{1}{2} \left(\tan x \sec x + \ln|\sec x + \tan x| \right) + C$$

$\int e^x \sin x + C$ → do integration by parts twice and solve like in $\int \sec^3 x \, dx$ example

Thats all the functions
- know which method to use

- Parts - have deriv and $\int$ that are no harder
- 1 - mix of trig/exp function w/ polynomial

In our class only consider elementry functions

- built from e^x, $\ln x$, sin, cos, x
- build up w/ $+, -, \times, \div$, inverses, etc.
- you will get elementry function back when deriv
- <u>but</u> not always when integrate

<u>Do not try to integrate</u>

$$e^{x^2},\ \frac{e^x}{x},\ \sin(x^2),\ \cos(e^x),\ \sqrt{x^3+1},\ \frac{1}{\ln x},\ \frac{\sin x}{x}$$

Lecture 30: Integration by Parts, Reduction Formulae

Integration by Parts

Remember the product rule:

$$(uv)' = u'v + uv'$$

We can rewrite that as

$$uv' = (uv)' - u'v$$

Integrate this to get the formula for integration by parts:

$$\int uv'\,dx = \boxed{uv - \int u'v\,dx}$$

Example 1. $\int \tan^{-1} x\,dx$.
At first, it's not clear how integration by parts helps. Write

$$\int \tan^{-1} x\,dx = \int \tan^{-1} x(1 \cdot dx) = \int uv'\,dx$$

with

$$u = \tan^{-1} x \quad \text{and} \quad v' = 1.$$

Therefore,

$$v = x \quad \text{and} \quad u' = \frac{1}{1+x^2}$$

Plug all of these into the formula for integration by parts to get:

$$\int \tan^{-1} x\,dx = \int uv'\,dx = (\tan^{-1} x)x - \int \frac{1}{1+x^2}(x)dx$$

$$= x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + c$$

Alternative Approach to Integration by Parts

As above, the product rule:

$$(uv)' = u'v + uv'$$

can be rewritten as

$$uv' = (uv)' - u'v$$

This time, let's take the *definite* integral:

$$\int_a^b uv'\,dx = \int_a^b (uv)'\,dx - \int_a^b u'v\,dx$$

By the fundamental theorem of calculus, we can say

$$\int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b u'v\,dx$$

Another notation in the indefinite case is

$$\int u\,dv = uv - \int v\,du$$

This is the same because

$$dv = v'\,dx \implies uv'\,dx = u\,dv \quad \text{and} \quad du = u'\,dx \implies u'v\,dx = vu'\,dx = v\,du$$

Example 2. $\int (\ln x)dx$

$$u = \ln x;\ du = \frac{1}{x}dx \quad \text{and} \quad dv = dx;\ v = x$$

$$\int (\ln x)dx = x\ln x - \int x\left(\frac{1}{x}\right)dx = x\ln x - \int dx = x\ln x - x + c$$

We can also use "advanced guessing" to solve this problem. We know that the derivative of *something* equals $\ln x$:

$$\frac{d}{dx}(??) = \ln x$$

Let's try

$$\frac{d}{dx}(x\ln x) = \ln x + x\cdot\frac{1}{x} = \ln x + 1$$

That's almost it, but not quite. Let's repair this guess to get:

$$\frac{d}{dx}(x\ln x - x) = \ln x + 1 - 1 = \ln x$$

Reduction Formulas (Recurrence Formulas)

don't think we did

Example 3. $\int (\ln x)^n\,dx$

Let's try:

$$u = (\ln x)^n \implies u' = n(\ln x)^{n-1}\left(\frac{1}{x}\right)$$

$$v' = dx;\ v = x$$

Plugging these into the formula for integration by parts gives us:

$$\int (\ln x)^n dx = x(\ln x)^n - \int n(\ln x)^{n-1}\cancel{x\left(\frac{1}{x}\right)}^{1}dx$$

Keep repeating integration by parts to get the full formula: $n \to (n-1) \to (n-2) \to (n-3) \to$ etc

Example 4. $\int x^n e^x\,dx$ Let's try:

$$u = x^n \implies u' = nx^{n-1};\quad v' = e^x \implies v = e^x$$

Putting these into the integration by parts formula gives us:

$$\int x^n e^x \, dx = x^n e^x - \int n x^{n-1} e^x \, dx$$

Repeat, going from $n \to (n-1) \to (n-2) \to$ etc.

Bad news: If you change the integrals just a little bit, they become impossible to evaluate:

$$\int \left(\tan^{-1} x\right)^2 dx = \text{impossible}$$

$$\int \frac{e^x}{x} \, dx = \text{also impossible}$$

Good news: When you can't evaluate an integral, then

$$\int_1^2 \frac{e^x}{x} dx$$

is an *answer*, not a question. This *is* the solution– you don't have to integrate it!

The most important thing is setting up the integral! (Once you've done that, you can always evaluate it numerically on a computer.) So, why bother to evaluate integrals by hand, then? Because you often get families of related integrals, such as

$$F(a) = \int_1^\infty \frac{e^x}{x^a} dx$$

where you want to find how the answer depends on, say, a.

Arc Length

This is very useful to know for 18.02 (multi-variable calculus).

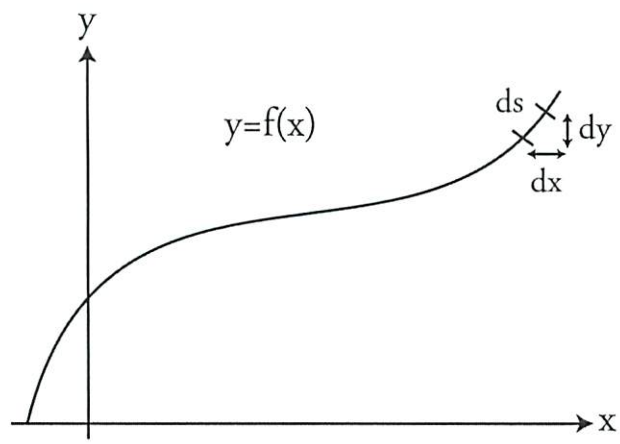


Figure 1: Infinitesimal Arc Length ds

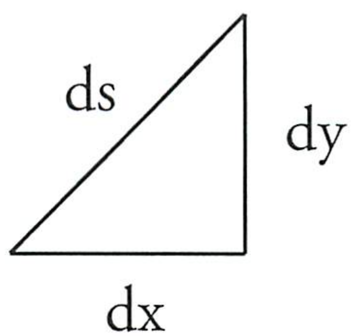


Figure 2: Zoom in on Figure 1 to see an approximate right triangle.

In Figures 1 and 2, s denotes arc length and $ds =$ the infinitesmal of arc length.

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (dy/dx)^2}\,dx$$

Integrating with respect to ds finds the length of a curve between two points (see Figure 3). To find the length of the curve between P_0 and P_1, evaluate:

$$\int_{P_0}^{P_1} ds$$

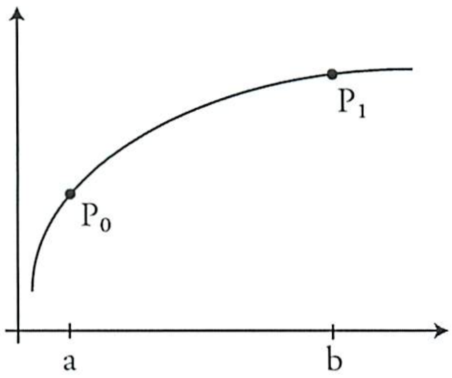


Figure 3: Find length of curve between P_0 and P_1.

We want to integrate with respect to x, not s, so we do the same algebra as above to find ds in terms of dx.

$$\frac{(ds)^2}{(dx)^2} = \frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2} = 1 + \left(\frac{dy}{dx}\right)^2$$

Therefore,

$$\int_{P_0}^{P_1} ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example 5: The Circle. $x^2 + y^2 = 1$ (see Figure 4).

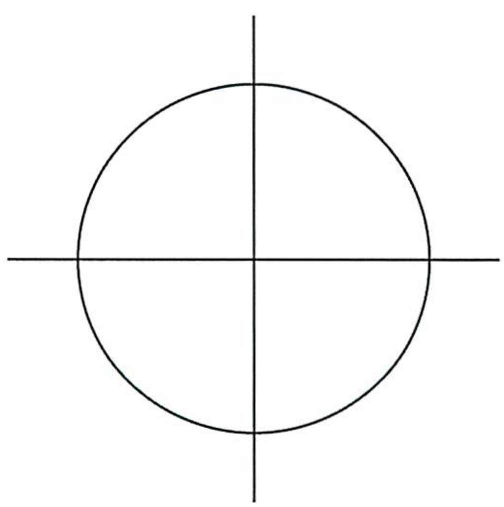


Figure 4: The circle in Example 1.

We want to find the length of the arc in Figure 5:

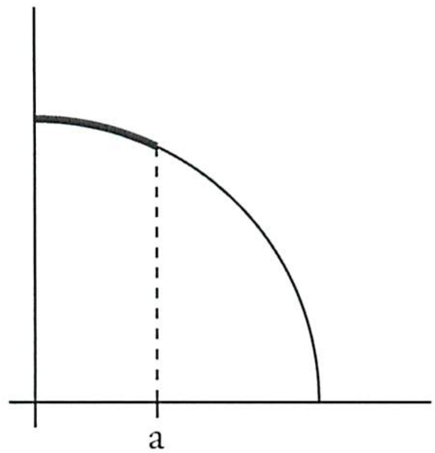


Figure 5: Arc length to be evaluated.

$$y = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{-2x}{\sqrt{1-x^2}}\left(\frac{1}{2}\right) = \frac{-x}{\sqrt{1-x^2}}$$

$$ds = \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx$$

$$1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2 = 1+\frac{x^2}{1-x^2} = \frac{1-x^2+x^2}{1-x^2} = \frac{1}{1-x^2}$$

$$ds = \sqrt{\frac{1}{1-x^2}}\,dx$$

$$s = \int_0^a \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big|_0^a = \sin^{-1} a - \sin^{-1} 0 = \sin^{-1} a$$

$$\sin s = a$$

This is illustrated in Figure 6.

18.01 Problem Set 7A

Michael Plasmeier

Due Friday 11/20/09, 1:45 pm in 2-106

+7B ENO sleet

35.5 / 63

+1 bonus

To be handed in together with problems from Problem Set 7B, to be posted on Friday, 11/13.

Part I (7 points)

Lecture 27. Thursday, Nov. 12 Trigonometric integrals. Direct substitution.
Read 10.2, 10.3 Work: 5B-9, 11, 13, 16; 5C-5, 7, 9, 11

Lecture 28. Friday, Nov. 13 Inverse substitution. Completing the square.
Read 10.4 Work: To be assigned on part 7B.

Part II (17 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Lec 25, 8pts) **On Beyond Simpson's Rule.**

In proving Simpson's rule, we showed that the area under an arbitrary parabola $P(x) = Ax^2 + Bx + C$ from $x = -w$ to $x = w$ could be expressed as integer linear combinations of $P(-w)$, $P(0)$, and $P(w)$ multiplied by $w/3$. That is, we showed

$$\int_{-w}^{w} (Ax^2 + Bx + C)dx = \frac{w}{3}\left[P(-w) + 4P(0) + P(w)\right]$$

and Simpson's rule followed. In this problem, we see if this is possible for cubic curves.

a) Draw a careful picture for the cubic problem like the one on p. 371 of Simmons used to illustrate Simpson's rule. Note, you'll need 3 intervals and 4 endpoints now that you're using cubic equations. Let's agree to write the cubic equation used to approximate our curve as $Q(x) = Ax^3 + Bx^2 + Cx + D$ and make the intervals on the x-axis have endpoints $x = 0, w, 2w, 3w$. (It seems harder to make them symmetric about the origin, since we don't get to use 0 as one of our endpoints in that case.)

b) Find the definite integral

$$\int_{0}^{3w} (Ax^3 + Bx^2 + Cx + D)dx$$

in terms of A, B, C, D. In other words, we're finding area under an arbitrary cubic curve. In mimicking the proof of Simpson's rule, factor out $w/8 = 3w/24$ from your answer so that the result is

$$w/8 \cdot (\text{SOMETHING})$$

18.01 Problem Set 7B

Due Friday 11/20/09, 1:45 pm in 2-106

To be handed in together with problems from Problem Set 7A.

Part I (15 points)

Lecture 28. Friday, Nov. 13 Inverse substitution. Completing the square.
Read 10.4 Work: 5D-1, 2, 7, 10

Lecture 29. Tuesday, Nov. 17 Integrating rational functions; partial fractions.
Read 10.6, Notes F Work: 5E-2, 3, 5, 6, 10h (complete the square)

Lecture 30. Thursday, Nov. 19 Integration by parts. Reduction formulas.
Read 10.7 Work: 5F-1a, 2d then 2b, 3

Lecture 31. Friday, Nov. 20 Parametric equations; arclength. Surface area
Read 17.1, 7.5 Work will be assigned on PS8.

Part II (24 points + 5 BONUS points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts - tallied on part 7a so not in above count) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Lec 28, 4pts: 2 + 2)

a) For any integer $n \geq 0$, use the substitution $\tan^2 x = \sec^2 x - 1$ to show that

$$\int \tan^{n+2} x\, dx = \frac{1}{n+1} \tan^{n+1} x - \int \tan^n x\, dx$$

b) Deduce a formula for $\int \tan^4 x\, dx$.

2. (Lec 29, 4pts: 3 + 1)

a) Derive a formula for $\int \sec x\, dx$ by writing $\sec x = \dfrac{\cos x}{1 - \sin^2 x}$ (verify this), and then making a substitution for $\sin x$ and using partial fractions. (Your final answer must be expressed in terms of x.)

b) Convert the formula into the more familiar one by multiplying the fraction in the answer on both top and bottom by $1 + \sin x$. (Note that $(1/2) \ln u = \ln \sqrt{u}$.)

3. (Lec 30, 3pts) Find the volume under the first hump of the function $y = \cos x$ rotated around the y-axis by the method of shells.

4. (13 pts: 2+2+2+2+2+3 and 5 pts BONUS) This problem explores Chebyshev polynomials T_n for $n = 0, 1, 2, \ldots$ defined by

$$T_n(x) = \cos(n \arccos x)$$

where $\arccos x$ is the inverse cosine function.

a) It is not hard to show that $T_0(x) = 1$ and $T_1(x) = x$. Express T_2 as a quadratic polynomial and T_3 as a cubic polynomial.

b) Show that for $n \geq 1$, $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

c) Use your answer in part (b) to show that T_n is a polynomial of degree n.

d) Use parts (a) and (b) to determine T_4 and T_5.

e) Notice that $T_n(\cos x) = \cos(nx)$. Use this fact and your answer from (d) to determine an identity for $\cos 5x$ as a polynomial in $\cos x$.

f) Compute the definite integral

$$\int_{-1}^{1} T_n(x) T_m(x) \frac{dx}{\sqrt{1 - x^2}}$$

(Hints: Your answer will depend on whether m and n are non-zero. You may want to take advantage of the identity in part (e).)

g) (BONUS) Explain a method for using your answer from part (f) to express any polynomial $p_n(x)$ of degree n in terms of Chebyshev polynomials. Demonstrate your method with a polynomial of degree 5. (Don't pick $T_5 = p_5$. Pick something interesting.)

PSet 7A

Michael Plasmeier

$\frac{35.5}{63}$ +1 bonus

11/20

Part A

Lecture 27 Trigonometric integrals, direct substitution

← not fancy method

5B-9 $\int e^x(1+e^x)^{-1/3}\,dx$

do not like these problems at all!

$u = 1+e^x$
$du = e^x\,dx$

$\int u^{-1/3}\,du$

$\frac{u^{2/3}}{\frac{2}{3}} = \frac{3}{2}u^{2/3} = \frac{3}{2}(1+e^x)^{2/3} + C$ ✓

11. $\int \sec^2 9x\,dx$ ← step given in 06 Lecture notes

$\frac{\tan 9x}{9} + C$

when step in hotte must differentiate inside?

$u = 9x$
$du = 9\,dx$

13. $\int \frac{x^2\,dx}{1+x^6}$ Hint $u = x^3$

$u = x^3$
$du = 3x^2\,dx$ ← how is that helpful
$\frac{1}{3}du = x^2\,dx$

$\int \frac{\frac{1}{3}\,du}{1+u^2} = \int \frac{1}{3+3u^2}\,du$

$$\cancel{\frac{1}{3u} + \frac{1}{\frac{3u^3}{3}} = \frac{1}{3u} + \frac{1}{u^3} = \frac{1}{x^3} + \frac{1}{x^9}}$$

$$\int \frac{du}{3(1+u^2)} = \frac{\tan^{-1}(u)}{3} + C = \frac{\tan^{-1}(x^3)}{3} + C$$

oh they wanted that!

16 Substitution (and change limits)

$$\int_{-1}^{1} \frac{\tan^{-1} x\,dx}{1+x^2}$$

$u = ?$ how do I know what to pick?
? always the trig

$u = \tan^{-1} x$

$du = \frac{1}{1+x^2}\,dx$ looked up, but would have otherwise not known

change bounds

$$\int_{\tan^{-1}(-1)}^{\tan^{-1}(1)} u\,du$$

$$\frac{u^2}{2}\Bigg|_{\tan^{-1}(-1) = -\frac{\pi}{4}}^{\tan^{-1}(1) = \frac{\pi}{4}}$$

$$\frac{(\tan^{-1}(1))^2}{2} - \frac{(\tan^{-1}(-1))^2}{2}$$

balances out since odd 0

5C-5
now typical

$$\int \sin^3 \cos^2 x\,dx$$

ask yourself what would make sense

$$\int (1-\cos^2 x)\cos^2 x \sin x\,dx$$

$u = \cos x$
$du = -\sin x\,dx$

$$\int -(1-u^2)(u^2)\,du$$

multiply out

$$-\int -(u^2 + u^4 \, du$$

$$-\frac{u^3}{3} + \frac{u^5}{5}$$

plug back in

$$\frac{-\cos x^3}{3} + \frac{\cos x^5}{5} + C$$

7. $\int \sin^2(4x) \cos^2(4x) \, dx$

$\cancel{u = \sin(4x)}$
$\cancel{du =}$

$\int \frac{\sin^2 8x \, dx}{4}$ ← how did they get this?

$$\int \frac{(1-\cos 16x)}{8} dx$$

$$\frac{1}{8} - \frac{\sin 16x}{128} + C$$

alt slow way $\left(\frac{1-\cos(8x)}{2}\right)\left(\frac{1+\cos(8x)}{2}\right)$ use similar trick to handle $\cos^2(8x)$

9. $\int \sin^3 x \sec^2 x \, dx$

$\int \frac{\sin^3 x}{\cos^2 x} dx = \int \tan^2 x \sin x \, dx$ ← not helpful?

split

make $\sin^2 x (1-\cos^2 x)$

$\int \sin^3(x)(\tan^2 x + 1)$??

$\int \frac{1-\cos^2 x}{\cos^2 x} \sin x \, dx$ ← want $\sin x$ left

how do you think of that

$u = \cos^2 x$
$du = -\sin x\, dx$

$-\int \frac{1-u^2}{u^2}\, du$

$\int -\frac{1}{u^2} + \int \frac{\cancel{u^2}}{\cancel{u^2}} 1$

$-\frac{u^{-1}}{-1} + u$

$+\frac{1}{u} + u \rightarrow + \frac{1}{\cos x} + \cos x + C$

$-$ $\underline{\sec x}$

11. $\int \sin x \cos(2x)\, dx$ (use double angles)

← one of 3

$\int \sin x \left(2\cos^2 x - 1\right) dx$

$u = \cos x$
$du = -\sin x\, dx$

$-\int 2u^2 - 1\, du$

$-\frac{2u^3}{3} + u$

$-\frac{2\cos^3 x}{3} + \cos x + C$

✓ Got one on my own!
~ w/ trig table

Part 2

0. See Sidebar + Sasha

1. On Beyond Simpson's Rule

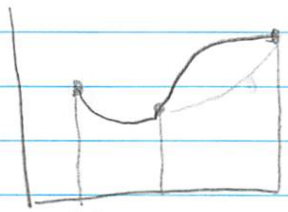
$P(x) = Ax^2 + Bx + C$ from $x = -w \Rightarrow w$
expressed as an integer linear combo of
$P(-w), P(0), P(w) \cdot \frac{w}{3}$

$$\int_{-w}^{w} (Ax^2 + Bx + C)\, dx = \frac{w}{3}\left[P(-w) + 4P(0) + P(w)\right]$$

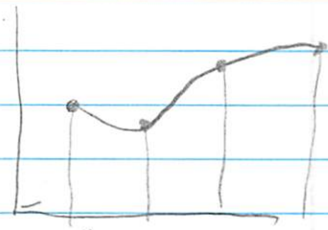
Simpsons rule

Does this work for cubic curves?

a) Draw p371



p371



Cubic

$Q(x) = Ax^3 + Bx^2 + Cx + D$
Endpoints $0, w, 2w, 3w$

? Don't get - is the function cubic or is the approx between intervals cubic?

b) Next page →

b Find the definite integral in terms of A, B, C, D

$$\int_0^{3w} (Ax^3 + Bx^2 + Cx + D)\,dx$$

- area under arbitrary curve
- factor out $\frac{w}{8} = \frac{3w}{24}$

$$\left. A\frac{x^4}{4} + B\frac{x^3}{3} + C\frac{x^2}{2} + Dx \right|_0^{3w}$$

so result starts w/ w/8

$$\frac{A}{4}(3w)^4 + B\frac{(3w)^3}{3} + \frac{C(3w)^2}{2} + D\,3w$$

$$20.25Aw^4 + 9Bw^3 + 4.5Cw^2 + 3Dw$$

$$\cancel{\frac{w}{8}\left(2.53125Aw^3 + 1.125Bw^2 + .5625Cw + .375D\right)}$$

• by 8 not divide!

$$\frac{w}{8}\left(162Aw^3 + 72Bw^2 + 36Cw + 24D\right)$$

? if we had only factored $\frac{w}{4}$ it would have worked but some reason taking extra factor of 2

c) Find integer constants k_0, k_1, k_2, k_3 so that

$$\frac{w}{8}\left(k_0Q(0) + k_1Q(w) + k_2Q(2w) + k_3Q(3w)\right) = \int_0^{3w}(Ax^3 + Bx^2 + Cx + D)\,dx$$

← integral from part B

Hint: this is an easy solving 4 unknowns w/ 4 equations

?? really confused

Sasha spent a lot of time on this

$k_0\, Q(0) = 162Aw^3 = 24D$
$k_1\, Q(w) = 72Bw^2$
$k_2\, Q(2w) = 36Cw$
$k_3\, Q(3w) = 24D$

$k_0 Q(0) = 162A(0)^3 + 72B(0)^2 + 36C(0) + 24D = 24D$
$k_1 Q(w) = 162A(w)^3 + 72B(w)^2 + 36C(w) + 24D$
$k_2 Q(2w) = 162A(2w)^3 + 72B(2w)^2 + 36C(2w) + 24D =$
$1296Aw^3 + 288Bw^2 + 72Cw + 24D$
$k_3 Q(3w) = 162A(3w)^3 + 72B(3w)^2 + 36C(3w) + 24D =$
$4374Aw^3 + 648Bw^2 + 108Cw + 24D$

? Don't get

?? Brendon office hrs

$\int_0^{3w} Q(x)\,dx = (162Aw^3 + 72Bw^2 + 36Cw + 24D)\frac{w}{8}$

$\frac{w}{8}\left(k_0 Q(0) + k_1 Q(w) + k_2 Q(2w) + k_3 Q(3w)\right)$

degree 4

=

compare coefficients w^3

4 eq - 8 unknowns - A on both sides
intercite but not sophisticated

$\frac{w}{8}\left(k_0 D + k_1 Aw^3 + k_1 Bw^2 + k_1 wC + k_1 D +\right.$
$k_2 8Aw^3 + k_2 4Bw^2 + k\,2w + k_2 D +$
...

$162\cancel{A} = k_1 A + 8k_2 A + 27k_3 \cancel{A}$ comparing cubic terms

d State a generalization of Simpson's rule for curves
Do you have any guesses about a rule for approximation by higher degree curves?

Wikipedia's $\frac{3}{8}$'s rule ~~$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$~~

not using cubics - but for cubic curves

web

$$A = \frac{h}{3}\left[y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + \dots + y_n\right]$$

$$\frac{h}{3}\left[y_0 + y_n + 4\left(y_1 + y_3 + \dots\ y_{n-1}\right) + 2\left(y_2 + y_4 + \dots\ y_{n-2}\right)\right]$$

$$\frac{h}{3}\left[(\text{first} + \text{last}) + 4\,(\text{sum odds}) + 2\,(\text{sum evens})\right]$$

Glinda
Gl second part

- what period would k curves have
- name some patterns says Brendan
- online somewhere says Brendan

2.a. Simmons 10.2 p340 49 abcd

49. Each of the following integrals is very easy to compute for a certain value of n. Find the value and carry out the integration.

Example $\int x^n \sin x^2 \, dx$ for $n=1$

$= -\frac{1}{2} \cos x^2 + C$

?? do it for $n=1$ or in general

a) $\int x^n e^{x^4} \, dx$

for $n=1$

$x e^{1^4} = \int x e \, dx$

$x^2 e + C$

for $n=n$

$\int x^n e^{n^4}$ ← ? did I do proper division?

$\frac{x^{n+1}}{n+1} e^{n^4} + C$

↑ same

b) $\int x^n \cos x^3 \, dx$

$\frac{x^{n+1}}{n+1} \sin x^3$

c $\int x^n \ln x \, dx$

$\frac{x^{n+1}}{n+1} \; x \ln x - x + C$

d $\int x^n \sec^2 \sqrt{x} \, dx$

~~$\int x^n (\tan^2 \sqrt{x} + 1) \, dx$~~ in notes

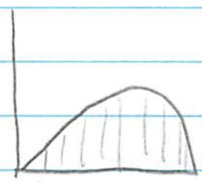
$\frac{x^{n+1}}{n+1} \tan \sqrt{x} + C$

b Simmons 10.3 p344 27ab

27 a. Find the volume of the solid of revolution generated when revolved around x -axis

last test!

$y = \sin x \qquad 0 \leq x \leq \pi$

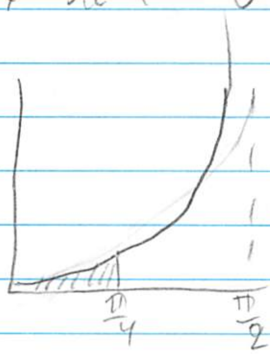


$$\int_0^{\pi} 2\pi x \cdot \sin x \, dx$$

$$2\pi \int x \sin x \, dx$$

Ti calc won't integrate - does not work

b $y = \sec x \qquad 0 \leq x \leq \pi/4$



$$\int_0^{\pi/4} 2\pi x \sec x \, dx$$

$$2\pi \int x \sec x \, dy$$

11 x

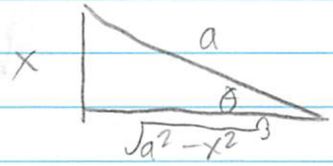
P-Set 7 B

See model from
– recitation 11/26

Part 1 Evaluate this is the triangle one

5D-1

$$\int \frac{dx}{(a^2-x^2)^{3/2}}$$



$\sin\theta = \frac{x}{a}$

$x = a\sin\theta$ ✓

$dx = a\cos\theta\, d\theta$

$$\int \frac{a\cos\theta\, d\theta}{(a^2-(a\sin\theta)^2)^{3/2}}$$

$$\int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}^{\,3}} = \cancel{\frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2(1-\cos^2\theta)}^{\,3}}}$$

wrong way

$$\int \frac{a\cos\theta\, d\theta}{\sqrt{a^2(1-\sin^2\theta)}^{\,3}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2\cos^2\theta}^{\,3}} = \int \frac{a\cos\theta\, d\theta}{a^3\cos^3\theta} =$$

$$\int \frac{d\theta}{a^2\cos^2\theta}$$

was close 1 more step ~~did not get~~ – ~~why~~

don't give up fast!

$$\frac{1}{a^2}\int \sec^2\theta\, d\theta = \frac{1}{a^2}\tan\theta + c$$

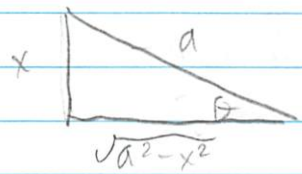
$$= \frac{x}{a^2\sqrt{a^2-x^2}^{\,3}} + C$$

Plug $\tan\theta = \frac{x}{\sqrt{a^2+x^2}}$

? look back at triangle

2.

$$\int \frac{x^3\, dx}{\sqrt{a^2-x^2}}$$



$\sin\theta = \frac{x}{a}$

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$$\int \frac{(a\sin\theta)^3\, a\cos\theta\, d\theta}{\sqrt{a^2-(a\sin\theta)^2}} = \int \frac{a^3\sin^3\theta\; a\cos\theta\, d\theta}{\sqrt{a^2(1-\sin^2\theta)}} =$$

$$\int \frac{a^4\sin^3\theta\cos\theta\, d\theta}{a\cos\theta} = \int a^3\sin^3\theta\, d\theta = a^3\int \sin^3\theta\, d\theta$$

now I guess its trigsub problem

$$a^3 \int \sin^3\theta \, d\theta = a^3 \int \sin^2\theta \sin\theta \, d\theta =$$

always forget what's next but as get it kinda easy

$$a^3 \int (1-\cos^2\theta) \sin\theta \, d\theta \rightarrow \begin{array}{l} u = \cos\theta \\ du = -\sin\theta \, d\theta \end{array}$$

lots of diff methods

$$-a^3 \int (1-u^2) \, du \Leftarrow$$

c copy right

$$-a^3 \left(u - \frac{u^3}{3}\right) = -a^3 \left(\cos\theta - \frac{\cancel{\sin}^{\cos}\,{}^3\theta}{3}\right) + C$$

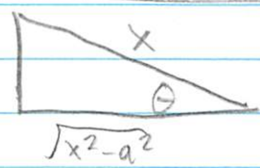
keep → forgetting this step!

Look back at triangle $\quad -a^3 \frac{\sqrt{a^2-x^2}}{a} + a^3 \left(\frac{\sqrt{a^2-x^2}}{a}\right)^3 \Big/ 3$

$$-a^2 \sqrt{a^2-x^2} + \frac{(a^2-x^2)^{3/2}}{3} + C$$

---

7. $\int \frac{\sqrt{x^2-a^2}}{x^2} dx$



$\sin\theta = \frac{a}{x}$

sec $\cancel{\csc}\,\theta = \frac{x}{a}$

$x = a\,\cancel{\csc}^{\sec}\theta$

change get

$dx = a\,\cancel{\csc}^{\sec}\theta \tan\theta \, d\theta$

~~perhaps since numerator???~~

$$\int \frac{\sqrt{a^2\sec^2\theta - a^2}}{a^2\sec^2} = \int \frac{\sqrt{a^2\tan^2\theta}}{a^2\sec^2\theta} = \int \frac{a \tan\theta}{a^2\sec^2\theta} d\theta$$

~~$u = \tan\theta$, $du = \sec^2\theta \, d\theta$~~ ~~$u = \sec\theta$, $du = \sec\theta\tan\theta \, d\theta$~~

dont - and try claim a's cancle

$$\cancel{\int \frac{u}{du}} \quad \int \frac{\sqrt{\cancel{a^2}(\sec^2\theta - 1)}}{\cancel{a^2}\sec^2\theta} = \int \frac{\sec^2\theta}{\sec\theta} - \frac{1}{\sec\theta}$$

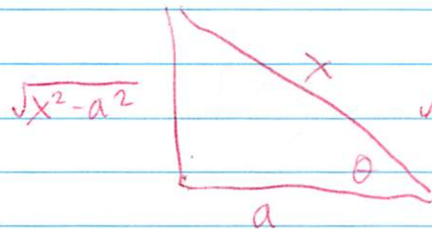
$$\int \sec\theta - \cos\theta \, d\theta$$

$$\boxed{\ln(\sec\theta + \tan\theta) - \sin\theta + C} \leftarrow \text{did by integrating}$$

Now look at triangle

$$\cancel{\ln\left(\frac{\sqrt{x^2-a^2}}{x} + \frac{a}{\sqrt{x^2-a^2}}\right) - \frac{a}{x} + C}$$

I think their triangle different



✓ they flipped where θ was
– so that is why I got csc
If I would have continued
that through it would have worked

$$\ln\left(\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right) - \frac{\sqrt{x^2-a^2}}{x} + C$$

little simplify ↓

$$\ln\left(x + \sqrt{x^2-a^2}\right) - \frac{\sqrt{x^2-a^2}}{x} + (C - \ln a)$$

10. $\displaystyle\int \frac{dx}{[x^2+4x+13]^{3/2}} \rightarrow x^2+4x+4+9$

Complete the square one – think never learned

$$\int \frac{dx}{((x+2)^2+3^2)^{3/2}}$$

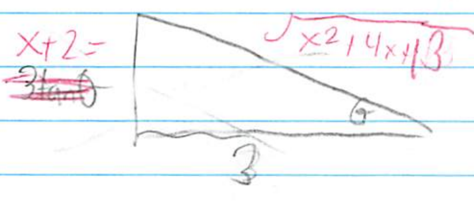
$x+2 = 3\tan\theta$ $\quad \tan\theta = \frac{x+2}{3}$

$dx = 3\sec^2\theta \, d\theta$

$$\int \frac{3\sec^2\theta \, d\theta}{((3\tan\theta)^2+3^2)^{3/2}}$$

← ? how does this make it any easier?

what going on here



see back

Briner office hrs

$$\int \frac{3\sec^2\theta}{[(3\tan\theta)^2+3^2)^{3/2}}\,d\theta$$

$$\int \frac{3\sec^2\theta}{(9\tan^2\theta+3^2)^{3/2}}\,d\theta$$

$$\int \frac{3\sec^2\theta}{(9(\tan^2\theta+1))^{3/2}}\,d\theta$$

Office hrs but did by self :)

$$\int \frac{3\sec^2\theta}{9^{3/2}(\sec^2\theta)^{3/2}}\,d\theta$$

$$\int \frac{\cancel{\sec^2\theta}}{9(\sec^2\theta)^{\cancel{3}/2}}\,d\theta$$

$$\frac{1}{9}\int \sec^2\theta\,d\theta$$

next pg

$\frac{1}{9}\int \cos\theta \, d\theta = \frac{1}{9}\sin\theta + C$

$\frac{(x+2)}{9\sqrt{x^2+4x+13}} + C$ ← back to triangle

Lecture 29 Rational Functions $\left(\frac{}{A} + \frac{}{B}\right)$

5E-2 $\int \frac{x\,dx}{(x-2)(x+3)}\,dx = \frac{A}{(x-2)} + \frac{B}{(x+3)}$

$x = A(x+3) + B(x-2)$

B is 0 at $x=2$

$2 = A(2+3)$ ← so A is $\boxed{\frac{2}{5}}$

$2 = \frac{2}{5} \cdot 5$

$x = \frac{2}{5}(x+3) + B(x-2)$

$x = \underbrace{\quad}_{\text{what value x makes this}=0}\; B(x-2)$

-3

$-3 = B(-3-2)$

$-3 = -5B$

$B = \boxed{\frac{3}{5}}$

$\int \frac{\frac{2}{5}}{x-2} + \frac{\frac{3}{5}}{x+3}$

$\frac{2}{5}\left(\ln x - \frac{x}{2}\right) + \frac{3}{5}\left(\ln(x) + \frac{x}{3}\right)$

$\frac{2}{5}\ln(x-2) + \frac{3}{5}\ln(x+3) + C$

learn the steps!
stumbled through

5E-10b

practice

$$\int \frac{(x+1)\,dx}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

$$x+1 = A(x-3) + B(x-2)$$

x so this is 0
$x=2$

$$2+1 = A(2-3)$$

$$1 = -1A$$

$$A = -3$$

← Close copy error grr

$$x+1 = (A)(x-3) + B(x-2)$$

so this = 0
$x=3$

$$3+1 = B(3-2)$$

$$4 = B$$

$$\int \frac{-3}{x-2} + \frac{4}{x-3}$$

$$-3\ln(x-2) + 4\ln(x-3) + C$$

on own - all except for copy error

5E-3

$$\int \frac{x\,dx}{(x^2-4)(x+3)}\,dx = \frac{Ax+B}{x^2-4} + \frac{C}{x+3}$$

can factor $\frac{A}{x+2} + \frac{B}{x-2}$

copy error

$$x = A(x-2)(x+3) + B(x+2)(x+3) + C(x-2)(x+2)$$

$$x = (x^2+x-6)A + (x^2+5x+6)B + (x^2+0-4)C$$

now group $(A+B+C)x^2 + (A+B+C)x + A+B+C$

and he didn't even do algebra in lecture.

- what algebra should we do?

Prince office hrs

She said coverup easier

$$x = (A+B+C)x^2 + (A+5B)x + (-6A+6B-4C)$$

2 polynomers are = only if all coefficients are =

$x^2)$ $A+B+C=0$

$x)$ $A+5B=1$ ← solve system 3 equations

$1)$ $-6A+6B-4C=0$

hard exactly because Heavyside works

$A + B + C = 0$

$A + 5B = 1$

$-6A + 6B - 4C = 0$

$-3A + 3B - 2C = 0$

$-3(1 - 5B) + 3B - 2C = 0$

$-3 + 15B + 3B - 2C = 0$

$18B - 2C = 3$

Says its really hard

Brier office hrs

$A = \frac{1}{10}$

$B = \frac{1}{2}$

$C = \frac{3}{5}$

$$\int \frac{.1}{x+2} + \int \frac{.5}{x-2} + \int \frac{.6}{x+3}$$

$$.1 \ln(x+2) + .5 \ln(x-2) + .6 \ln(x+3)$$

5. $$\int \frac{3x+2}{x(x+1)^2}\,dx = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)}$$

- case 2 since repeating term
- can get some things from Heavy sides
- then solve

? am not clear on exactly what

did not do example in class

$$3x+2 = A(x+1)(x+1) + B(x)(x+1) + C(x+1)(x)$$
$$3x+2 = (x^2+2x+1)A + (x^2+x)B + (x^2+x)C$$

$x^2)\ 0 = A + B + C$
$x)\ 3 = 2A + B + C$
$1)\ 2 = 1A$

$A = 2$
$0 = 2 + B + C \qquad 3 = 2(2) + B + C$
$-2 = B + C \qquad 1 = B + C$

why???

Can coverup for $x = 1$

$$\frac{5}{4} = 2 + \frac{B}{2} + \frac{1}{4} \Rightarrow B = -2$$

$$\int \frac{3x+2}{x(x+1)^2}\,dx = 2\ln x - 2\ln(x+1) - \frac{1}{x+1} + C$$

6. $\int \frac{2x-9}{(x^2+9)(x+2)} \, dx$

do w/ coverup

$\frac{A}{(x+3)} + \frac{B}{(x-3)} + \frac{C}{(x+2)}$

Multiply all sides by $x+3$

$A + \frac{B(x+3)}{(x-3)} + \frac{C(x+3)}{(x+2)} = \frac{2x-9}{(x-3)(x+2)}$

to get this to $=0$

$x=-3$

$A + 0 + 0 = \frac{2(-3)-9}{(-3-3)(-3+2)} = \frac{-15}{-6 \cdot -1} = \frac{-15}{6}$

$= \frac{-5}{2}$

so now

$\frac{A(x-3)}{(x+3)} + B + \frac{C(x-3)}{(x-2)} = \frac{2(3)-9}{(3+3)(3+2)} = \frac{-3}{6 \cdot 5} = \frac{-3}{30} = \frac{-1}{10}$

$x=3$ $x=3$

$\frac{A(x+2)}{(x+3)} + \frac{B(x+2)}{(x-3)} + C = \frac{2(-2)-9}{(-2+3)(-2-3)} = \frac{-13}{1 \cdot -5} = \frac{13}{5}$

$x=-2$

$\int \frac{-5/2}{(x+3)} + \int \frac{-1/10}{(x-3)} + \int \frac{13/5}{(x+2)}$

10h.

$$\int \frac{(x^2+1)\,dx}{x^2+2x+2} = \cancel{\frac{A}{x+1}} \cancel{+ \frac{B}{x+1}}$$ not factorable

what here ??

this is another complete the square one

$\cancel{x^2+1 = A(}$

$$\int 1 - \left(\frac{1+2x}{x^2+2x+1}\right) dx$$

$$x - \int \frac{(2y-1)\,dy}{y^2+1}$$

$y = x+1$

$dy = 1\,dx$

$$x - \ln(y^2+1) + \tan^{-1}(y) + C$$

$$x - \ln(x^2+2x+2) + \tan^{-1}(x+1) + C$$

Part 2

0. See P-Set 7A Part 2 Qustion 0 ✓ 3

1. For any integer $n \geq 0$ use the substitution $\tan^2 x = \sec^2 x - 1$ to show that

$$\int \tan^{n+2} x\,dx = \frac{1}{n+1} \tan^{n+1} x - \int \tan^n x\,dx$$

$\int \tan^2 x = \int \sec^2 x$
$\int \tan x = \ln(\sec x)$ ← but how w/ multiplying

ex $n=4$ $\int \tan^6 x\,dx = \frac{1}{5} \tan^5 x - \int \tan^4 x\,dx$

2/2

peel off $\int \tan^n x$ by subtracting

$\int \tan^6 - \int \tan^4 x = \int \tan^2 x$

$\int \sec^2 x$

$\tan x + C$

but why $\frac{1}{5}$?

→ to get rid of n??

1

4

b. Deduce a formula for $\int \tan^4 x \, dx$

$\cancel{\int \tan^2 x \cdot \tan^2 x}$

or like a

$n = 2$

$$\int \tan^4 x = \frac{1}{2+1} \tan^3 x - \int \tan^2 x \, dx$$

$$\frac{1}{3} \tan^3 x - \int \tan^2 x \, dx$$

$\int \sec^2 x$

$\tan x + C$

where did I get this from

$\tan(\sec^2 - 1)$

sub

directly

2. Derive a formula for $\int \sec x \, dx$ by writting $\frac{\cos x}{1-\sin^2 x}$ e verify and then making a substitution for $\sin x$ and using partial fraction Express final ans in terms of x

ans
$\ln(\sec u + \tan u)$

you have to know what to do

$$\int \sec x \, dx = \frac{\cos x}{1-\sin^2 x} dx$$

$u = \sin x$
$du = \cos x \, dx$

$$\int \frac{1}{1-u^2} du = \int 1 - \int u^{-2} du$$

$$u - \frac{u^{-1}}{-1} = u - \frac{1}{u} = \sin x - \frac{1}{\sin x} = \sin x - \csc x$$

? why partial fractions and wrong

7, 0.5

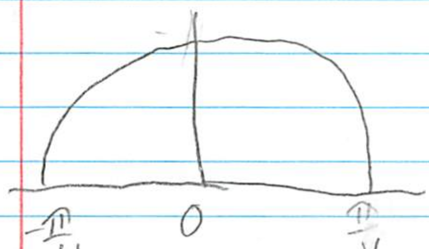
2b. Convert the formula into a more Familliar one by multiplying the fraction in the anser on both top and bottom by $1+\sin x$

Note) $\frac{1}{2}\ln u = \ln\sqrt{u}$

I did not get #2 ans

3. Find the volume under the first hump of the function $y = \cos x$ rotated around the y-axis by method of shells.

Blast from past!



← only 180°

$$\int_{-\frac{\pi}{4}}^{\pi/4} \pi x \cdot \cos x \, dx$$

$$\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x \cos x \, dx$$

$$\pi \left(\cos x + x \sin x \right) \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$\pi \left[\left(\cos \tfrac{\pi}{4} + \tfrac{\pi}{4} \sin \tfrac{\pi}{4} \right) - \left(\cos \tfrac{-\pi}{4} - \tfrac{\pi}{4} \sin \tfrac{-\pi}{4} \right) \right]$$

Integration by parts $\int u \, dv = uv - \int v \, du$

what day oh todays lecture I have not gone to - no wonder!

$u = x$ $\quad dv = \cos x \, dx$

$du = 1 \, dx$ $\quad \int dv = \int \cos x \, dx$

$du = dx$ $\quad v = \sin x$

$$\int u \, dv = uv - \int v \, du$$

$$x \sin x - \int \sin x$$

$$x \sin x - \cos x$$

answer?

$= 1$

4. Chebyshev polynomials T_n for $n = 0, 1, 2, \ldots$

$$T_n = \cos(n \arccos x)$$

inverse cos function $\cos^{-1}(x)$

a) It's not hard to show that $T_0(x) = 1$
$T_1(x) = x$

T_2 = quadratic polynomial
T_3 = cubic polynomial

$2x^2 - 1 \to$ Brendon office hrs

$$T_2 = \cos(2\cos^{-1}(x))$$

double angle $\cos(2\theta) = 2\cos^2\theta - 1$

special case angle sum formula

$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

$$2(\cos(\cos^{-1}(x)))^2 - 1$$
$$2x^2 - 1$$

$$T_3 = \cos(3\cos^{-1}(x))$$
$$\cos(2\cos^{-1}(x) + \cos^{-1}(x))$$
$$\cos(2\cos^{-1}(x))\cos(\cos^{-1}(x)) - \sin(2\cos^{-1}(x))\sin(\cos^{-1}(x))$$

another double angle $\sin(2\theta) = 2\sin\theta\cos\theta$

$$(2x^2 - 1)x - 2\sin(\cos^{-1}(x))\cos(\cancel{\cos^{-1}(x)})\sin(\cos^{-1}(x))$$

x

then finish off

// 1

b) Show that for $n \geq 1$, $T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$

$T_n(x) = \cos(n\cos^{-1}(x))$

Brendon off ice hrs

$T_{n+1}(x) \stackrel{?}{=} 2x\, T_n(x) - T_{n-1}(x)$

$\cos((n+1)\cos^{-1}(x)) \stackrel{?}{=} 2x\cos(n\cos^{-1}(x)) - \cos((n-1)\cos^{-1}(x))$

– have angle sum + difference formulas

algebra →

↑ Left ↑ right

See Wikipedia for formulas

left) $\cos((n+1)\cos^{-1}(x)) = \cos(n\cos^{-1}(x))\cos(\cancel{\cos^{-1}(x)}) - \sin(n\cos^{-1}(x))\sin(\cos^{-1}(x))$

x

right) $2x\cos(n\cos^{-1}(x)) - \cos((n-1)\cos^{-1}(x))$ 2 expand

$2x\cos(n\cos^{-1}(x)) - \cos(n\cos^{-1}(x))\cos(\cancel{-\cos^{-1}(x)})$ 2

x

$+\sin(n\cos^{-1}(x))\sin(-\cos^{-1}(x))$

grind through - show left + right =

can be proven by taking derivitive

angle formulas

to prove the recursion

1

11/19

Briner office hrs

Induction - show something is true for all n

(1) Show true n=1
  - starting value

(2) Assume statement true for $j \leq n$

(3) Use (2) to show the $j = n+1$ case is true

T_0 ✓ T_1 ✓ T_2 ✓ T_3 T_4 T_5 → Induction

c) Use your answer in Part b to show T_n is polynomial of degree n

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x) \quad \lceil n \geq 1 \rfloor$$

Brendon Office Hrs

- use induction to prove (jumps out at you)
- know $T_0(x) = 1$, $T_1(x) = x$
- can plug in what you know recursively
- have 0, 1 → find 2
- have 1, 2 → find 3 ... lots of work
- prob how program computer

- induction - good for recursion
  - subtraction won't kill top efficent

- degree of answer = n
- use recursion to prove 1 more

$T_0 = 1$ ← base case

$T_1 = x$ need to prove

$T_2 = 2x^2 - 1$ ←prove since 2 term recursion

- Assume for $k \leq n$, $T_k(x)$ is a polynomial of degree k
- Want to show $T_{n+1}(x)$ is a polynomial of degree $n+1$
- Because $T_n(x)$, $T_{n-1}(x)$ are polynomials by assumption, $T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$ is a polynomial

By assumption T_n has degree n, T_{n-1} has degree $n-1$

On Own

$n=1$ $\quad T_2(x) = 2x\,T_1(x) - T_0(x)$

1, 2 $\quad T_2(x) = 2x \cdot x - 1$ ✓

$$T_3(x) = 2x\,T_2(x) - T_1(x)$$
$$= 2x \cdot (2x^2 - 1) - x$$
$$= 4x^3 - 2x - x \quad ✓$$

↓ Can prove via induction ? what more to write

d) Use answer from a + b to find T_4 + T_5

- So by recursion

$T_0 = 1$
$T_1 = x$
$T_2 = 2x^2 - 1$
$T_3 = 4x^3 - 2x - 1$

$T_4 = 2x\, T_3(x) - T_2(x)$
$= 2x(4x^3 - 2x - 1) - (2x^2 - 1)$
$8x^4 - 4x^2 - 2x - 2x^2 + 1$
$8x^4 - 6x^2 - 2x + 1$ X $\frac{=}{1}$

$T_5 = 2x\, T_4(x) - T_3(x)$
$2x(8x^4 - 6x^2 - 2x + 1) - (4x^3 - 2x - 1)$
$16x^5 - 12x^3 - 4x^2 + 2x - 4x^3 + 2x + 1$
$16x^5 - 16x^3 - 4x^2 + 4x + 1$

e) Notice that $T_n(\cos x) = \cos(nx)$, Use this fact and answer from d to determine an identity for $\cos 5x$ as a polynomial in $\cos x$

$\cos(5x) = T_n(\cos x)$
+ but I found just x

$T_n(\cos x) = \cos(n\, \cancel{\cos^{-1}}(\cos x))$
$\cos(nx)$

$T_n(\cos x) = 16\cos^5 x - 16\cos^3 x - 4\cos^2 x + 4\cos x + 1$ X $\frac{=}{1}$
just that

$\frac{=}{2}$

f) Compute the definite integral

$$\int_{-1}^{1} T_n(x)\, T_m(x)\, \frac{dx}{\sqrt{1-x^2}}$$

Hint - Your answer will depend on whether m and n are non zero. See identity in part E.

Brendan office Hrs

$x = \cos\theta$ || ← show

$$\int_0^{\pi} \cos(mx)\cos(nx)\,dx$$

↑ integral is difficult
Olga says look up online
or Complex # and Oiler Formula
or kimberly's method

0 $m \neq n$
π $m = 0, n = 0$

- Substitution
- change bounds

Wollfram Alpha (is so cool)

- use trig idenity $\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha-\beta) + \cos(\alpha+\beta)$ $a = mx$ $b = nx$

$$\frac{1}{2}\int (\cos(x(m-n)) + \cos(x(m+n)))\,dx$$

- expand integrand

$$\frac{1}{2}\int \cos(x(m-n)) + \frac{1}{2}\int \cos(x(m+n))\,dx$$

$(m+n)x$

$v \quad ds = m+n \, d$

$$\frac{1}{2(m-n)} \int \cos v \, dv + \frac{1}{2(m+n)} \int \cos v \, dv$$

$$\frac{\sin v}{2(m-n)} + \frac{\sin v}{2(m+n)} + C$$

$$\frac{\sin(x(m-n))}{2(m-n)} + \frac{\sin(x(m+n))}{2(m+n)} + C$$

$$\frac{\sin(\pi(m-n))}{2(m-n)} + \frac{\sin(\pi(m+n))}{2(m+n)} -$$

$$\left[\frac{\sin(0)}{2(m-n)} + \frac{\sin(0)}{2(m+n)}\right]$$

? what does that prove

$\frac{1}{2}$

Bonus 9 Explain a method for using your answer from part f to express any polynomial $P_n(x)$ of degree n in terms of Chebyshev Polynomials. Demonstrate, but not w/ $T_5 = P_5$

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, \frac{dx}{\sqrt{1-x^2}}$$

?? $P_n(\cos x) =$

wikipedia

$$\sum_{n=0}^{\infty} P_n(x)\, p^x = \frac{1-px}{1-2px+p^2}$$

whole lot of stuff in article I don't get

1

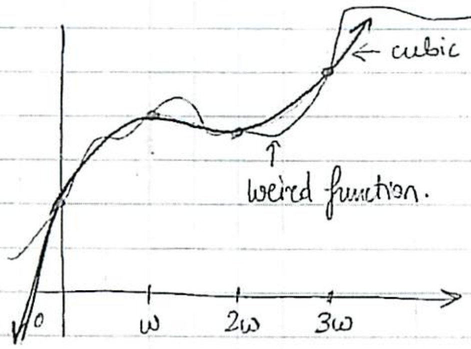
PSet 7 · 18.01
Fall 2009

①

7A

Problem 1 [On Beyond Simpson's Rule] (Graders: do this 2+2+2+2.)

a) Draw a picture for a 3-interval cubic approximation



b) Find the definite integral $\int_0^{3w} Ax^3+Bx^2+Cx+D\,dx$

$$\int_0^{3w} Ax^3+Bx^2+Cx+D = \frac{Ax^4}{4}+\frac{Bx^3}{3}+\frac{Cx^2}{2}+Dx\Big|_0^{3w} = \frac{A(3w)^4}{4}+\frac{B(3w)^3}{3}+\frac{C(3w)^2}{2}+D(3w)$$

$$= \frac{w}{8}\left(162Aw^3+72Bw^2+36Cw+24D\right)$$

c) Find integer constants k_0, k_1, k_2, k_3 so that

$$\frac{w}{8}\left(k_0Q(0)+k_1Q(w)+k_2Q(2w)+k_3Q(3w)\right) = \int_0^{3w}(Ax^3+Bx^2+Cx+D)\,dx$$

$$\sum k_iQ(iw) = k_0D + k_1(Aw^3+Bw^2+Cw+D) + k_2(8Aw^3+4Bw^2+2Cw+D) + k_3(27Aw^3+9Bw^2+3Cw+D)$$

// LHS.

$$= w^3(Ak_1+8Ak_2+27Ak_3) + w^2(Bk_1+4Bk_2+9Bk_3) + w(Ck_1+2Ck_2+3Ck_3) + Dk_0.$$

match this with RHS $= 162Aw^3+72Bw^2+36Cw+24D.$

Get a system of equations

$\Rightarrow k_0+k_1+k_2+k_3 = 24.$

$$\begin{cases} k_1+8k_2+27k_3=162 \\ k_1+4k_2+9k_3=72 \\ k_1+2k_2+3k_3=36. \end{cases}$$

$2k_2+6k_3=36$

$4k_2+18k_3=90$

$4k_2+12k_3=72$

$6k_3=18 \Rightarrow k_3=3.$

$4k_2+36=72 \Rightarrow 4k_2=36$

$\Rightarrow k_2=9.$

$k_1 = 36-2\cdot9-3\cdot3 =$

$= 36-18-9=9.$

$k_0 = 24-18-3=3$

$$\boxed{k_0=3,\quad k_1=9,\quad k_2=9,\quad k_3=3}$$

Problem 1, cont.

(d) State a generalization of Simpson's rule for cubic curves.

$$\int_0^a f(x)\,dx = \frac{a/3}{8}\left(3f(0) + 9f(a/3) + 9f(2a/3) + 3f(a)\right)$$

$$= \frac{a}{8}\left(f(0) + 3f(a/3) + 3f(2a/3) + f(a)\right).$$

$$= \frac{3\Delta x}{8}\left(f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + \dots + 3f(x_{3k-1}) + f(x_{3k})\right)$$

$[n = 3k]$

Problem 2

a) Simmons 10.2 : 49 abcd.

49 a) $n=3$ $\int x^3 e^{x^4} dx = \frac{1}{4} e^{x^4} + c$ b) $n=2$ $\int x^2 \cos x^3 dx = \frac{1}{3}\sin x^3 + c$

c) $n=-1$ $\int \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2 + c$ d) $n = -\frac{1}{2}$ $\int \frac{\sec^2\sqrt{x}}{\sqrt{x}} dx = 2\tan\sqrt{x} + c.$

b) Simmons : 10.3 : 27 ab

[Find volume of solid of revol. obt. by rotating indicated region ∂ x-axis.]

27 a) $y = \sin x$, $0 \le x \le \pi$.

$$\int_0^\pi \pi \sin^2 x\,dx = \pi\left(\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right)\Big|_0^\pi = \boxed{\pi^2/2}$$

b) $y = \sec x$ $0 \le x \le \pi/4$

$$\int_0^{\pi/4} \pi \sec^2 x\,dx = \pi \tan x\Big|_0^{\pi/4} = \boxed{\pi}.$$

Part II, 7B

Problem 1

a) For any $n \ge 0$ show $\int \tan^{n+2} x\,dx = \frac{1}{n+1}\tan^{n+1} x - \int \tan^n x\,dx$

$$\int \tan^{n+2} x\,dx = \int \tan^n x \cdot \tan^2 x\,dx = \int \tan^n x\,(\sec^2 x - 1)\,dx = \int \tan^n x \sec^2 x\,dx - \int \tan^n x\,dx$$

$$= \int \tan^n x\,d(\tan x) - \int \tan^n x\,dx = \frac{1}{n+1}\tan^{n+1} x - \int \tan^n x\,dx \quad //.$$

7B

Problem 1, cont.

e) Deduce a formula for $\int \tan^4 x\, dx$

$$\int \tan^2 x\, dx = \int \sec^2 x\, dx - \int 1\, dx = \tan x - x + c'.$$

$$\Rightarrow \int \tan^4 x\, dx = \boxed{\tfrac{1}{3}\tan^3 x - \tan x + x + c.}$$

Problem 2

a) Derive a formula for $\int \sec x\, dx$ by writing $\sec x = \frac{\cos x}{1-\sin^2 x}$, and then making a subst. for $\sin x$ and using partial fractions. (express your final answer in terms of x)

$$\sec x = \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1-\sin^2 x}$$

$u = \sin x$

$$\Rightarrow \int \sec x\, dx = \int \frac{\cos x}{1-\sin^2 x}\, dx = \int \frac{d(\sin x)}{1-\sin^2 x} = \int \frac{du}{1-u^2} = \int \frac{1}{1-u^2}\, du$$

$$\frac{1}{1-u^2} = \frac{1}{(1-u)(1+u)}$$

$$= \int \frac{1/2}{1-u} + \frac{1/2}{1+u}\, du = -\frac{1}{2}\ln(1-u) + \frac{1}{2}\ln(1+u) + c.$$

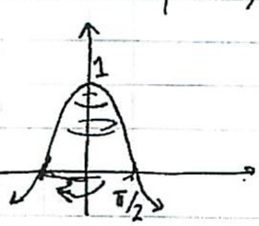
$$= \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right) + c = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right) + c.$$

b) Convert the formula above into a more familiar one,

$$\frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right) + c = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\right) = \frac{1}{2}\ln\left(\frac{(1+\sin x)^2}{\cos^2 x}\right) + c.$$

$$= \ln\left(\frac{1+\sin x}{\cos x}\right) + c = \ln(\sec x + \tan x) + c.$$

Problem 3 Find the vol. under the first hump of $y = \cos x$, rotated abt the y-axis, by the method of shells.



$$V = \int_0^{\pi/2} 2\pi r h\, dx = \int_0^{\pi/2} 2\pi x \cos x\, dx = 2\pi\,(x\sin x + \cos x)\Big|_0^{\pi/2}$$

$$= 2\pi\left(\tfrac{\pi}{2} + 0 - 0 - 1\right) = 2\pi\left(\tfrac{\pi}{2} - 1\right) = \boxed{\pi^2 - 2\pi}.$$

Problem 4 [Chebyshev polynomials] $T_n(x) = \cos(n \arccos x)$ 4

a) Express T_2 as a quadratic pol. and T_3 as a cubic

$$T_2(x) = \cos(2\arccos x) = 2\cos^2(\arccos x) - 1 = \boxed{2x^2 - 1.}$$

$$T_3(x) = \cos(3\arccos x) = \cos(2\arccos x)\cos(\arccos x) - \sin(2\arccos x)\sin(\arccos x) =$$

$$= (2x^2-1)(x) - 2\sin(\arccos x)\cos(\arccos x)\sin(\arccos x) =$$

$$= (2x^2-1)(x) - 2x\sin^2(\arccos x) =$$

$$= (2x^2-1)(x) - 2x(1-x^2) = \boxed{4x^3 - 3x}$$

b) Show that for $n \geq 1$, $T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$

$$T_{n+1}(x) = \cos((n+1)\arccos x) = \cos(n\arccos x)\,x - \sin(\overset{((n-1)+1)\arccos x}{n\arccos x})\sin(\arccos x) =$$

$$= xT_n - \Big[\Big(\sin((n-1)\arccos x)\cos(\arccos x) + \cos((n-1)\arccos x)\sin(\arccos x)\Big)\sin(\mathrm{ar.}x)\Big]$$

$$= xT_n - \big[x\sin((n-1)\arccos x)\sin(\arccos x) + (1-x^2)\cos((n-1)\arccos x)\big]$$

$$= xT_n - \big[-x\cos(n\arccos x) + x\cos((n-1)\arccos x)\cos(\arccos x) + T_{n-1}(x) - x^2T_{n-1}(x)\big] =$$

$$= xT_n - \big[-xT_n + \cancel{x^2T_{n-1}(x)} + T_{n-1}(x) - \cancel{x^2T_{n-1}(x)}\big] =$$

$$= 2xT_n(x) - T_{n-1}(x). \quad //$$

c) Show that T_n is a pol. of degree n. ~~[Rather T_{n+1} of deg~~

Rather: T_{n+1} of deg $n+1$.

Show by induction. Base cases: T_0 - of deg. 0, T_1 of deg 1, assume k of deg k for $k \leq n$.

Then $\deg(T_{n+1}) = \deg(2x\,T_n - T_{n+1}) = \max(\deg(x\,T_n), \deg(T_{n+1})) =$

$$= \max(n+1, n+1) = n+1 \;//.$$

#4d) Determine T_4, T_5

$$T_4 = 2x\,T_3(x) - T_2(x) = 2x\,(4x^3 - 3x) - (2x^2 - 1) = \boxed{8x^4 - 8x^2 + 1}$$

$$T_5 = 2x\,T_4(x) - T_3(x) = 2x\,(8x^4 - 8x^2 + 1) - (4x^3 - 3x) = 16x^5 - 20x^3 + 5x.$$

e) $T_n(\cos x) = \cos nx$. Determine an identity for $\cos 5x$ as pol. in $\cos x$

$$\cos 5x = T_n(\cos x) = 16\cos^5 x - 20\cos^3 x + 5\cos x$$

f) Compute $\int_{-1}^{1} T_n(x)\, T_m(x) \frac{dx}{\sqrt{1-x^2}}$ If $n=m=0$, this is $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{-1}^{1} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

Let $x = \cos y$ $dx = -\sin y\, dy$ $-1 \le x \le 1 \Rightarrow \pi \le y \le 0$. on this interval, $\sqrt{1-\cos^2 y} = \sin y$.

assume $n \neq 0$ or $m \neq 0$

$$\int_{\pi}^{0} T_n(\cos y)\, T_m(\cos y) \frac{-\sin y\, dy}{\sin y} =$$

$$= -\left(-\int_{0}^{\pi} T_n(\cos y)\, T_m(\cos y)\, dy\right) = \int_{0}^{\pi} \cos ny \cos my\, dy =$$

$$= \begin{cases} \left.\dfrac{\sin(n-m)y}{2(n-m)} + \dfrac{\sin(n+m)y}{2(n-m)}\right|_{0}^{\pi} = 0, & n \neq m \\ \displaystyle\int_0^{\pi} \cos^2 ny\, dy = \int_0^{\pi} \frac{\cos 2ny + 1}{2}\, dy = \int_0^{\pi} \frac{1}{2}\, dy = \frac{1}{2}\pi & \text{if } n=m. \end{cases}$$

$$\boxed{= \begin{cases} 0, & n \neq m \\ \frac{1}{2}\pi, & n = m. (\neq 0) \\ \pi, & n = m = 0. \end{cases}}$$

Lecture 31
Parametric Equation

11/20

Pset 8 Due Tue Dec 1

Exam 4 Dec 3

<u>Today</u> Parametric Equations

- totally different way to represent a function

t : parameter
graph is terms of coordinates functions of t

$x = f(t)$
$y = g(t)$

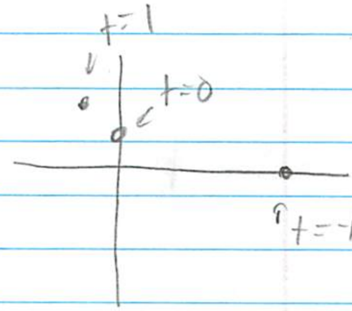
somtimes range for $t \in [a, b]$
- if not specified - could be everything

Eg $x(t) = t^2 - 2t$
$y(t) = t + 1$ } set of parametric equations

What does graph look like?

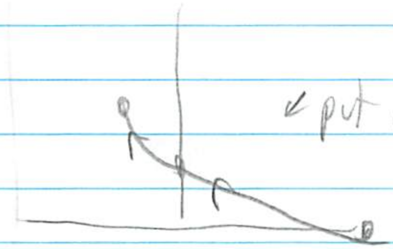
<u>Simple approch</u> - plot points

t	x	y
-1	3	0
0	0	1
1	-1	2



→ more info since have "time" also
t = "time"

(-) time is the past
0 t is now
(+) time is future



↙ put arrows to represent time

Second Approach

try to eliminate t variable in an equation relating x and y

$x = t^2 - 2t$ careful substitution

$y = t + 1 \longrightarrow t = y - 1$

$x = (y-1)^2 - 2(y-1)$

$x = y^2 - 2y + 1 - 2y + 2$

$x = y^2 - 4y + 3$

Example

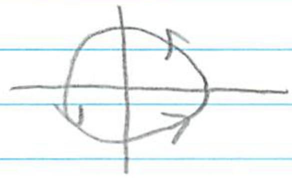
$x = \cos t \qquad t \in [0, 2\pi]$

$y = \sin t$

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

↑ parametric equation for unit circle

at $t=0$ we begin at $x=1$ $y=0$

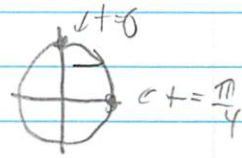


If $\in [0, 3\pi]$ ← retrace over

What about $x = \sin 2t$ $y = \cos 2t$ $t \in [0, 2\pi]$

$$x^2 + y^2 = \sin^2(2t) + \cos^2(2t) = 1$$

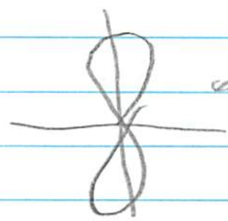
Still a circle
- but start at $(0,1)$
- and go clockwise
- get there twice as fast



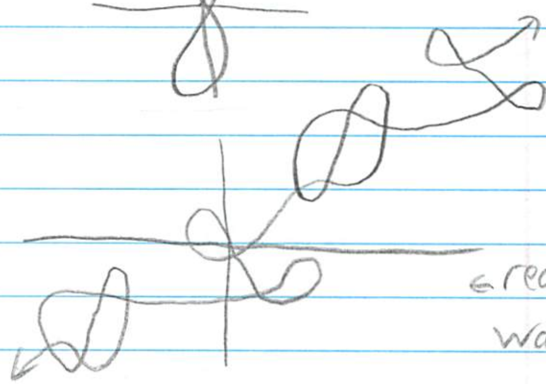
$\boxed{\text{* So } \underline{\text{way}} \text{ is important (time, etc)}}$

Harder Example

$x = \sin 2t$
$y = \cos t$



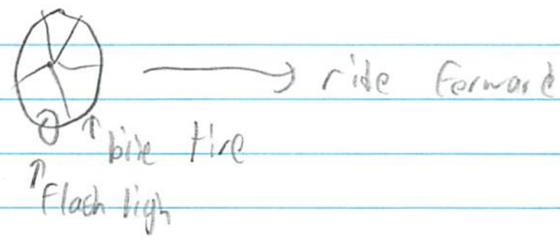
$x = t + 2\sin 2t$
$y = t + 2\cos 5t$



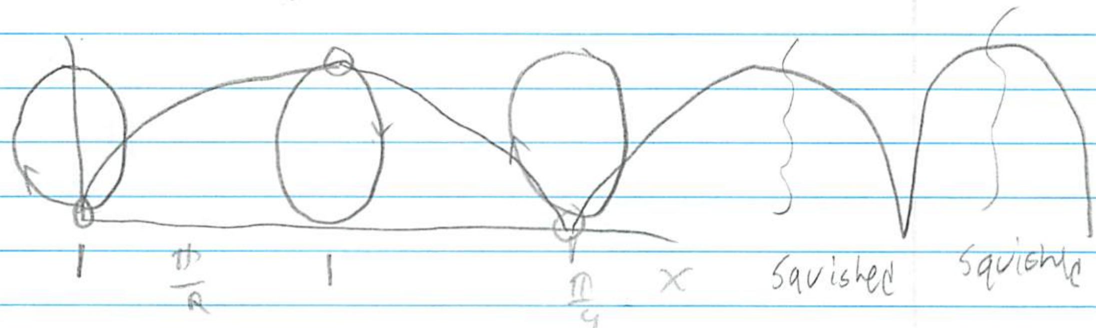
may be hard to solve for function
or impossible
- but can find tangent line at 1 pt by picking a little chunk

Hardest example

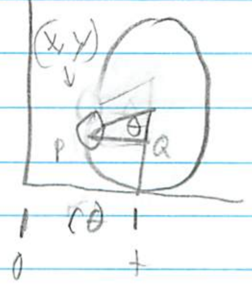
- physical situation in terms of parametric equations
- cycloid



path of light



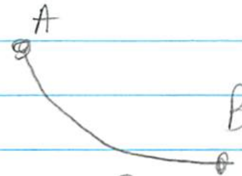
Write parametric equation for this curve



$$x(\theta) = |OT| - |PQ| = \boxed{r\theta - r\sin\theta}$$
$$y(\theta) = r - |CQ| = \boxed{r - r\cos\theta}$$

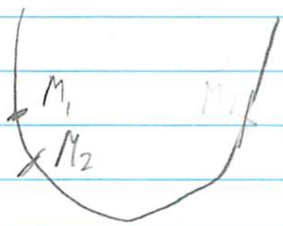
$\theta = 0 \Rightarrow (0,0)$

Cycloid is key to Brachistochrone Problem
- find curve between 2 pt in space A, B such that a marble rolls from A, B in the shortest amt of time
  - gravity only action



cycloid is fastest shape

2. Tautochrone problem - Find the curve on which no matter where marble is placed, it arrives at bottom of curve at same time



← meet at bottom
- any position of curve

Cool proof in Simmon's appendix uses Snell's Law
textbook

Calculus w/ Parametric Equations

$x = f(t)$
$y = g(t)$

Want to know equation for tangent line to curve at some t_0

$x_0 = x(t_0)$
$y_0 = y(t_0)$

Pretend we know function $y = F(x)$
Then substitute $g(t) = F(f(t))$

Differentiate $g'(t) = F'(f(t))f'(t)$

Wanted $F'(x) = F'(f(t)) = \boxed{\dfrac{g'(t)}{f'(t)}}$ provided $f'(t) \neq 0$

Looks eaiser in Liebniz notation

$$\frac{dy}{dt} = \frac{dy/dt}{dx/dt}$$

Find the tangent line to cycloid when $\theta = \frac{\pi}{3}$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\sin\theta}{r - r\cos\theta} = \frac{\sin\theta}{1-\cos\theta}$$

when $\theta = \frac{\pi}{3}$ $= \dfrac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}} = \sqrt{3} = \left.\dfrac{dy}{dx}\right|_{\theta=\frac{\pi}{3}}$

tangent line at $\theta = \frac{\pi}{3}$

$$y - y\left(\tfrac{\pi}{3}\right) = \sqrt{3}\left(x - x\left(\tfrac{\pi}{3}\right)\right)$$

$y\left(\frac{\pi}{3}\right) = r - r\cos\frac{\pi}{3}$ w/ example

$= r/2$

$$\int_a^b y\,dx = \int_\alpha^\beta g(t)\,f'(t)\,dt$$

bounds mean $x \in [a,b]$

$t = \alpha \to x = a$

$t = \beta \to x = b$

$x = f(t)$

want to solve $a = f(t)$

$b = f(t)$

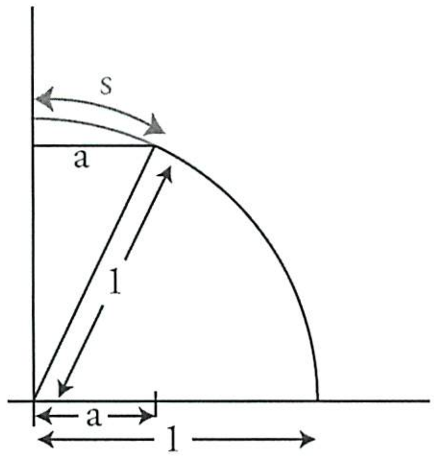


Figure 6: s = angle in radians.

Parametric Equations

Example 6.

$$x = a\cos t$$
$$y = a\sin t$$

Ask yourself: what's constant? What's varying? Here, t is variable and a is constant. Is there a relationship between x and y? Yes:

$$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$$

Extra information (besides the circle):
At $t = 0$,

$$x = a\cos 0 = a \quad \text{and} \quad y = a\sin 0 = 0$$

At $t = \dfrac{\pi}{2}$,

$$x = a\cos\frac{\pi}{2} = 0 \quad \text{and} \quad y = a\sin\frac{\pi}{2} = a$$

Thus, for $0 \le t \le \pi/2$, a quarter circle is traced counter-clockwise (Figure 7).

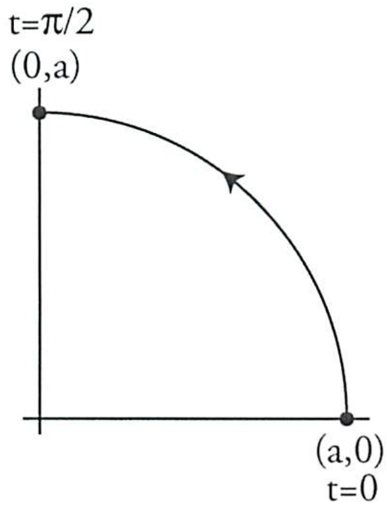


Figure 7: Example 6. $x = a\cos t,\ y = a\sin t$; the particle is moving counterclockwise.

Example 7: The Ellipse See Figure 8.

$$x = 2\sin t; \quad y = \cos t$$

$$\frac{x^2}{4} + y^2 = 1 (\implies (2\sin t)^2/4 + (\cos t)^2 = \sin^2 t + \cos^2 t = 1)$$

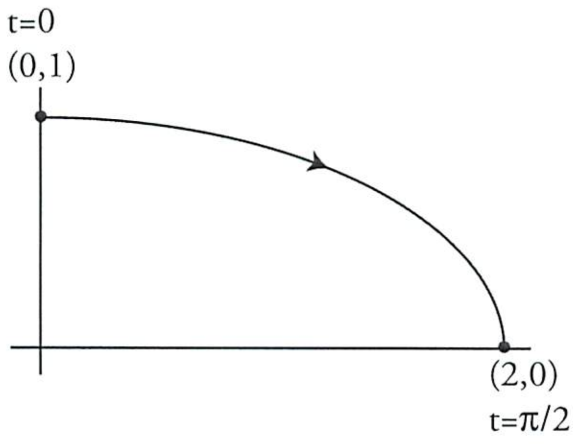


Figure 8: Ellipse: $x = 2\sin t,\ y = \cos t$ (traced clockwise).

Arclength ds **for Example 6.**

$$dx = -a\sin t\,dt, \quad dy = a\cos t\,dt$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(-a\sin t\,dt)^2 + (a\cos t\,dt)^2} = \sqrt{(a\sin t)^2 + (a\cos t)^2}\,dt = a\,dt$$

Recitation

Integration by Parts + Parametric 11/23

Relecture

$\int u\,dv = uv - \int v\,du$

ex $\int x \sec^2 x\,dx$

$u = x$ $dv = \sec^2 x\,dx$

$du = dx$ $v = \tan x$

$x\tan x - \int \tan x\,dx$

$x\tan x - \ln|\sec x|$

$\int \underbrace{\ln x}_{u}\,\underbrace{dx}_{dv}$

$du = \frac{1}{x}dx$

$x\ln x - \int dx$

$x\ln x - x$

$du = \frac{1}{x}dx$

$v = x$

$\int_1^2 x^2 \ln x\,dx$

~~$u = x^2$ $dv = \ln x\,dx$~~

~~$du = 2x\,dx$ $v = x\ln x - x$~~

~~↑want simpler ↑~~

$u = \ln x$ $dv = x^2$

$du = \frac{1}{x}dx$ $v = \frac{x^3}{3}$

$\ln x \cdot \frac{x^3}{3}\Big|_1^2 - \int_1^2 \frac{x^3}{3}\frac{1}{x}dx$

$\int \frac{x^2}{3}dx$ ← do 1st

$\frac{(2)^3}{3}\ln(2) - \frac{(2)^3}{9} - \frac{1}{9} - \left(\frac{(1)^3 \ln(1)}{3} - \frac{1^3}{9} - \frac{1}{1}\right)$

$8/3 \ln 2 - 7/9$

$$\boxed{\int e^x \sin x\,dx}$$

$u = e^x$ $\qquad$ $dv = \sin x\,dx$

$du = e^x\,dx$ $\qquad$ $v = -\cos x$

$$-e^x \cos x + \int \cos x\, e^x\,dx$$

*now integration by parts on this

=

$u = e^x$ $\qquad$ $dv = \cos x\,dx$

$du = e^x\,dx$ $\qquad$ $v = \sin x$

$$\boxed{-e^x \cos x + e^x \sin x - \int \sin x\, e^x\,dx}$$

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int \sin x\, e^x\,dx$$

$$2\int e^x \sin x\,dx = e^x(\sin x - \cos x)$$

- treat like algebra
- treat integral as unknown

$$\frac{e^x(\sin x - \cos x)}{2}$$ ← can evaluate to fin

1. What strategy would be sucessful

1. U-substition
2. Trig - identies sin cos
3. Trig substitution - triangle $\sqrt{a^2-x^2}$ ...
4. Partial fractions $\frac{P(x)}{Q(x)}$ $\quad \frac{A}{x} + \frac{B}{x+1}$

4 b. Complete the square ←

5. by parts $uv - \int v\,du$

Which system should I use?

$\int u^{-1/2}\,du$

should be 2 if trig

1. $\int \frac{\cos x}{\sqrt{1+\sin x}}\,dx$ ② ~~trig identities~~ ~~substitution~~ ① u substitution $u = 1+\sin x$ ↑complex denom and its deriv is there

2. $\int \frac{x+2}{\sqrt{1-x^2}}\,dx$ ③ trig substitution - or could do u split $\frac{x}{\sqrt{1-x^2}} + \frac{2}{\sqrt{1-x^2}}$

3. $\int \tan^3 t\,dt$ ② trig identities - splitting $\int \tan^2 t \tan t\,dt$

$\int \underbrace{\sec^2 - 1}_{u}\ \underbrace{\tan t}_{du} - \int \tan t\,dt$ ↑integrate $\frac{\sin}{\cos}$

4. $\int \frac{\ln x}{x^2}\,dx$ ~~① u substitution?~~ ⑤ by parts? ~~$u=\frac{1}{x}$~~ ~~$du = \ln x$~~ ~~$\frac{1}{u^2}\,du$~~ $u = \ln x$ $dv = x^{-2}$ $du = \frac{1}{x}$ $v = -x^{-1}$

5. $\int \frac{2}{x^2+4x+5}\,dx$ ④ partial fractions complete the square $(x+2)(x+2)+1$ x^2+4x+4

$\int \frac{dx}{1+x^2} = \arctan$

6. $\int \frac{x}{(x-1)^2}$ ④ partial fraction case 2

$\frac{A}{(x-1)} + \frac{B}{(x-1)^2} = \frac{x}{(x-1)^2}$

$A(x-1) + B = x$

$A = 1$

- could do u substitution

Parametric Equations

$x(t) = (t+1)^{1/2}$
$y(t) = t^{1/2}$ ← example

- you have etch a sketch
- 2 knobs
- does not matter how fast you turn it

$y = x$ ← turn both exactly the same

$x = t^3 \quad -\infty < t < \infty$

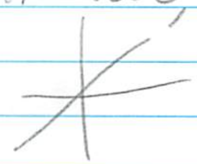
$y = t^3$

? will plot $y = x$

plot each point at t

$t = 10$ at 1000, 1000

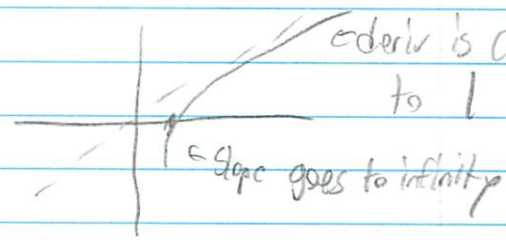
→ same line much faster



~~x~~ t tells you how fast you are turning it

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$ can't cancle out ts
how to find 1st deriv

$$= \frac{(t+1)^{1/2}}{t^{1/2}}$$



Exam Thur
HW Due Tue

Lecture 32
Parametric Equations 2

11/24

Last time

easy parts of parametric equations

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{if} \quad \frac{dx}{dt} \neq 0$$

$y = g(t) \qquad x = f(t)$

$\frac{dy}{dt} = g'(t) \qquad \frac{dx}{dt} = f'(t)$ don't need to eliminate /dt

This time

new application : arc lenght / surface area

Area

usually $y = f(x)$

and integrating $\int_a^b f(x)\,dx \rightarrow$ By substitution $= \int_a^b y\,dx$

for area under curve from $x = a$ to $x = b$

$$= \int_\alpha^\beta g(t)\, f'(t)\, dt$$

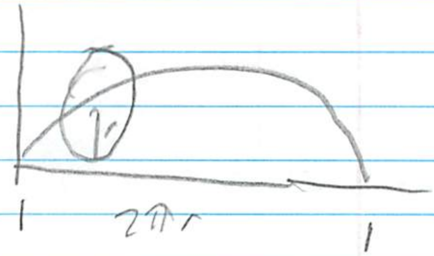
$y = g(t) \qquad x = f(t) \qquad x = a$

$dx = f'(t)\,dt \qquad$ find $t = \alpha$

so that $f(\alpha) = a$

Set up area under 1 hump cycloid

$y = r(1-\cos\theta)$
$x = r\theta - \sin\theta$
$dx = r(1-\cos\theta)\,d\theta$ ← find



$$\int_0^{2\pi r} y\,dx$$

$$\int_0^{2\pi} r^2(1-\cos\theta)^2\,d\theta$$

$x=0$ find θ so that
$r\theta - r\sin\theta = 0$
$\theta = 0$

$r\theta - r\sin\theta = 2\pi r$
$\theta = 2\pi$

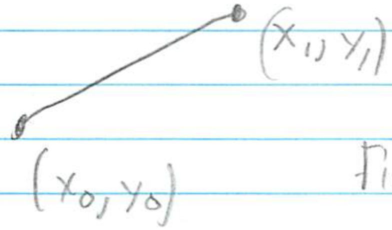
$$\int_0^{2\pi} r^2\Big(1-2\cos\theta+\underbrace{\cos^2\theta}_{\frac{1}{2}(1+\cos 2\theta)}\Big)\,d\theta$$

$= 3\pi r^2$

- major thought problems centuries ago
- people spent whole life on this
-

Arc lenght

0.



Find distance

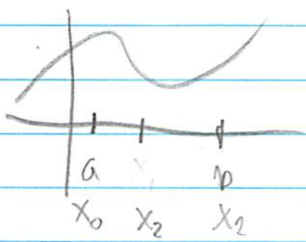
- distance formula

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

1.

add up each line individually using distance formula

2.



hard
approximate w/ pair of lines
- piecewise linear curve

Define the arc lenght

$$\lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{|P_i P_{i-1}|}_{\text{distance between the 2 points } (x_i, y_i)\ (x_{i-1}, y_{i-1})}$$

As make more and more slices it gets better
Can we solve w/ integrals - yes
- since Riemann sum?

$$|P_i P_{i-1}| = g(x_i^*)\,\Delta x$$

$$\lim_{n \to \infty} \sum_{i=1} g(x_i^*)\,\Delta x \quad \leftarrow \Delta x = x_i - x_{i-1}$$

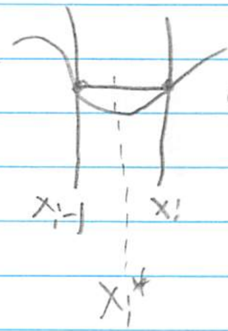
some function
evaluated at the point
x_i^* in the ith interval

$$\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

$$y_i = f(x_i)$$

$$= \underbrace{(x_i - x_{i-1})}_{\Delta x} \cdot \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}$$

slope of secent line
b/w 2 pts



← Mean Value theorm
a point equals it
$f(x_i^*)$

$$\Delta x \sqrt{1 + f(x_i^*)^2}$$

Therefore arc lenght $= \int_a^b \sqrt{1 + f'(x)^2}\,dx$

ugly and hard
so only limited #
of examples
can do

Eg Arc lenght of 1 hump of sine function

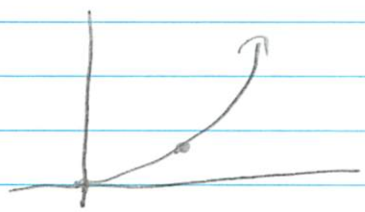
$y = \sin x$
$y'(x) = \cos x$

$$\underbrace{\int_0^{\pi} \sqrt{1+\cos^2 x}\, dx}_{\text{impossible integral}}$$

← no antideriv in terms of elementary functions

Eg2 Arc lenght along parabola doable but ugly

$y = x^2$ from $(0,0)$ to $(1,1)$



$$= \int_0^1 \sqrt{1+4x^2}\, dx$$

$x = \frac{1}{2}\tan\theta$

we can solve it but ugly

$$\boxed{= \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5}+2)}{4}}$$

On test would have to only set up unless really easy

Can do in parametric equations - no harder

$$AL = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

$dx = \frac{dx}{dt} dt$

$x = f(t)$ can also write $dx = f'(t)\,dt$
$y = g(t)$

Lecture 31: Parametric Equations, Arclength, Surface Area

Arclength, continued

Example 1. Consider this parametric equation:

$$x = t^2 \quad y = t^3 \quad \text{for } 0 \le t \le 1$$

$$x^3 = (t^2)^3 = t^6; \quad y^2 = (t^3)^2 = t^6 \quad \implies x^3 = y^2 \implies y = x^{2/3} \qquad 0 \le x \le 1$$

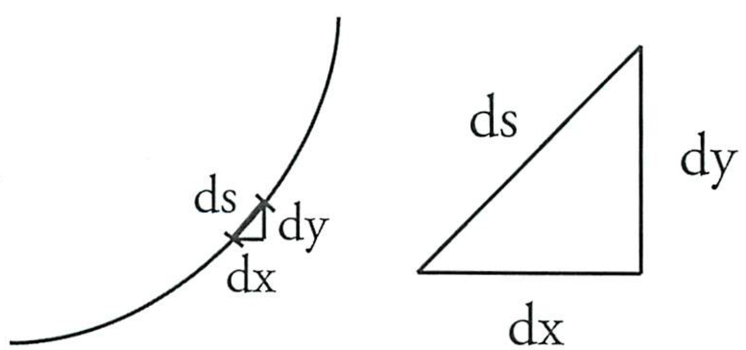


Figure 1: Infinitesimal Arclength.

$$(ds)^2 = (dx)^2 + (dy)^2$$

$$(ds)^2 = \underbrace{(2t\,dt)^2}_{(dx)^2} + \underbrace{(3t^2\,dt)^2}_{(dy)^2} = (4t^2 + 9t^4)(dt)^2$$

$$\text{Length} = \int_{t=0}^{t=1} ds = \int_0^1 \sqrt{4t^2 + 9t^4}dt = \int_0^1 t\sqrt{4 + 9t^2}dt$$

$$= \frac{(4 + 9t^2)^{3/2}}{27}\Big|_0^1 = \frac{1}{27}(13^{3/2} - 4^{3/2})$$

Even if you can't evaluate the integral analytically, you can always use numerical methods.

Surface Area (surfaces of revolution)

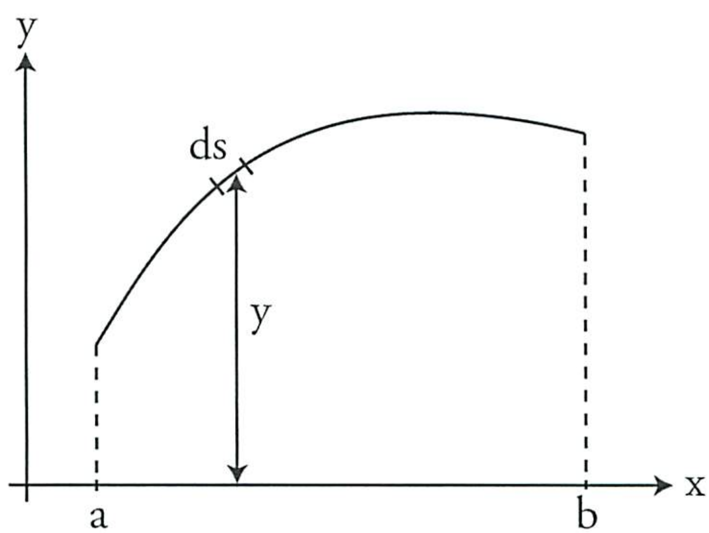


Figure 2: Calculating surface area

ds (the infinitesimal curve length in Figure 2) is revolved a distance $2\pi y$. The surface area of the thin strip of width ds is $2\pi y\, ds$.

Example 2. Revolve Example 1 ($x = t^2, y = t^3, 0 \le t \le 1$) around the x-axis. Refer to Figure 3.

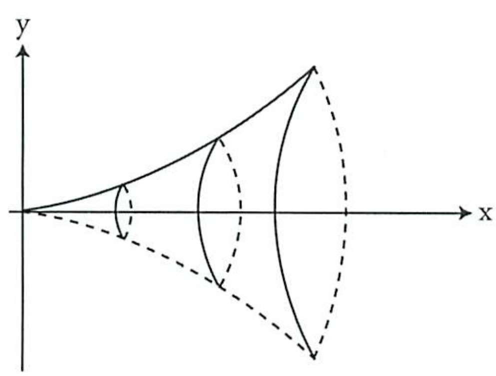


Figure 3: Curved surface of a trumpet.

$$\text{Area} = \int 2\pi y\, ds = \int_0^1 2\pi \underbrace{t^3}_{y} \underbrace{t\sqrt{4+9t^2}\,dt}_{ds} = 2\pi \int_0^1 t^4\sqrt{4+9t^2}\,dt$$

Now, we discuss the method used to evaluate

$$\int t^4(4+9t^2)^{1/2}dt$$

We're going to ignore the factor of 2π. You can reinsert it once you're done evaluating the integral. We use the trigonometric substitution

$$t = \frac{2}{3}\tan u; \quad dt = \frac{2}{3}\sec^2 u\, du; \quad \tan^2 u + 1 = \sec^2 u$$

Putting all of this together gives us:

$$\begin{aligned}
\int t^4(4+9t^2)^{1/2}\,dt &= \int \left(\frac{2}{3}\tan u\right)^4 \left(4+9\left(\frac{4}{9}\tan^2 u\right)\right)^{1/2} \left(\frac{2}{3}\sec^2 u\, du\right) \\
&= \left(\frac{2}{3}\right)^5 \int \tan^4 u(2\sec u)(\sec^2 u\, du)
\end{aligned}$$

This is a tan − sec integral. It's doable, but it will take a long time for you to work the whole thing out. We're going to stop evaluating it here.

Example 3 Let's use what we've learned to find the surface area of the unit sphere (see Figure 4).

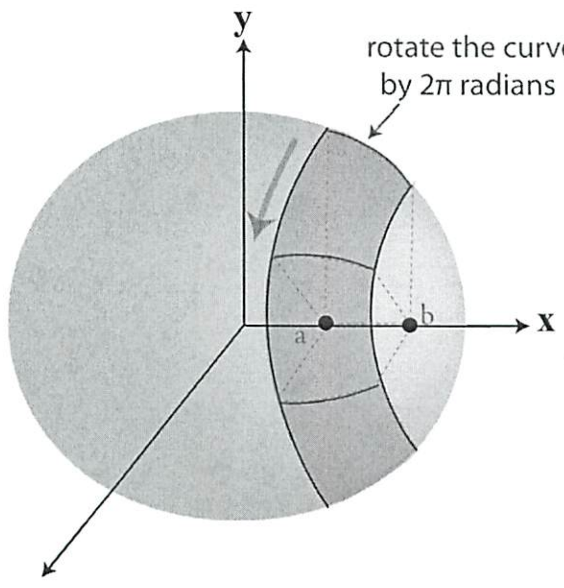


Figure 4: Slice of spherical surface (orange peel, only, not the insides).

For the top half of the sphere,

$$y = \sqrt{1-x^2}$$

We want to find the area of the spherical slice between $x = a$ and $x = b$. A spherical slice has area

$$A = \int_{x=a}^{x=b} 2\pi y \, ds$$

From last time,

$$ds = \frac{dx}{\sqrt{1-x^2}}$$

Plugging that in yields a remarkably simple formula for A:

$$A = \int_a^b 2\pi\sqrt{1-x^2}\frac{dx}{\sqrt{1-x^2}} = \int_a^b 2\pi \, dx$$
$$= 2\pi(b-a)$$

Special Cases

For a whole sphere, $a = -1$, and $b = 1$.

$$2\pi(1-(-1)) = 4\pi$$

is the surface area of a unit sphere.

For a half sphere, $a = 0$ and $b = 1$.

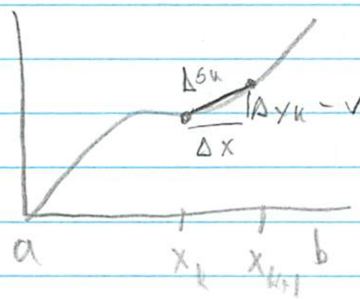
$$2\pi(1-0) = 2\pi$$

Recitation

Arc Lenght + Surface Area 11/30

Integrating + Parametrics
- arc lenght
- surface area

Can do in parametric form or function form



$\Delta x^2 + \Delta {y_k}^2 = \Delta {s_k}^2$ take Δ infetesimal small

$ds^2 = dx^2 + dy^2$

arc lenght
- add a bunch of Δs_k
- let Δ go to 0

$$= \int_a^b ds = \sum_{n=1}^{n} \Delta s_k$$

(some theory behind this not explaining 18.014)

Function form $\quad \frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2}$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

solve for ds

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Parametric form

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\sqrt{x'^2 + y'^2}\, dt$$

for $d\theta$ does not matter

Arc lenght problems

ex 1 Find the arc lengh of

$y = \ln(\cos x)$ for x between 1 and 2

$\to dx \to \frac{1}{\cos x} \cdot \sin x = \tan x$

$\int_1^2 \sqrt{1 + \tan^2 x}$

$\int_1^2 \sqrt{\sec^2 x}$

$\int_1^2 |\sec x|$

$\left. \ln|\sec x + \tan x| + C \right|_1^2$ ← stop here

$\ln(\sec 2 + \tan 2) - \ln(\sec 1 + \tan 1)$

ex 2

Find lenght of ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x}{a} = \cos\theta$ $\quad \frac{y}{b} = \sin\theta$ ←

$x = a\cos\theta$ $\quad y = b\sin\theta$

known

circle → $x^2 + y^2 = r^2$

$\underbrace{\left(\frac{x}{r}\right)^2}_{\cos} + \underbrace{\left(\frac{y}{r}\right)^2}_{\sin} = 1$

$\cos^2\theta + \sin^2\theta = 1$

What are the bounds? $0, 2\pi$ give you an ellipse

$2\pi, 4\pi$

$-\pi, \pi$ ← same curve - start different

- just set up
  - don't have to do it all

$$\int_0^{2\pi} \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\, d\theta$$

Find ds for

$$y = \frac{1}{4}x^4 + \frac{1}{8x^2}$$

don't forget how to take a simple deriv

$\leftarrow \cancel{y' = \frac{3}{4}x^3 + \frac{1}{16x}}$ $y' = x^3 - \frac{1}{4x^3}$

$\cancel{\frac{3}{4}x^3 + \frac{1}{16x}}$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1 + \cancel{\left(\frac{3}{4}x^3 + \frac{1}{16}x\right)^2}}\, dx \qquad y'^2 = x^6 - \frac{1}{2} + \frac{1}{16}x^6$$

$$= \cancel{\sqrt{1 + \frac{9}{16}x^5 + \frac{1}{16\cdot 16}x^2}}$$

$$= \sqrt{1 + x^6 - \frac{1}{2} + \frac{1}{16}x^6}\, dx$$

$$= \sqrt{x^6 + \frac{1}{2} + \frac{1}{16}x^6}\, dx$$

↑ see perfect square
- middle term just different w/ sign

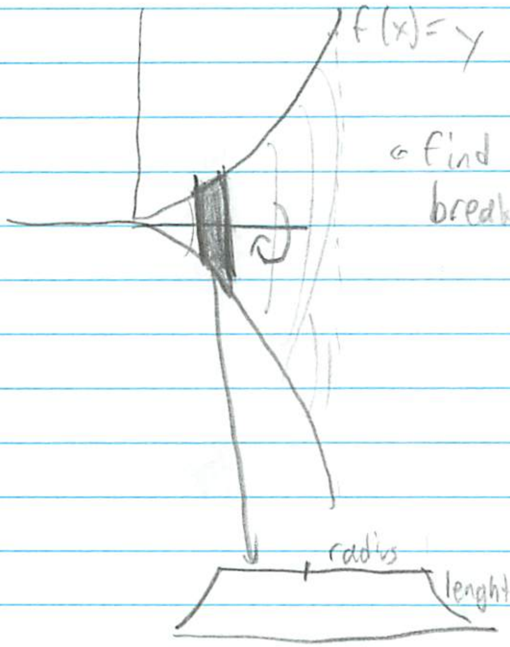
$$= \sqrt{\left(x^3 + \frac{1}{4x^3}\right)^2}\, dx$$

$$= \left| x^3 + \frac{1}{4x^3} \right| dx$$

- if $\ominus$ somewhere when you integrate must split $\oplus$ $\ominus$ portions

Surface area

- add up lateral area of cylinders
  - truncated cones



← find surface area break up pieces + look at it

$SA = 2\pi \text{ radius} \cdot \text{lenght}$

$2\pi \text{ radius } ds$

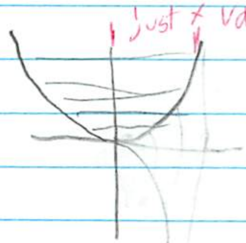
lose since function or parametric

Total $SA = \int 2\pi \text{ radius } ds$

↑ height of original function is height in this case

Find SA $y = x^2$ rotated about x-axis

$0 \le x \le 2$



just x value

$\int_0^2 2\pi (x) \sqrt{1 + (2x)^2}\, dx$ ← $\frac{dx}{dx}$

$\int_0^2 2\pi x \sqrt{1 + 4x^2}\, dx$ ← stop

~~$\int_0^2 2\pi x \,(2x + 1)$ +2x~~

Lecture 32
Infinite Series + Exam 4 Review

12/1

One final topic: infinite series
and exam 4 Thur
- no makeup

Linear approximation to $f(x)$ at $x = a$

$$f(a) + f'(a)(x-a)$$

Quadratic approx

$$f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

why this form?
Wanted the derivitves of the approximation to match the derivitves of $f(x)$

$L(a) = f(a)$ $\quad$ $Q(a) = f(a)$
$L'(a) = f'(a)$ $\quad$ $Q'(a) = f'(a)$
$L''(a) = 0$ $\quad$ $Q''(a) = f''(a)$
$\quad$ $Q'''(a) = 0$

Basic principle: To approximate $f(x)$ choose a polynomial whose first few derivitives at $x=a$ match the derivitives of $f(x)$ at $x=a$

Best aprox to $f(x)$ at $x=a$ using degree n polynomial
- want first n derivitves to match

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)}a}{n!}(x-a)^n$$

Taylor

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

the more derivs you match the better it will be

So take $\lim_{n \to \infty} T_n(x) = \sum_{n \to \infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$

kinda like Riemann sums

Questions

Fri → 1. Does the limit exist?
- Do there exist values for X so that limit exists?

2. Can we ever write down formulas for $f^{(i)}(a)$ for our function $f(x)$?
- Can you write the #s in?

Tue → 3. Does the $\lim_{n \to \infty} T_n =$ function?

Write Down

$T_n(x)$ for $f(x) = e^x$ at $x = 0$

Need to compute ith derivatives evaluated at $0 = a$

$f^{(i)}(0) = e^x \big|_0 = 1$

$$T_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!} \quad \text{for } e^x \text{ at } a=0$$

$$\boxed{\underline{\text{Claim}} \quad \lim_{n\to\infty} \sum_{i=0}^{n} \frac{x^i}{i!} = e^x}$$

<u>Test</u>

$x = 1) \quad \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{i!} = e$

$x=0 \quad \frac{0^0}{0!} = 1$

<u>Exam 4</u>

① $\int \tan^{-1} x \, dx$ — by parts (use if nothing left) <u>or</u> u sub. ← not on its own

② $\int \tan^3 x \sec^5 x \, dx$ — trig identity + substitution

③ $\int \frac{\sqrt{a^2-x^2}}{x} dx$ — trig substitution triangle
<u>- inverse</u>

④ $\int \frac{2x^3 + x^2 + 2x - 1}{x^4 - 1} dx$ — partial fraction

⑤ Surface area / arc lenght / tangent lines
- in parametric and function

1. $\int \tan^{-1} x \, dx$

$u = \tan^{-1} x \qquad dv = dx$
$du = \dfrac{1}{1+x^2} \qquad v = x$

$x \tan^{-1} x - \int \dfrac{x}{1+x^2} \, dx$

$w = 1+x^2$
$dw = 2x \, dx$

$\dfrac{1}{2} \int \dfrac{1}{w} \, dw$ ← be able to recognize

$= \frac{1}{2} \ln|1+x^2| + c$

2. $\int \tan^3 x \sec^5 x$

↑ trying to save some trig values for substitution

* memorize identitys

$\cancel{u = \tan x}$
$\cancel{du = \sec^2 x}$ or $u = \sec x$
$du = \sec x \tan x$

↑ pick

$\int \tan x \, (\tan^2 x) \sec^5 x \, dx$
$\int \tan x$

↑ know identity $\dfrac{\sin^2 x}{\cos^2 x} + \dfrac{\cos^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x}$

$\tan^2 x + 1 = \sec^2 x$

$\int \tan x \, (\sec^2 x - 1) \sec^5 x \, dx$

$\int \sec^6 x \, (\sec x \tan x) dx - \int \sec^4 x \, (\sec x \tan x) \, dx$

$u = \sec x \qquad du = \sec x \tan x$

$$= \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

<u>Arc lenght under 1 hump cycloid</u>

$y = r(1-\cos\theta)$
$x = r\theta - r\sin\theta$
$dx = r(1-\cos\theta)\,d\theta$
$dy = r\sin\theta\, d\theta$

$$\int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

$$\int_0^{2\pi} r\sqrt{1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1}}\, d\theta$$

$$\int_0^{2\pi} r\sqrt{2(1-\cos\theta)}\, d\theta$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$r\int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$$

$$\int_0^{2\pi} 2r\left|\sin\frac{\theta}{2}\right| d\theta$$

know sin is (+) since ... $[0, 2\pi]$

$$2r\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta$$

$$\boxed{8r}$$

3. $\int \tan^2 x \sec x \, dx$

↑ bad problem

can convert to all sec

$\sec^2 x - 1$

$$\int \underbrace{\sec^3 x \, dx}_{\text{parts}} - \int \underbrace{\sec x \, dx}_{\text{memorize!}}$$

4.

$$\boxed{\begin{array}{ll} \sqrt{a^2 - x^2} & x = a\sin\theta \\ \sqrt{a^2 + x^2} & x = a\tan\theta \\ \sqrt{x^2 - a^2} & x = a\sec\theta \end{array}}$$

$$\int \frac{2x^3 + x^2 + 2x - 1}{x^4 - 1}$$

↑ diff of squares $(x^2 - 1)(x^2 + 1)$

$(x-1)(x+1)(x^2+1)$

$$\frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx + D}{x^2 + 1}$$

↑ ↑

can solve w/ coverup method

Solutions $A = 1$

$B = 1$

$C = 0$

$D = 1$

$$\int \left(\frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x^2+1} \right) dx$$

Arc Lenght

- memorize formula
- won't have to re derive

$$\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{if} \quad y = f(x)$$

$$\int \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \quad \text{in} \quad x = g(y)$$

Parametric

$$\int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

for T

Surface Area

- know it

We are not doing Polar Coords

Lecture 32: Polar Co-ordinates, Area in Polar Co-ordinates

Polar Coordinates

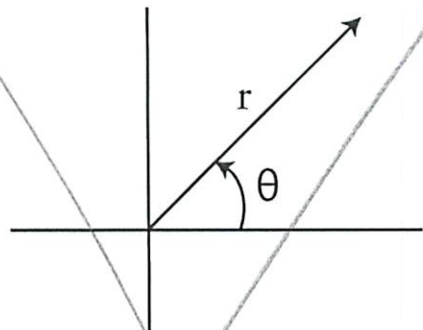


Figure 1: Polar Co-ordinates.

In polar coordinates, we specify an object's position in terms of its distance r from the origin and the angle θ that the ray from the origin to the point makes with respect to the x-axis.

Example 1. What are the polar coordinates for the point specified by $(1, -1)$ in rectangular coordinates?

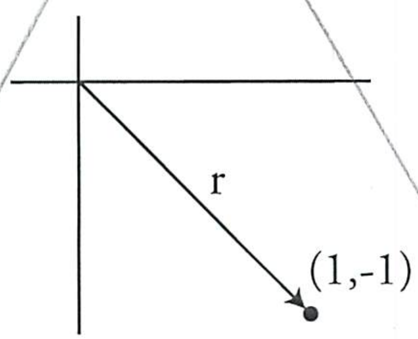


Figure 2: Rectangular Co-ordinates to Polar Co-ordinates.

$$\begin{aligned} r &= \sqrt{1^2 + (-1)^2} = \sqrt{2} \\ \theta &= -\frac{\pi}{4} \end{aligned}$$

In most cases, we use the convention that $r \geq 0$ and $0 \leq \theta \leq 2\pi$. But another common convention is to say $r \geq 0$ and $-\pi \leq \theta \leq \pi$. All values of θ and even negative values of r can be used.

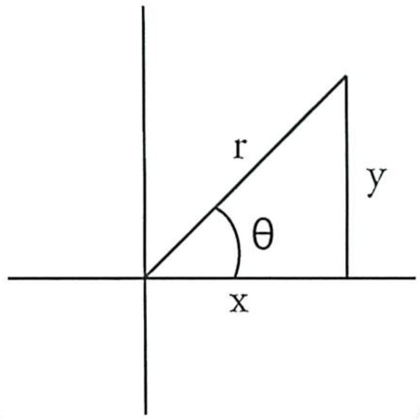


Figure 3: Rectangular Co-ordinates to Polar Co-ordinates.

Regardless of whether we allow positive or negative values of r or θ, what is *always* true is:

$$\boxed{x = r\cos\theta \quad \text{and} \quad y = r\sin\theta}$$

For instance, $x = 1$, $y = -1$ can be represented by $r = -\sqrt{2}$, $\theta = \dfrac{3\pi}{4}$:

$$1 = x = -\sqrt{2}\cos\frac{3\pi}{4} \quad \text{and} \quad -1 = y = -\sqrt{2}\sin\frac{3\pi}{4}$$

Example 2. Consider a circle of radius a with its center at $x = a$, $y = 0$. We want to find an equation that relates r to θ.

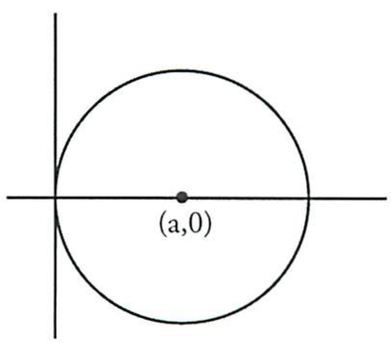


Figure 4: Circle of radius a with center at $x = a, y = 0$.

We know the equation for the circle in rectangular coordinates is

$$(x-a)^2 + y^2 = a^2$$

Start by plugging in:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

This gives us

$$(r\cos\theta - a)^2 + (r\sin\theta)^2 = a^2$$
$$r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta = a^2$$
$$r^2 - 2ar\cos\theta = 0$$
$$\boxed{r = 2a\cos\theta}$$

The range of $0 \le \theta \le \frac{\pi}{2}$ traces out the top half of the circle, while $-\frac{\pi}{2} \le \theta \le 0$ traces out the bottom half. Let's graph this.

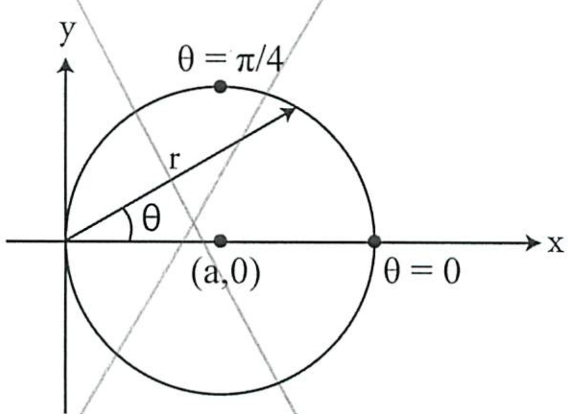


Figure 5: $r = 2a\cos\theta, \quad -\pi/2 \le \theta \le \pi/2$.

At $\theta = 0$, $r = 2a \implies x = 2a,\ y = 0$
At $\theta = \frac{\pi}{4}$, $r = 2a\cos\frac{\pi}{4} = a\sqrt{2}$

The main issue is finding the range of θ tracing the circle once. In this case, $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

$$\theta = -\frac{\pi}{2} \text{ (down)}$$
$$\theta = \frac{\pi}{2} \text{ (up)}$$

Weird range (avoid this one): $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. When $\theta = \pi$, $r = 2a\cos\pi = 2a(-1) = -2a$. The radius points "backwards". In the range $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, the same circle is traced out a second time.

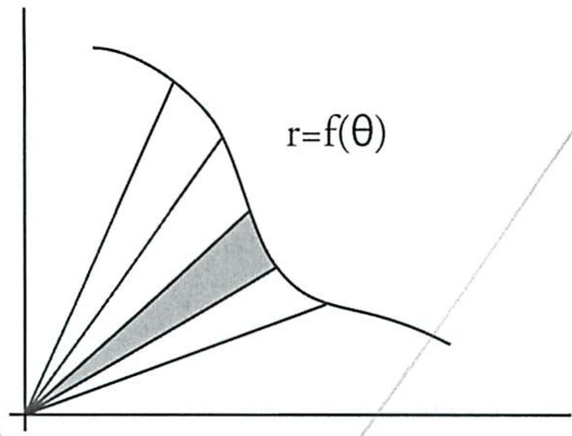


Figure 6: Using polar co-ordinates to find area of a generic function.

Area in Polar Coordinates

Since radius is a function of angle ($r = f(\theta)$), we will integrate with respect to θ. The question is: what, exactly, should we integrate?

$$\int_{\theta_1}^{\theta_2} ??\, d\theta$$

Let's look at a very small slice of this region:

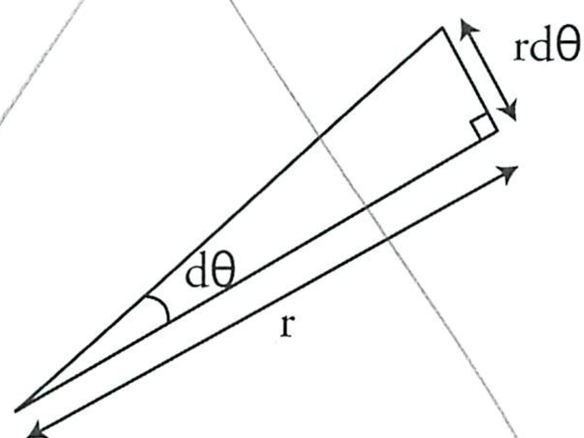


Figure 7: Approximate slice of area in polar coordinates.

This infinitesimal slice is approximately a right triangle. To find its area, we take:

$$\text{Area of slice} \approx \frac{1}{2}(\text{base})\,(\text{height}) = \frac{1}{2}r(r\, d\theta)$$

So,

$$\text{Total Area} = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2\, d\theta$$

Example 3. $r = 2a\cos\theta$, and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (the circle in Figure 5).

$$A = \text{area} = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(2a\cos\theta)^2\, d\theta = 2a^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

Because $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$, we can rewrite this as

$$A = \text{area} = \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\, d\theta = a^2 \int_{-\pi/2}^{\pi/2} d\theta + a^2 \int_{-\pi/2}^{\pi/2} \cos 2\theta\, d\theta$$

$$= \pi a^2 + \frac{1}{2}\sin 2\theta \Big|_{-\pi/2}^{\pi/2} = \pi a^2 + \frac{1}{2}[\cancelto{0}{\sin\pi - \sin(-\pi)}]$$

$$A = \text{area} = \pi a^2$$

Example 4: Circle centered at the Origin.

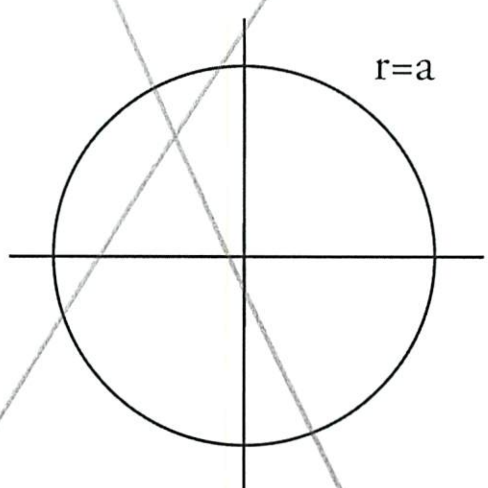


Figure 8: Example 4: Circle centered at the origin

$$x = r\cos\theta; \;\; y = r\sin\theta$$

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

The circle is $x^2 + y^2 = a^2$, so $r = a$ and

$$x = a\cos\theta; \;\; y = a\sin\theta$$

$$A = \int_0^{2\pi} \frac{1}{2}a^2\, d\theta = \frac{1}{2}a^2 \cdot 2\pi = \pi a^2.$$

Example 5: A Ray. In this case, $\theta = b$.

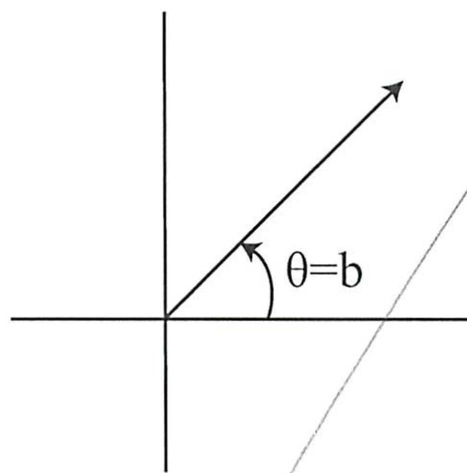


Figure 9: Example 5: The ray $\theta = b, 0 \le r < \infty$.

The range of r is $0 \le r < \infty$; $x = r\cos b$; $y = r\sin b$.

Example 6: Finding the Polar Formula, based on the Cartesian Formula

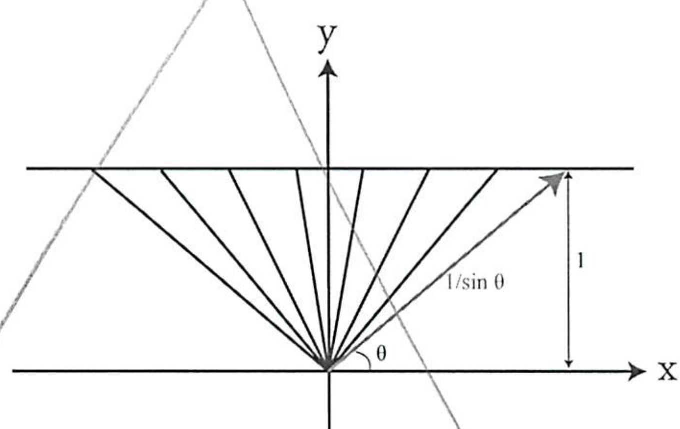


Figure 10: Example 6: Cartesian Form to Polar Form

Consider, in cartesian coordinates, the line $y = 1$. To find the polar coordinate equation, plug in $y = r\sin\theta$ and $x = r\cos\theta$ and solve for r.

$$r\sin\theta = 1 \implies r = \frac{1}{\sin\theta} \quad \text{with} \quad 0 < \theta < \pi$$

Example 7: Going back to (x, y) coordinates from $r = f(\theta)$.
Start with

$$r = \frac{1}{1 + \frac{1}{2}\sin\theta}.$$

Hence,

$$r + \frac{r}{2}\sin\theta = 1$$

Plug in $r = \sqrt{x^2 + y^2}$:

$$\sqrt{x^2 + y^2} + \frac{y}{2} = 1$$

$$\sqrt{x^2 + y^2} = 1 - \frac{y}{2} \quad \Longrightarrow \quad x^2 + y^2 = \left(1 - \frac{y}{2}\right)^2 = 1 - y + \frac{y^2}{4}$$

Finally,

$$x^2 + \frac{3y^2}{4} + y = 1$$

This is an equation for an ellipse, with the origin at one focus.

Useful conversion formulas:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Example 8: A Rose $r = \cos(2\theta)$
The graph looks a bit like a flower:

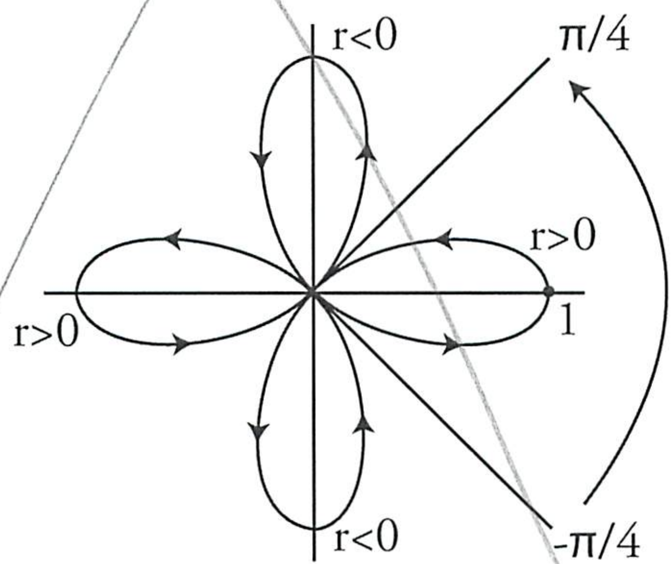


Figure 11: Example 8: Rose

For the first "petal"

$$-\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

Note: Next lecture is Lecture 34 as Lecture 33 is Exam 4.

Exam 4 Review

1. Trig substitution and trig integrals.
2. Partial fractions.
3. Integration by parts.
4. Arc length and surface area of revolution
5. Polar coordinates
6. Area in polar coordinates.

Questions from the Students

- Q: What do we need to know about parametric equations?
- A: Just keep this formula in mind:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Example: You're given

$$x(t) = t^4$$

and

$$y(t) = 1 + t$$

Find s (length).

$$ds = \sqrt{(4t^3)^2 + (1)^2}dt$$

Then, integrate with respect to t.

- Q: Can you quickly review how to do partial fractions?
- A: When finding partial fractions, first check whether the degree of the numerator is greater than or equal to the degree of the denominator. If so, you first need to do algebraic long-division. If not, then you can split into partial fractions.

Example.

$$\frac{x^2 + x + 1}{(x-1)^2(x+2)}$$

We already know the *form* of the solution:

$$\frac{x^2 + x + 1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

There are two coefficients that are easy to find: B and C. We can find these by the cover-up method.

$$B = \frac{1^2 + 1 + 1}{1 + 2} = \frac{3}{3} \qquad (x \to 1)$$

To find C,

$$C = \frac{(-2)^2 - 2 + 1}{(-2-1)^2} = \frac{1}{3} \qquad (x \to -2)$$

To find A, one method is to plug in the easiest value of x other than the ones we already used $(x = 1, -2)$. Usually, we use $x = 0$.

$$\frac{1}{(-1)^2(2)} = \frac{A}{-1} + \frac{1}{(-1)^2} + \frac{1/3}{2}$$

and then solve to find A.

The Review Sheet handed out during lecture follows on the next page.

Michael Plasmeier

$\frac{30}{43}$

18.01 Problem Set 8 – Fall 2009

Due Tuesday, 12/01/09, 1:45 pm in 2-106

Part I (12 points)

Lecture 31. Friday, Nov. 20 Calculus using parametric equations.
Read: Read 17.1, 17.2 Work: 4E-2, 3, 8.

Lecture 32. Tuesday, Nov. 24 Arclength and Surface area
Read 7.5, 7.6 Work: 4F-1d, 4, 5, 8; 4G-2, 5.
If a curve is given by $x = x(t)$, $y = y(t)$, to find its arclength, use ds in the form

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\ ,$$

and integrate from start to finish: from $t = t_0$ to $t = t_1$.

THANKSGIVING BREAK – No recitation on Wednesday, Nov. 25

Lecture 33. Tuesday, Dec. 1 Review for Exam 4.

Thursday, Dec. 3 **Exam** covering techniques of integration, parametric equations, arc length and surface area. Location to be announced.

Part II (31 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Lec 30, 12 pts: 3 + 3 + 2 + 2 + 2) This question involves integration by parts. Let

$$I_n = \int_0^{\pi/2} \sin^n x\, dx$$

a) Use integration by parts to show that, for odd powers (e.g. $2n + 1$) of sine,

$$I_{2n+1} = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}$$

b) Use integration by parts to show that, for even powers (e.g. $2n$) of sine,

$$I_{2n} = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}\,\frac{\pi}{2}$$

c) Show that $I_{2n+2} \le I_{2n+1} \le I_{2n}$

d) Use parts (a) and (b) to show that

$$\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$$

and hence deduce that $\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$.

e) Use parts (a),(b), and (d) to find

$$\lim_{n\to\infty} \frac{2}{1} \cdot \frac{2}{1} \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}.$$

2. (Lec 31, 7 pts: 3 + 1 + 3)

a) Find the algebraic equation in x and y for the curve

$$x = a\cos^k t, \quad y = a\sin^k t.$$

Draw the portion of the curve $0 \leq t \leq \pi/2$ in the three cases $k = 1$, $k = 2$, $k = 3$.

b) Without calculation, find the arclength in the cases $k = 1$ and $k = 2$.

c) Find a definite integral formula for the length of the curve for general k. Then evaluate the integral in the three cases $k = 1$, $k = 2$, and $k = 3$. (Your answer in the first two cases should match what you found in part (b), but the calculation takes more time.)

3. (Lec 32, 9 pts: 3 + 1 + 3 + 2) The hyperbolic sine and cosine are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

a) Show that

i) $\frac{d}{dx}\sinh x = \cosh x$ and $\frac{d}{dx}\cosh x = \sinh x$.

ii) $\cosh^2 x = 1 + \sinh^2 x$ and

iii) $\cosh^2 x = \frac{1+\cosh 2x}{2}$

b) What curve is described parametrically by $x = \cosh t$, $y = \sinh t$? (Give the equation and its name.)

c) The curve $y = \cosh x$ is known as a *catenary.* It is the curve formed by a chain whose two ends are held at the same height.

i) Sketch the curve

ii) Find its arclength from the lowest point to the point $(x_1, \cosh x_1)$ for a fixed $x_1 > 0$.

d) Find the area of the surface of revolution formed by revolving the portion of the curve from part (c) around the x-axis. This surface is known as a catenoid. It is interesting because it is the surface of least area connecting the two circles that form its edges. If you dip two circles of wire in a soap solution, then (with some coaxing) a soap film will form in this shape. In general, the soap films try to span a frame of wires with a surface with the least area possible.

Michael Plasmeier

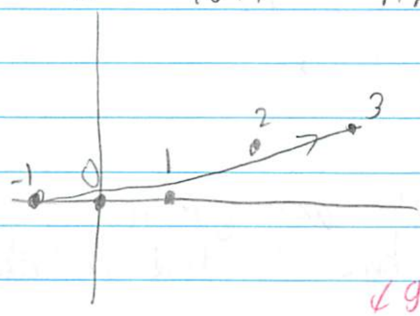
PSet 8

12/1

4E-2 Find the rectangular expression for

$x = t + \frac{1}{t}$ $y = t - \frac{1}{t}$ compute (x^2 and y^2)

t	x	y
-1	-2	0
0	0	8
1	2	0
2	2.5	1.5
3	3.33	2.666
10	10.1	9.9



I guess they solved for t + plugged in

$x^2 = t^2 + 2 + \frac{1}{t^2}$

$y^2 = t^2 - 2 + \frac{1}{t^2}$

make t cancle out

subtract $x^2 - y^2 = 4$

3. $x = 1 + \sin t$ → $x - 1 = \sin t$

$\sin^{-1}(x-1)$

$y = 4 + \cos t$ → $y - 4 = \cos t$

$t = \cos^{-1}(y-4)$

$x = 1 + \sin(\cos^{-1}(y-4))$

$y = 4 + \cos(\sin^{-1}(x-1))$

See back

$(x-1)^2 + (y-4)^2 = \sin^2\theta + \cos^2 t = 1$ circle

~~why one θ, other t ??~~ mistake in ans says Brimer w/ sahar

$x = 1 + \sin t$

$\sin t = x - 1$

$\sin^2 t = (x-1)^2$

$1 - \cos^2 t = x^2 - 2x + 1$

$-1 \qquad\qquad -1$

$-\cos^2 t = x^2 - 2x$

$\cos^2 t = 2x - x^2$

$y = 4 + \cos t$

$y - 4 = \cos t$

$(y-4)^2 = \cos^2 t$

$2x - x^2 = y^2 - 8y + 16$

$\cancel{x}\,(x-1)^2 = (y-4)^2$

$(1, 4)$

- but that is not ans given
- so you just hope to find the right thing

8. At noon a small snail starts at center of open clock face. Creeps along hour hand reaching end at 1 pm. Hand is 1 m long

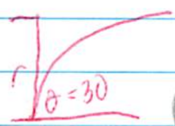
Parametric equation for time

$Q = (\cos\theta, \sin\theta)$

$\downarrow$

in terms of hours

$\theta = \begin{pmatrix} \pi/2, & t=0 \\ \pi/3 & t=1 \end{pmatrix}$ $\theta \downarrow$ linearly w/ time



$$\theta - \frac{\pi}{2} = \frac{\frac{\pi}{3} - \frac{\pi}{2}}{1-0}(t-0)$$

$$\theta = \frac{\pi}{2} - \frac{\pi}{6}t$$

Position

$P = (t\cos\theta, t\sin\theta)$ where t increases from $0 \to 1$

$$x = t\cos\left(\frac{\pi}{2} - \frac{\pi}{6}t\right) = t\sin\frac{\pi}{6}t$$

$$y = t\sin\left(\frac{\pi}{2} - \frac{\pi}{6}t\right) = t\cos\frac{\pi}{6}t$$

Lecture 32 | Arclenght and Surface area

4F - 1d | Find the arclenght $y = \frac{1}{3}(2+x^2)^{3/2}$ $1 \le x \le 2$

- find distance w/ distance formula
- add each line
- riemann sum as you get closer + closer
- limit ...
- end with

$\frac{1}{3} \cdot \frac{3}{2}(2+x^2)^{1/2} \cdot 2x$

~~plus right?~~

no multiply

$x(2+x^2)^{1/2}$ w/ chain rule - always do that

$$\int_a^b \sqrt{1+f'(x)^2}\,dx$$

$$\int_1^2 \sqrt{1+\left(\frac{1}{2}\sqrt{2+x^2} \cdot 2x\right)^2}\,dx$$

pain to integrate

$$\frac{2}{3}\left(\left(1+\left(\cancel{4}x\sqrt{2+x^2}\right)^2\right)\right)^{3/2}$$

trig sub w/ triangle and like u sub

- Wolfram says no answer to integral

multiply out - trig rule.

$ds = \sqrt{1+2x^2+x^4}\,dx$

whole thing

$(1+x^2)\,dx$ and factor

$$\int_1^2 (1+x^2)\,dx$$ oh simplifies real simple

$$x + \frac{x^3}{3}\Big|_1^2$$

$\frac{10}{3}$

4. Find the lenght of the curve

$x = t^2$ for $0 \le t \le 2$
$y = t^3$

This is from '86 notes

$x^3 = (t^2)^3 = t^6$ $x^3 = y^2$
$y^2 = (t^3)^2 = t^6$ $y = x^{2/3}$ ← get y in terms of x
– by various reasons...

$(ds)^2 = (dx)^2 + (dy)^2$

$(ds)^2 = \underbrace{(2t\,dt)^2}_{(dx)^2} + \underbrace{(3t^2\,dt)}_{(dy)^2} = (4t^2 + 9t^4)(dt)^2$

$\int_0^2 ds$

$\int_0^2 \sqrt{4t^2 + 9t^4}\,dt$

ans not in back

$\int_0^2 t\sqrt{4 + 9t^2}\,dt$

$\left.\frac{(4 + 9t^2)^{3/2}}{27}\right|_0^2$

$\frac{1}{27}\left(40^{3/2} - 4^{3/2}\right)$

5. Find an integral for the lenght of the curve given parametrically in 4E-2 for $1 \le t \le 2$, Simplify but don't evaluate.

$x = t + \frac{1}{t}$ $y = t - \frac{1}{t}$

From Recitation 11/30

$ds \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$x = t + t^{-1}$ $dy = t - t^{-1}$

$dx = 1 + t^{-2}$ $dy = 1 - t^{-2}$ differentiate correctly

$dx = 1 + \frac{1}{t^2}$ $dy = 1 - \frac{1}{t^2}$

$ds = \sqrt{\left(1 + \frac{1}{t^2}\right)^2 + \left(1 - \frac{1}{t^2}\right)^2}\, dt$

multiply out

$ds = \sqrt{1 + \cancel{\frac{2}{t^2}} + \frac{1}{t^4} + 1 - \cancel{\frac{2}{t^2}} + \frac{1}{t^4}}\, dt$

$\frac{1}{t^2} \cdot \frac{1}{t^2} = \frac{1}{t^4}$

$\sqrt{2 + \frac{2}{t^4}}\, dt$

* remember not to lose terms

$\int_1^2 \sqrt{2 + \frac{2}{t^4}}\, dt$

8. Find the lenght of the curve $x = e^t \cos t$ $0 \le t \le 10$

$y = e^t \sin t$

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$dx = e^t \cos t - \sin t\, e^t$

$dy = e^t \sin t + \cos t\, e^t$

always try for $\cos^2\theta + \sin^2\theta + 1$ -1 on 1 side or $\cos^2\theta$

$\sqrt{(e^t \cos t - \sin t\, e^t)^2 + (e^t \sin t + \cos t\, e^t)^2}$

factor calculator

$\sqrt{(\cos t - \sin t)^2 e^{2t} + (\cos t + \sin t)^2 e^{2t}}$

ans sheet

$e^t \sqrt{2\cos^2 t + 2\sin^2 t}$

$\sqrt{2}\, e^t$

how supposed to do on own
- clever algebra
- factor cleverly
- just need to see

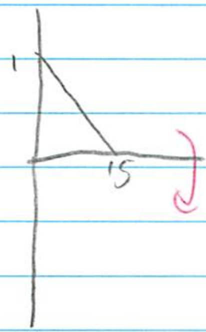
$\int_0^{10} \sqrt{2}\, e^t \, dt$

$\sqrt{2}\, e^{10} - \sqrt{2}\, e^0$

$\sqrt{2}(e^{10} - 1)$

Surface Area

46-2 Find area of segment of $y=1-2x$ in 1st quad around x axis



$\int_0^{.5} ds \cdot 2\pi \text{radius}$ $\quad dy = -2\,dx$

-I did y

x axis

$\int_0^{.5} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot 2\pi \cancel{y}(1-2x)$

↑ copy formula right

$\int_0^{.5} \sqrt{1+(-2)^2} \cdot 2\pi \cancel{x}(1-2x)$

$\int (2\pi x)\sqrt{5}$ $(1-2)$ 5

$\int_0^{.5} 2\sqrt{5}\pi(1-2x)$

$\frac{2\sqrt{5}\pi\, x-2x^2}{2}\Big|_0^{.5}$

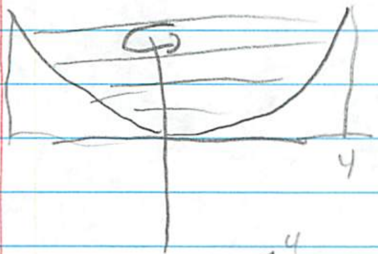
$\cancel{\sqrt{5}\pi(.5)^2 - \sqrt{5}\pi 0^2}$

$\cancel{.25\sqrt{5}\pi \qquad 0}$

$\frac{\sqrt{5}\pi}{2}$

2 errors
- copy error
- confused x y axis

46-5 Find the area of $y = x^2$ $0, \le x \le 4$ revolved around y axis



$$\int_0^4 2\pi x \sqrt{1 + (2x)^2}$$

$$\int_0^4 2\pi x \sqrt{1 + 4x^2}\, dx$$

Think ans in back is really wrong

$$u = 1 + 4x^2\, dx$$
$$du = 8x\, dx$$
$$\frac{du}{8} = x\, dx$$

$$\int_0^4 2\pi \sqrt{u}\, \frac{1}{8} du$$

$$\int \frac{\pi}{4} \sqrt{u}\, du$$

$$\frac{\pi}{4} \int \sqrt{u}\, du$$

$$\frac{\pi}{4} \cdot \frac{2}{3} u^{3/2}$$

$$\frac{2\pi (1+4x^2)^{3/2}}{12} = \frac{\pi (1+4x^2)^{3/2}}{6} \Bigg|_0^4$$

$$\frac{\pi (1 + 4(4)^2)^{3/2}}{6} - \frac{\pi \cdot 1^{3/2}}{6}$$

$$\frac{(65 \cdot \sqrt{65} - 1)\pi}{6} = 273.866$$

$=$

12

Part 2

0. See Side bar ✓

So little time to work on this one

1. This question is on integration by parts

$$I_n = \int_0^{\pi/2} \sin^n x\,dx$$

Some one said this was really long

a) Use integration by parts to show that, for odd powers (eg 2n+1) of sin

$$I_{2n+1} = \frac{2 \cdot 4 \cdot 6 \cdot \ldots\ 2n}{3 \cdot 5 \cdot 7 \cdot \ldots\ (2n+1)}$$

$$\int_0^{\pi/2} \sin^{2n+1} x\,dx$$

but what is $\sin^n$

try a # → 2

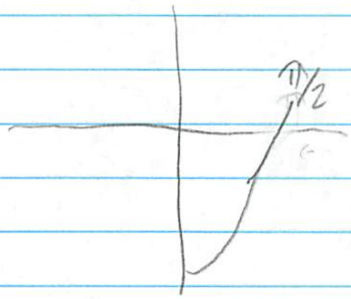
$$\int_0^{\pi/2} \sin^5 x\,dx$$

Wolfram says

indefinite $-\frac{5\cos x}{8} + \frac{5}{48}\cos(3x) - \frac{1}{80}\cos(5x) + c$

definite $\frac{8}{15} \rightarrow \frac{2 \cdot 4}{3 \cdot 5}$ 2n / 2n+1 so works ✓

but why in all world



also $\cancel{\frac{5}{16} e^{-ix} - \frac{5}{16} e^{ix}}$ ← did not do i

typing in n - no help

What does this have to do w/ parametrics?

no - integration by parts (lecture 30)

$$uv - \int v \, du$$

$u = \sin^{2n+1} x \qquad dv = x$

$du = \cos x \sin^2 x^{2n}(2n+1) \qquad v = \frac{1}{2}x^2$

↑can't integrate easily, differentiation bad too

$$\sin^{2n+1} \cdot \frac{1}{2}x^2 - \int x \cos x \sin^2 x^{2n} (2n+1)$$

that does not really work either

won't integrate

?

1

b. Now for even powers

$$I_{2n} = \frac{1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots 2n} \frac{\pi}{2}$$

c Show that $I_{2n+2} \leq I_{2n+1} \leq I_{2n}$

adding a value to the n you are computing

ex $n = 2$

$I_8 = .429 = \frac{35\pi}{256}$ but how does that work out?

$I_9 = .406 = \frac{128}{315}$

$I_{10} = .3865 = \frac{63\pi}{512}$

divisor rises more rapidly than numerator?
larger number always in denom

?

0

d) Use parts a and b to show

$$\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$$

and hence deduce that $\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$

yeah as gets infantolly small does not matter

1

e) Use parts a, b, d to find

$$\lim_{n \to \infty} \frac{2}{1} \cdot \frac{2}{1} \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$

$$\frac{2n}{2n-1} \cdot \frac{2n}{2n+1}$$

now lect 31

2.a. Find the algebraic equation in x and y for curve

$x = a\cos kt$
$y = a\sin kt$

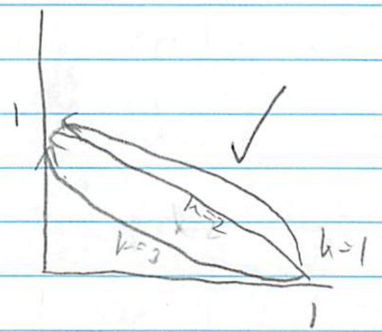
Draw a portion of the curve $0 \le t \le \pi/2$ in $k = 1, 2, 3$

$\left(\frac{x}{a}\right)^{1/k} = \cos t$

$\left(\frac{y}{a}\right)^{1/k} = \sin t$

$\left(\frac{x}{a}\right)^{2/k} + \left(\frac{y}{a}\right)^{2/k} = 1$

kimberly



✓

$k=1$ a^2 →

b. Find arc lenght for $k = 1$, $k=2$

$\int \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}$

$\sqrt{1^2 + 1^2}$

$\sim \sqrt{1}$ → same/similar for $k=2$

1 $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$

2/ $\frac{x}{a} + \frac{y}{a} = 1$

3/ $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$

1 $\frac{\pi a^2}{2}$

2 $\sqrt{2a^2} = a\sqrt{2}$ ✓ 3

1

/4

c Find a definite integral formula for the lenght of a curve for general k then eval for $k=1,2,3$
- You should confirm ans from b

$$\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$dx = a \cdot k \cos^{k-1} t \cdot \sin t$$
$$dy = a \cdot k \sin^{k-1} t \cdot \cos t$$

$$\int \sqrt{a^2 k^2 \left[(\cos^{k-1} t \sin t)^2 + (\sin^{k-1} t \cos t)^2\right]}\, dx$$

for $k=1$

$$\int \sqrt{a^2 1^2 \left[(\cos^0 t \cdot \sin t)^2 + (\sin t^0 \cos t)^2\right]}\, dt$$

$$\int \sqrt{a^2 (\sin^2 + \cos^2 t)}\, dt$$

$$\int |a| \rightarrow \frac{a^2}{2}\Big|_0^{\frac{\pi}{2}} \qquad \frac{\pi a}{2}$$

for $k=2$

$$2a \int \sqrt{a^2 2^2 \left[(\cos^1 t \sin^2 t)^2 + (\sin t \cos^2 t)^2\right]}\, dt$$

same

$$\int \sqrt{4a^2 (\cos^2 t \sin^4 t) + (\sin^2 t \cos^4 t)}\, dt$$

$$\int \sqrt{4a^2 \sin^2 t \cos^2 t}\, dt$$

← another $\sqrt{2}$ somewhere

$$\int 2a \sin t \cos t\, dt \quad \sqrt{2}$$

$$\sqrt{2}\ -\frac{1}{2} a \, \cancel{\cos(2t)}\ \sin^2 t + c \Big|_0^{\frac{\pi}{2}}$$

$a\sqrt{2}$

$k=3$

$$\int \sqrt{9a^2 (\cos^2 t \sin^3 t)^2 + (\sin^2 t \cos^3 t)^2}$$

$(\cos^2 t + \sin^2 t)(\cos^2 t - \sin^2 t)$

$$\int \sqrt{9a^2 \cos^4 t \sin^6 t + \sin^4 t \cos^6 t}$$

$$\int \sqrt{9a^2 \cos^4 t \sin^4 t}$$

$$\int 3a \cos^2 t \sin^2 t$$

$\frac{1}{2}\sin^2 t$

$$3a\left(\cancel{\frac{t}{8} - \frac{1}{32}\sin 4t}\right) + c$$

$\frac{3\pi a}{16}$ $\frac{3a}{2}$ ✓ $\frac{3a}{2}[1-0]$

"3

3
Lec 32

The hyperbolic sin and cos defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

a) Show that

$$\frac{d}{dx} \sinh x = \cosh x \quad \text{and} \quad \frac{d}{dx} \cosh x = \sinh x$$

$$\frac{d}{dx} \frac{e^x + e^{-x}}{2}$$

$$\frac{1}{2}\left(\frac{d}{dx} e^x + \frac{d}{dx} e^{-x}\right)$$

$$\frac{1}{2}\left(e^x + - e^{-x}\right)$$ just a given I think or chain rule -remember to differentiate inside

$$\frac{e^x - e^{-x}}{2}$$

$$\frac{d}{dx} \frac{e^x - e^{-x}}{2}$$

$$\frac{1}{2}\left(\frac{d}{dx} e^x - \frac{d}{dx} e^{-x}\right)$$

$$\frac{1}{2}\left(e^x - - e^{-x}\right)$$

$$\frac{e^x + e^{-x}}{2}$$

aii $\cosh^2 x = 1 + \sinh^2 x$

$$\left(\frac{e^x + e^{-y}}{2}\right)^2 = 1 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}\left(e^{x^2} + 2e^{-x}e^{x} + e^{-2x}\right)$$

kimberly

$$\frac{e^{2x} + e^{-2x}}{4} = 1 + \frac{e^{2x} - e^{-2x}}{4} = \frac{1}{4}\left(e^{x^2} + 2 + e^{-2x}\right)$$

— that

$$\frac{e^{2x} + e^{-2x} - e^{2x} + e^{-2x}}{4}$$

$$\frac{2e^{-2x}}{4} = \frac{e^{-2x}}{2}$$

? how does this help

expand

$$= 1 + \frac{e^{2x} - 2e^{x}e^{-x} + e^{-2x}}{4}$$

$$= 1 + \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$= \frac{1}{4}\left(4 - 2 + e^{2x} + e^{-2x}\right)$$

$$= \frac{1}{4}\left(2 + e^{2x} + e^{-2x}\right)$$

$$\frac{1}{4}\left(2 + e^{2x} + e^{-2x}\right) = \frac{1}{4}\left(2 + e^{2x} + e^{-2x}\right) \checkmark$$

weird
just go
through
and do it
out
till it
=

a iii $\cosh^2 x = \dfrac{1+\cosh 2x}{2}$

kimberly

$$\left(\frac{e^{x}+e^{-x}}{2}\right)^2 = \frac{1+\left(\frac{e^{2x}+e^{-2x}}{2}\right)}{2}$$

$$\frac{e^{2x}+e^{-2x}}{4} = \frac{1}{2}+\frac{e^{2x}+e^{-2x}}{4}$$

again what went wrong?

$$= \tfrac{1}{4}\left(e^{2x}+2+e^{-2x}\right)$$

$$= 2\cosh^2 x - 1$$

$$= \tfrac{1}{2}\left(e^{2x}+2+e^{-2x}-1\right)$$

$$= \tfrac{1}{2}\left[e^{2x}+2-2+e^{-2x}\right]$$

$$= \tfrac{1}{2}\left[e^{2x}+e^{-2x}\right] = \cos 2x$$

$$\frac{1+\cosh 2x}{2} = \frac{1+\frac{1}{2}e^{2x}+\frac{1}{2}e^{-2x}}{2}$$

$$= \tfrac{1}{2}+\tfrac{1}{4}e^{2x}+\tfrac{1}{4}e^{-2x}$$

$$= \tfrac{1}{4}\left[e^{2x}+2+e^{-2x}\right] = \cosh^2 x$$

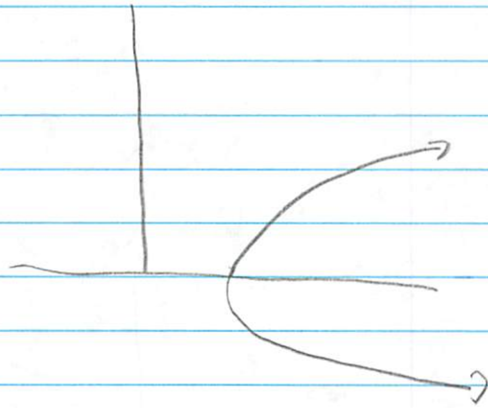
/

3

b. What curve is described by

$x = \cosh t$
$y = \sinh t$

$x^2 = \cosh^2 t$
$y^2 = \sinh^2 t$
$x^2 - y^2 = 1$



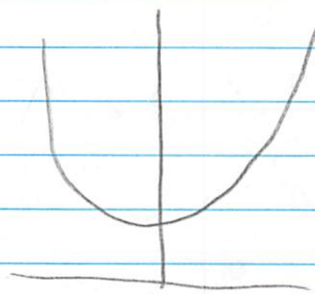
$y^2 = x$
$y = \sqrt{x}$

sideways parabola = hyperbola

1

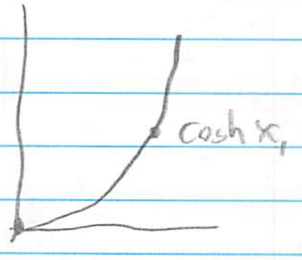
c) The curve $y = \cosh x$ is known as a catenary
It is a curve formed by a chain whose 2 ends are held at the same height.

i) Sketch



✓

ii) Find arclength from the lowest pt to the point $(x_1, \cosh x_1)$ for a fixed $x_1 > 0$



$$\int_{x_1}^{\cosh x_1} \sqrt{1 + \left(\frac{d}{dx} \cosh x\right)^2}$$

$$\int_{x_1}^{\cosh x_1} \sqrt{1 + \sinh^2 x}$$

$$\int_{x_1}^{\cosh x_1} \sqrt{\cosh^2 x}$$

Kimberly

$$\int_{x_1}^{\cosh x_1} |\cosh x|$$

?, from x_1 to 0

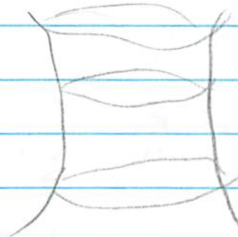
$$\sinh x + c \Big|_{x_1 \ 0}^{\cancel{\cosh x_1} \ x_1}$$

$$\sinh(\cancel{\cosh}(x_1)) - \sinh(x_1)$$

$\sinh(x_1)$ ✓

3

d) Find the area of the surface of revolution formed by revolving the portion of the curve from part c around the x axis. This is know as a catenoid.



Alpha

Likely

It is interesting because it is the surface of the lease area connecting the 2 circles that form its edges. If you dip soap into water, it will assume this shape since soap tries to minimize surface area.

$$\int 2\pi x_1 \stackrel{\checkmark \cosh}{|\cosh x|}$$

$$2\pi \int \cosh^2 x \, dx$$

$$\cancel{2\pi \quad x \sinh x - \cosh x} \Big|_{x_1}^{\cosh x_1}$$

$$\cancel{2\pi\left(-\sinh(x) - \cosh(\cosh(x)) + \cosh(x)\sinh(\cosh(x)) + 1\right)}$$

$$2\pi \int \frac{1 + \cosh 2x}{2} \, dx$$

$$\pi x_1 + \pi \left[\frac{\sinh 2x}{2}\right]_0^{x_1}$$

$$\pi\left[x_1 + \frac{\sinh 2x_1}{2}\right] \checkmark \quad 2$$

Exam 4 Review Handout

1. Integrate by **trigonometric substitution**; evaluate the **trigonometric integral** and work backwards to the original variable by evaluating **trig(trig^{-1})** using a right triangle:

a) $a^2 - x^2$ use $x = a\sin u$, $dx = a\cos u\, du$.

b) $a^2 + x^2$ use $x = a\tan u$, $dx = a\sec^2 u\, du$

c) $x^2 - a^2$ use $x = a\sec u$, $dx = a\sec u\tan u\, du$

2. Integrate rational functions P/Q (ratio of polynomials) by the method of **partial fractions**: If the degree of P is less than the degree of Q, then factor Q completely into linear and quadratic factors, and write P/Q as a sum of simpler terms. For example,

$$\frac{3x^2+1}{(x-1)(x+2)^2(x^2+9)} = \frac{A}{x-1} + \frac{B_1}{(x+2)} + \frac{B_2}{(x+2)^2} + \frac{Cx+D}{x^2+9}$$

Terms such as $D/(x^2+9)$ can be integrated using the trigonometric substitution $x = 3\tan u$.

This method can be used to evaluate the integral of any rational function. In practice, the hard part turns out to be factoring the denominator! In recitation you encountered two other steps required to cover every case systematically, namely, completing the square[1] and long division.[2]

3. **Integration by parts:**

$$\int_a^b uv'dx = uv\Big|_a^b - \int_a^b u'vdx$$

This is used when $u'v$ is simpler than uv'. (This is often the case if u' is simpler than u.)

4. **Arclength:** $ds = \sqrt{dx^2 + dy^2}$. Depending on whether you want to integrate with respect to x, t or y this is written

$$ds = \sqrt{1 + (dy/dx)^2}\, dx; \quad ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt; \quad ds = \sqrt{(dx/dy)^2 + 1}\, dy$$

5. **Surface area for a surface of revolution:**

a) around the x-axis: $2\pi y ds = 2\pi y\sqrt{1 + (dy/dx)^2}\, dx$ (requires a formula for $y = y(x)$)

b) around the y-axis: $2\pi x ds = 2\pi x\sqrt{(dx/dy)^2 + 1}\, dy$ (requires a formula for $x = x(y)$)

6. **Polar coordinates:** $x = r\cos\theta$, $y = r\sin\theta$ (or, more rarely, $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$)

a) Find the polar equation for a curve from its equation in (x, y) variables by substitution.

b) Sketch curves given in polar coordinates and understand the range of the variable θ (often in preparation for integration).

7. **Area in polar coordinates:**

$$\int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta$$

(Pay attention to the range of θ to be sure that you are not double-counting regions or missing them.)

[1] For example, we rewrite the denominator $x^2 + 4x + 13 = (x+2)^2 + 9 = u^2 + a^2$ with $u = x + 2$ and $a = 3$.

[2] Long division is used when the degree of P is greater than or equal to the degree of Q. It expresses $P(x)/Q(x) = P_1(x) + R(x)/Q(x)$ with P_1 a quotient polynomial (easy to integrate) and R a remainder. The key point is that the remainder R has degree less than Q, so R/Q can be split into partial fractions.

The following formulas will be printed with Exam 4

$$\sin^2 x + \cos^2 x = 1; \quad \sec^2 x = \tan^2 x + 1$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x; \quad \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\cos 2x = \cos^2 x - \sin^2 x; \quad \sin 2x = 2\sin x \cos x$$

$$\frac{d}{dx}\tan x = \sec^2 x; \quad \frac{d}{dx}\sec x = \sec x \tan x; \quad \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}; \quad \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\int \tan x \, dx = -\ln(\cos x) + c; \quad \int \sec x \, dx = \ln(\sec x + \tan x) + c$$

See the next page for a review on integration of rational functions.

Postscript: Systematic integration of rational functions

For a general rational function P/Q, the first step is to express P/Q as the sum of a polynomial and a ratio in which the numerator has smaller degree than the denominator.

For example,

$$\frac{x^3}{x^2-2x+1} = x+2+\frac{3x-2}{x^2-2x+1}$$

(To carry out this long division, do not factor the denominator $Q(x) = x^2 - 2x + 1$, just leave it alone.) The quotient $x + 2$ is a polynomial and is easy to integrate. The remainder term

$$\frac{3x-2}{(x-1)^2}$$

has a numerator $3x - 2$ of degree 1 which is less than the degree 2 of the denominator $(x - 1)^2$. Therefore there is a partial fraction decomposition. In fact,

$$\frac{3x-2}{(x-1)^2} = \frac{(3x-3)+1}{(x-1)^2} = \frac{3}{x-1} + \frac{1}{(x-1)^2}$$

In general, if P has degree n and Q has degree m, then long division gives

$$\frac{P(x)}{Q(x)} = P_1(x) + \frac{R(x)}{Q(x)}$$

in which P_1, the quotient in the long division, has degree $n - m$ and R, the remainder in the long division, has degree at most $m - 1$.

Evaluation of the "simple" pieces

The integral

$$\int \frac{dx}{(x-a)^n} = \frac{-1}{n-1}(x-a)^{1-n} + c$$

if $n \neq 1$ and $\ln|x - a| + c$ if $n = 1$. On the other hand the terms

$$\int \frac{xdx}{(Ax^2+Bx+C)^n} \quad \text{and} \quad \int \frac{dx}{(Ax^2+Bx+C)^n}$$

are handled by first completing the square:

$$Ax^2 + Bx + C = A(x - B/2A)^2 + \left(C - \frac{B^2}{4A}\right)$$

Using the variable $u = \sqrt{A}(x - B/2A)$ yields combinations of integrals of the form

$$\int \frac{udu}{(u^2+k^2)^n} \quad \text{and} \quad \int \frac{du}{(u^2+k^2)^n}$$

The first integral is handled by the substitution $w = u^2 + k^2$, $dw = 2udu$. The second integral can be worked out using the trigonometric substitution $u = k\tan\theta$ $du = k\sec^2\theta d\theta$. This then leads to sec-tan integrals, and the actual computation for large values of n are long.

There are also other cases that we will not cover systematically. Examples are below:

1. If $Q(x) = (x - a)^m(x - b)^n$, then the expression is

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m} + \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2} + \cdots + \frac{B_n}{(x-b)^n}$$

2. If there are quadratic factors like $(Ax^2 + Bx + C)^p$, one gets terms

$$\frac{a_1x + b_1}{Ax^2 + Bx + C} + \frac{a_2x + b_2x}{(Ax^2 + Bx + C)^2} + \cdots + \frac{a_px + b_p}{(Ax^2 + Bx + C)^p}$$

for each such factor. (To integrate these quadratic pieces complete the square and make a trigonometric substitution.)

18.01 REVIEW FOR EXAM 4

IVAN LOSEV

Contents

1. Integration techniques

Table 1: Basic antiderivatives

no	Function $f(x)$	Antiderivative $\int f(x)dx$
1	$x^a, a \neq -1$	$\frac{x^{a+1}}{a+1} + C$
2	x^{-1}	$\ln\lvert x\rvert + C$
3	$\sin(x)$	$-\cos(x) + C$
4	$\cos(x)$	$\sin(x) + C$
5	e^x	$e^x + C$
6	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + C$
7	$\frac{1}{1+x^2}$	$\arctan(x) + C$

u substitution

1.1. **Substitutions.**
Recall that if $u = u(x)$, then $\int f(u(x))u'(x)dx = \int f(u)du$ (follows from the chain rule). Then one can computes $\int f(u)du$ and plugs $u = u(x)$ in the result.

When we compute definite integrals we can make substitution both for the integrand and for the bounds: $\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$.

This is convenient when we make multiple substitutions. Indeed, we do not need to remember which substitution we made, we can just compute the integral on the right-hand side.

Problems to practice:

- Practice questions to exam 3, problem 1.
- Exam 3, problem 1, problem 5d.

1.2. **Trigonometric integrals.** This includes integrals of the form $\int \sin^n x \cos^m x dx$, where n and m are integers (positive or negative). One also can have some polynomial combinations of sin's and cos's. These integrals are computed by substitutions of the form $u = \sin x, \cos x, \tan x$, etc., or by using double angle identities. A right technique for computing the integral depends on values of m and n.

Case A1. n is odd and positive. Then one can substitute $u = \cos x$ (so that $du = -\sin x dx$). The integral becomes $-\int (1-u^2)^{(n-1)/2} u^m du$.

Case A2. m is odd and positive. Then one can substitute $u = \sin x$ (so that $du = \cos x dx$). The integral becomes $\int u^n (1-u^2)^{(m-1)/2} du$.

Case B. m, n are both $\geqslant 0$ and even. Then one can use the identities

$$\sin^2 x = \frac{1-\cos 2x}{2}, \cos^2 x = \frac{1+\cos 2x}{2}$$

to rewrite the integrand as

$$\left(\frac{1-\cos 2x}{2}\right)^{n/2} \left(\frac{1+\cos 2x}{2}\right)^{m/2}.$$

Expanding brackets we get a linear combination of functions $\cos^k 2x$ with $k \leqslant \frac{m+n}{2}$. Multiples of $\cos^k 2x$ with odd k are integrated as in case A2 above, while to integrate the multiples of $\cos^k 2x$ with even k one again needs to apply the double angle identities (leading to $\cos 4x$) and so on.

Case C1. The integral can be rewritten as $\int \tan^k x \sec^l x dx$, where $k = n, l = -n - m$. Suppose l is even and positive. Then, using the substitution $u = \tan x$ (so that $du = \sec^2 x dx$), we get $\int u^k (1+u^2)^{(l-2)/2} du$.

Case C2. The integral can be rewritten as $\int \cot^k x \csc^l x dx$, where $k = m, l = -n - m$. Suppose l is even and positive. Use $u = \cot x$ (so that $du = -\csc^2 x dx$).

Some cases are not covered by the list above, e.g., $\int \sec x dx$. In general, if we have the integral $\int \sin^n x \cos^m x dx$ with n odd but negative, we still can apply the substitution $u = \cos x$. However, the integrand we obtain is no longer a polynomial, it is a rational function (fraction of two polynomials).

Problems to practice.

- Practice questions, problem 6 (trig. integrals also appear in some other problems after making a trig. substitution, see the next subsection).

1.3. **Trigonometric substitutions.** These substitutions are used when an integral involves $\sqrt{a+bx+cx^2}$ (and also expressions like $\frac{1}{(ax^2+bx+c)^n}$, where ax^2+bx+c has no zeroes). First of all, we complete the square and make a linear substitution to get one of the following expressions under the root:

(1) $\sqrt{a^2+x^2}$ (substitute $x = a\tan\theta$),
(2) $\sqrt{a^2-x^2}$ (substitute $x = a\sin\theta$),
(3) $\sqrt{x^2-a^2}$ (substitute $x = a\sec\theta$).

In the first case, for instance, the expression under the square root becomes $a^2 \sec^2 \theta$ and so extracting the root we get just $a \sec \theta$.

Then the integral becomes a trigonometric integral considered above. One should compute it and then express the answer in terms of x (instead of θ).

Problems to practice.

- Practice questions, problem 2.
- Practice exam, problem 1.
- Exam, problem 3 (the denominator should be $(4 + x^2)^{3/2}$).

1.4. **Rational functions.** We want to compute $\int \frac{p(x)}{q(x)} dx$, where $p(x), q(x)$ are polynomials.

$\frac{A}{x} + \frac{B}{x}$

First of all, we need to compare degrees of $p(x)$ and $q(x)$. If degree of $p(x)$ is bigger or equal than that of $q(x)$ we need to divide $p(x)$ by $q(x)$ (with remainder). The result of the division is written as $p(x) = s(x)q(x) + r(x)$, where $s(x)$ is the (partial) quotient and $r(x)$ is the remainder, whose degree is strictly less than that of $q(x)$. Then we can write $\frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}$. So $\int \frac{p(x)}{q(x)} dx = \int s(x) dx + \int \frac{r(x)}{q(x)}$.

The division procedure is carried out using the long division algorithm.

factor bottom

-cover up

-then solving system

Example. Divide $p(x) = 2x^4$ by $q(x) = x^2 + 2x + 1$. The difference between the degrees is 2 so the partial quotient $s(x)$ must have degree 2. The coefficient of x^2 in $s(x)$ is the quotient of the leading coefficients of $p(x)$ and $q(x)$ so it is 2. Then we multiply $q(x)$ by $2x^2$ and subtract the result from $p(x)$: $2x^4 - 2x^2(x^2 + 2x + 1) = -4x^3 - 2x^2$. Repeat the same procedure with $-4x^3 + 2x^2$ instead of $2x^4$. We get: the x-coefficient of $s(x)$ is -4 and $-4x^3 - 2x^2 - (-4)x(x^2+2x+1) = 6x^2+4x$. The same one more time: $6x^2+4x-6(x^2+2x+1) = -8x - 6$. So $s(x) = 2x^2 - 4x + 6, r(x) = -8x - 6$.

Below we will assume that degree of $p(x)$ is less than that of $q(x)$. In this case $\frac{p(x)}{q(x)}$ is decomposed into the sum of certain elementary fractions. To determine these fractions one should represent $q(x)$ as the product of irreducible factors ("irreducible" means "cannot be decomposed further", it is either a linear polynomial or a quadratic polynomial without zeroes). Then the decomposition of $\frac{p(x)}{q(x)}$ is made using the cover-up method. This method is explained below for cubic $q(x)$.

We have exactly one of the following 4 cases.

Case 1. $q(x) = (x - a)(x - b)(x - c)$, where all a, b, c are different. Then $\frac{p(x)}{q(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$. To determine A use "cover-up": multiply both sides by $(x - a)$ and plug $x = a$. The l.h.s. becomes $\frac{p(a)}{(a-b)(a-c)}$ and the r.h.s. is just A. The coefficients B and C are found analogously.

Case 2. $q(x) = (x - a)(x^2 + bx + c)$, where $x^2 + bx + c$ has no zeroes. Then $\frac{p(x)}{q(x)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$. We can find A as before: $A = \frac{p(a)}{a^2+ba+c}$. Then we rewrite our equality subtracting $\frac{A}{x-a}$ from both sides. We get

$$\frac{p(x) - A(x^2 + bx + c)}{(x - a)(x^2 + bx + c)} = \frac{Bx + C}{x^2 + bx + c}$$

Now the numerator is 0 for $x = a$ (from the choice of A) and so is divisible by $x - a$: $p(x) - A(x^2 + bx + c) = (x - a)p_1(x)$ for some polynomial $p_1(x)$ (that again can be found using the division algorithm). So $\frac{p_1(x)}{x^2+bx+c} = \frac{Bx+C}{x^2+bx+c}$. We can find B and C from the equality $p_1(x) = Bx + C$.

Case 3. $q(x) = (x-a)^2(x-b)$, where $a \neq b$. Then $\frac{p(x)}{q(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{B}{x-b}$. We can determine B as above. Using the similar idea we can determine A_2: multiply both sides by $(x-a)^2$ and plug $x = a$ to get $\frac{p(a)}{a-b} = A_2$. Then

$$\frac{A_1}{x-a} + \frac{B}{x-b} = \frac{p(x)}{q(x)} - \frac{A_2}{(x-a)^2} = \frac{p(x) - A_2(x-b)}{(x-a)^2(x-b)}$$

As in the previous case, $p(x) - A_2(x-b) = (x-a)p_1(x)$. So we get $\frac{p_1(x)}{(x-a)(x-b)} = \frac{A_1}{x-a} + \frac{B}{x-b}$. Now we can determine A_1 using cover-up.

Case 4. $q(x) = (x-a)^3$. No need in cover-up, just rewrite $p(x)$ as a polynomial of $x - a$.

Integration.

Once we have decomposed $\frac{p(x)}{q(x)}$ to the sum as explained above, it is not very difficult to integrate it. For example, for cubic $q(x)$ only the following summands can arise: a polynomial, $\frac{A}{(x-a)^k}, k = 1, 2, 3$ and $\frac{Bx+C}{x^2+bx+c}$. All but the last one are standard.

To compute $\int \frac{Bx+C}{x^2+bx+c}dx$ we first complete the square. This reduces the computation to integrals of the form:

$$\int \frac{1}{x^2+a^2}dx = a^{-1}\arctan(x/a), \int \frac{x}{x^2+a^2}dx = \frac{1}{2}\ln(x^2+a^2).$$

Problems to practice.

- Practice questions, problem 1.
- Practice exam, problem 3.
- Exam, problem 1.

1.5. **Integration by parts.** The formulas are as follows:

$$\int udv = \boxed{uv - \int vdu}$$

or, for definite integrals,

$$\int_a^b udv = uv]_a^b - \int_a^b vdu.$$

The crucial ingredient in the solution of such problems is to present the integrand, say $f(x)dx$, in the form $f(x)dx = udv$. There is no general recipe to do this, however, one should try to find a presentation $f(x)dx = udv$ such that dv is easy to integrate. Another hint, which is often helpful, is that vdu should be easier to integrate than udv.

Problems to practice.

- Practice questions, problems 3,4.
- Practice exam, problem 2.
- Exam, problem 2.

2. Other

2.1. Parametric calculus. By parametric calculus we mean the study of curves given parametrically such as the cycloid: $x(\theta) = r(\theta - \sin\theta), y(\theta) = r(1 - \cos\theta)$, where r is some constant and θ is a parameter.

Tangents. The tangent to a parametric curve given by $(x(t), y(t))$ at $t = t_0$ is given by

$$y - y(t_0) = \frac{y'(t_0)}{x'(t_0)}(x - x(t_0))$$

Areas. Let $(x(t), y(t))$ be a parametric curve such that $x(t)$ is increasing for $t \in [t_1, t_2]$. Then the area under the curve is given by $\int_{t_1}^{t_2} y(t)dx(t)$. If $x(t)$ is decreasing, then the area will be given by $-\int_{t_1}^{t_2} y(t)dx(t)$.

Finally, the area inclosed by a curve $(x(t), y(t)), t \in [t_1, t_2]$ (without self-intersections) is $\pm\int_{t_1}^{t_2} y(t)dx(t)$. The easiest observation that helps to distinguish btw. plus and minus is that the area is always positive. A more subtle observation is that we should put "+" if the point $(x(t), y(t))$ moves clockwise as t increases and "−" otherwise.

In practice exams parametric curves appear in problems concerning arc-length and surface area.

2.2. Arc-length. The length of the piece of the graph $y = f(x)$ enclosed btw. $x = a$ and $x = b$, where $a < b$, is $\int_a^b ds$, where $ds := \sqrt{dx^2 + dy^2} = \sqrt{1 + f'(x)^2}dx$. The length of the parametric curve $(x(t), y(t))$, where $t \in [t_1, t_2]$ is still given by $\int_{t_1}^{t_2} ds$, where now $ds = \sqrt{dx^2 + dy^2} = \sqrt{x'(t)^2 + y'(t)^2}dt$.

Problems to practice.

- Practice questions, problems 6a,7.
- Practice exam, 4a.
- Exam, problem 4a.

2.3. Surface area. The area of the surface obtained by rotating $y = f(x), a \leqslant x \leqslant b$, about x-axis is given by $\int_a^b 2\pi y ds$, where ds is computed as above. Here we suppose that $y = f(x) \geqslant 0$ for all $x \in [a, b]$.

The similar formula works for parametric curves. More precisely, if we rotate $(x(t), y(t)), t \in [t_1, t_2]$ about the x-axis we get the area equal to $\int_{t_1}^{t_2} yds = \int_{t_1}^{t_2} 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}dt$. Here again we suppose that $y(t) \geqslant 0$.

Problems to practice:

- Practice questions, problem 6b.
- Exam, problem 4b.

(Pr Exam #1)

Exam 4 will not include polar coordinates.
Some problems have been removed from the previous years' practice exams.

18.01 Practice Questions for Exam 4 – Fall 2006

Problem 1. Evaluate $\displaystyle\int \frac{x-4}{(x+1)(x^2+4)}\,dx.$

Problem 2. Evaluate $\displaystyle\int_0^2 \frac{dx}{(x^2+4)^2}$ by making the substitution $x = 2\tan u$.

Problem 3.

a) Derive a reduction formula relating $\displaystyle\int_0^1 x^{2n}e^{-x^2}\,dx$ to $\displaystyle\int_0^1 x^{2n-2}e^{-x^2}\,dx.$

b) Let $F(x) = \displaystyle\int_0^1 e^{-x^2}\,dx$. Express $\displaystyle\int_0^1 x^2e^{-x^2}\,dx$ in terms of values of $F(x)$.

Problem 4. Find the volume of the solid obtained by rotating about the y-axis the finite region bounded by the positive x- and y-axes and the graph of $y = \cos x$.

~~**Problem 5.** Make a reasonable sketch of one loop of the polar curve $r = \sin 3\theta$, and find the area inside it.~~

Problem 6. Let $x(t) = \cos^3 t$, $y(t) = \sin^3 t$, $0 \le t \le \pi/2$ be a parametric representation of a curve.

a) Compute the arclength of the curve.

b) Compute the surface area of the surface formed by rotating the curve around the x-axis.

Problem 7. Set up an integral for the length of one arch of the curve $y = \sin x$, and by estimating the integral, tell how this length compares with $\pi\sqrt{2}$.

~~**Problem 8.** A circular metal disc of radius a has a non-constant density δ (units: gms/cm^2); the density at a point P on the disc is given by $\delta = r^2$, where r is the distance of the point from the center of the disc. Set up and evaluate a definite integral giving the total mass of the disc.~~

~~**Problem 9.**~~

~~a) Sketch the curve given in polar coordinates by $r = 1 + \cos\theta$~~

~~b) Find the polar coordinates of the following two points (show work):~~

~~(i) where the curve in part (a) intersects the circle of radius 3/2 centered at the origin;~~

~~(ii) where the above curve intersects the circle of radius 3/2 centered at the point $x = 3/2$ on the x-axis.~~

Other kinds of problems:

- Other kinds of partial fractions decompositions; ! + trig subst., integration by parts, etc.
- sketching curves given parametrically, finding their arclength;
- finding surface area for rotated curves in xy-coordinates;
- [~~deriving polar equations of curves given geometrically, changing from rectangular equations to polar and vice versa.~~]
- Chapter 10 !

Be able to use all the methods of integration covered in class.
Furthermore, you should be able to tell which method of integration would work best for a given integral.
For this you should work through the integrals on pp. 368 - 369.
Read § 10.8 (at least.)

– Seems like most of exam is the different integrations + one parametric

Practice Questions for Exam 4

1) $\dfrac{x-4}{(x+1)(x^2+4)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+4} = \dfrac{(A+B)x^2+(B+C)x+4A+C}{(x+1)(x^2+4)}$

$\left.\begin{array}{l} A+B=0 \\ B+C=1 \\ 4A+C=-4 \end{array}\right\}$ $\begin{array}{l} A=-1 \\ B=1 \\ C=0 \end{array}$

$\displaystyle\int \frac{x-4}{(x+1)(x^2+4)}dx = \int\left[\frac{-1}{x+1}+\frac{x}{x^2+4}\right]dx = -\ln|x+1| + \int \frac{x}{x^2+4}dx$

$\boxed{= -\ln|x+1| + \frac{1}{2}\ln|x^2+4| + C}$

2) $\displaystyle\int_0^2 \frac{dx}{(x^2+4)^2} = \int_0^{\pi/4} \frac{2\sec^2 u}{(4\tan^2 u+4)^2}du = \frac{1}{8}\int_0^{\pi/4}\cos^2 u\,du = \frac{1}{8}\int_0^{\pi/4}\frac{1}{2}(1+\cos(2u))du$

$x = 2\tan u$ $\qquad 1+\tan^2 u = \sec^2 u$

$dx = 2\sec^2 u\,du$

$\displaystyle = \frac{1}{16}\left(u+\frac{1}{2}\sin(2u)\right)\Big|_0^{\pi/4} = \boxed{\frac{1}{64}(\pi+2)}$

3) a) $\displaystyle\int_0^1 \underbrace{x^{2n}}_{g'}\ \underbrace{e^{-x^2}}_{f}\,dx = \frac{1}{2n+1}x^{2n+1}e^{-x^2}\Big|_0^1 - \int_0^1(-2xe^{-x^2})\frac{1}{2n+1}x^{2n+1}dx$

$\boxed{n=k-1}$

$\displaystyle\int_0^1 x^{2k-2}e^{-x^2}dx = \frac{1}{2k-1}\cdot\frac{1}{e} + \frac{2}{2k-1}\int_0^1 x^{2k}e^{-x^2}dx$

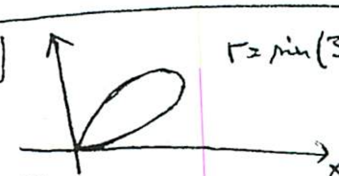
$\boxed{\displaystyle\int_0^1 x^{2k}e^{-x^2}dx = \frac{-1}{2e} + \frac{2k-1}{2}\int_0^1 x^{2k-2}e^{-x^2}dx}$

b) $\displaystyle\int_0^1 x^2e^{-x^2}dx = \frac{-1}{2e} + \frac{1}{2}\int_0^1 e^{-x^2}dx$

4) $\displaystyle V = \int_0^{\pi/2} 2\pi x\cos x\,dx = 2\pi\left[x\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx\right]$

$= \boxed{\pi(\pi-2)}$

5) $r = \sin(3\theta)$



$\displaystyle A = \int_0^{\pi/3}\frac{1}{2}\sin^2(3\theta)d\theta$

$\displaystyle = \frac{1}{4}\int_0^{\pi/3}(1-\cos(6\theta))d\theta$

$\displaystyle = \frac{\theta}{4} - \frac{1}{24}\sin(6\theta)\Big|_0^{\pi/3} = \boxed{\pi/12}$

7) $\boxed{\displaystyle L = \int_0^{\pi}\sqrt{1+\cos^2 x}\,dx}$

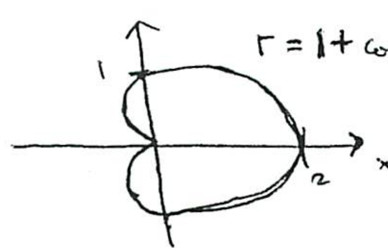
$\cos^2 x \le 1$, so $\sqrt{1+\cos^2 x} \le \sqrt{2}$

so $\boxed{\displaystyle L = \int_0^{\pi}\sqrt{1+\cos^2 x} \le \pi\sqrt{2}}$

8) $\displaystyle M = \int_0^a 8\cdot 2\pi r\,dr = 2\pi\int_0^a r^3dr = \frac{\pi}{2}r^4\Big|_0^a = \boxed{\frac{\pi}{2}a^4}$ grams

9) a)

$r = 1+\cos\theta$



$r = 3\cos\theta$
$r = 1+\cos\theta$
$3\cos\theta = 1+\cos\theta$
$\cos\theta = 1/2$
$\boxed{(\theta=\frac{\pi}{3}, r=\frac{3}{2})\quad(\theta=\frac{5\pi}{3}, r=3/2)}$

9) b) i. circle of radius $\frac{3}{2}$ centered at origin:

$r = \frac{3}{2}$, $\quad \frac{3}{2} = 1+\cos\theta$, $\cos\theta = \frac{1}{2}$

$\theta = \pi/3, \frac{5\pi}{3}$ $\quad\boxed{(\theta=\frac{\pi}{3}, r=\frac{3}{2});\ (\theta=\frac{5\pi}{3}, r=3/2)}$

$r = 3/2$

ii. $(x-\frac{3}{2})^2+y^2 = (\frac{3}{2})^2$

$x^2-3x+y^2=0$

$r^2\cos^2\theta - 3r\cos\theta + r^2\sin^2\theta = 0$

$r^2 - 3r\cos\theta = 0$

$r = 3\cos\theta$

$\underline{6\rfloor}$ a) $L = \int ds = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^{\pi/2} \left((-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2 \right)^{1/2} dt$

$= 3\int_0^{\pi/2} \cos t \sin t \, \underbrace{(\cos^2 t + \sin^2 t)}_{1}^{1/2} dt$

$= \frac{3}{2} \int_0^{\pi/2} \sin(2t)\, dt = -\frac{3}{4}\cos(2t)\Big|_0^{\pi/2}$

$= \boxed{3/2}$

b) $A = \int 2\pi y \, ds = \int_0^{\pi/2} 2\pi \sin^3 t \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right)^{1/2} dt$

$= 2\pi \int_0^{\pi/2} \sin^3 t \cdot 3\cos t \sin t \, dt = 6\pi \int_0^{\pi/2} \sin^4 t \cos t \, dt$

$u = \sin t$

$du = \cos t \, dt$

$= 6\pi \int_0^1 u^4 \, du = \boxed{\frac{6\pi}{5}}$

(Pr Exam #2)

See comments on Practice Exam #1.

18.01 Practice Exam 4

Problem 1. (15) Evaluate $\displaystyle\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$ by making a trigonometric substitution; remember to change the limits also. U( )

Problem 2. (15) Find the volume of the solid obtained by rotating about the y-axis (that's the y-axis) the area under the graph of $y = e^x$ and over the interval $0 \le x \le 1$.
(Suggestion: use cylindrical shells.)

Problem 3. (20) Evaluate $\displaystyle\int \frac{4x}{(x^2-1)(x-1)}\,dx.$
(Begin by factoring the denominator completely; don't forget the final integration.)

Problem 4. (15: 8, 7)

a) Write down the definite integral of the form $\displaystyle\int_a^b f(x)\,dx$ which represents the arclength of the curve $y = \sin^2 x$ for $0 \le x \le \pi/4$.

b) Show that the arclength is less than 1.2 by estimating the integral.
(Indicate how you are estimating it. It helps to write the integral in the simplest-looking form.)

~~**Problem 5.**~~ (15)

A circle in the xy-plane has radius a and center at the point $(b, 0)$ on the x-axis; assume $a < b$.

a) Write its equation in rectangular coordinates, and change the equation to polar coordinates.

Problem 5b can be skipped; no question similar to 5b will appear on the 2006 exam.

b) Check your answer by deriving it directly in polar coordinates, using a geometric formula.

Problem 6. (20)

Once every century the Klingon moon Bosok completely eclipses the moon Yan-ki. As seen from the city Mitek, the two moons appear as circular discs, with radii respectively $\sqrt{2}$ and 1 (units: dorks).

When the center of Bosok is exactly on the circumference of Yan-ki, the problem is to find the area (in square dorks) of the region of the Yan-ki disc that is not yet covered.

~~a)~~ Draw a diagram in polar coordinates, placing the center of Bosok at the origin, and the other center on the positive x-axis. Set up a definite integral representing the area of the desired region. Indicate how the significant limit on the integral was determined.

b) Evaluate the integral.

(If you have time, check your answer by elementary geometry, but no credit for this, just a warm glow of satisfaction and a smug feeling of superiority.)

18.01 Practice Exam 4 Solutions

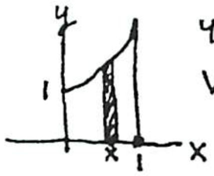
1 $\displaystyle\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} \frac{\sin^2 u}{\cancel{\cos u}} \cdot \cancel{\cos u}\,du$

Put $x = \sin u$ (or $x = \cos u$) $\quad \frac{1}{2} = \sin \pi/6$

$$= \int_0^{\pi/6} \frac{1-\cos 2u}{2}\,du$$

$$= \frac{u}{2} - \frac{\sin 2u}{4}\Big]_0^{\pi/6} = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

2



$y = e^x$

$$\text{Volume} = \int_0^1 2\pi x e^x\,dx$$

$$\int x e^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x$$

$$\therefore \text{volume} = 2\pi(xe^x - e^x)\Big|_0^1 = 2\pi(0-(-1)) = 2\pi$$

3 $\displaystyle\frac{4x}{(x^2-1)(x-1)} = \frac{4x}{(x-1)^2(x+1)}$

$$= \frac{2}{(x-1)^2} + \frac{B=1}{x-1} + \frac{-1}{x+1}$$

by coverup $\qquad$ by coverup

Put $x=0$: $\quad 0 = 2 - B - 1; \quad B = 1$

Integrating:

$$\int \frac{4x\,dx}{(x^2-1)(x+1)} = \frac{-2}{x-1} + \ln(x-1) - \ln(x+1) + c$$

$$\underset{\text{OR}}{=} \frac{-2}{x-1} + \ln\left(\frac{x-1}{x+1}\right) + c$$

4 $\displaystyle\int_a^b \sqrt{1+y'^2}\,dx \qquad y = \sin^2 x$

$y' = 2\sin x\cos x = \sin 2x$

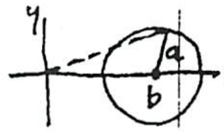
Arclength

$$= \int_0^{\pi/4} \sqrt{1+\sin^2 2x}\,dx$$

Since $0 \le \sin^2 2x \le 1$ on the interval,

$$\int_0^{\pi/4} \sqrt{1+\sin^2 2x}\,dx \le \underbrace{\pi/4}_{\text{length of interval}} \cdot \sqrt{2} < \frac{3.2}{4}\cdot\frac{3}{2} = 1.2$$

5



a) $(x-b)^2 + y^2 = a^2$ (by translation of $x^2+y^2=a^2$ to $(b,0)$ center)

$x^2 + y^2 - 2bx + b^2 = a^2$

$$\boxed{r^2 - 2br\cos\theta = a^2 - b^2}$$

b) Applying law of cosines to

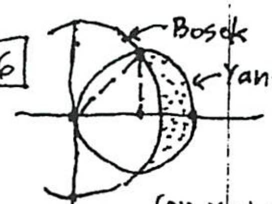
we get

$$a^2 = r^2 + b^2 - 2br\cos\theta,$$

same as part (a).

DON'T HAVE TO DO!

6



shows this is a right Δ, $\alpha = \pi/4$

(or you can solve $\begin{cases} r = 2\cos\theta \\ r = \sqrt{2} \end{cases}$ simultaneously, for θ)

Using symmetry:

$$\text{Area} = 2\cdot\frac{1}{2}\int_0^{\pi/4} (2\cos\theta)^2 - (\sqrt{2})^2\,d\theta$$

b) $\displaystyle = 2\int_0^{\pi/4} (\underbrace{2\cos^2\theta - 1}_{=\cos 2x})\,d\theta = 2\cdot\frac{\sin 2x}{2}\Big]_0^{\pi/4} = 1$

[By elem. geometry:

$= \frac{1}{2} + \frac{\pi}{4}$

$= \frac{\pi(\sqrt{2})^2}{8} + A \quad \therefore A = 1/2,\ 2A = 1$]

(1) # Calc Exam 4 Review 12/2

Double angle Formulas (memorize)

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta$

$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

$\int \sin(\sqrt{x})\,dx$ ← christee thinks really hard

$\int \frac{\sqrt{x}\sin\sqrt{x}}{\sqrt{x}}\,dx$ ← needed it in denom

$x^{1/2} = u$

$\frac{1}{2}x^{-1/2}\,dx = du$

$2\int u\sin u\,du$ → classic by parts

$w = u$ $\quad dv = \sin u\,du$

$dw = du$ $\quad v = -\cos u$

$2\left(-u\cos u + \int \cos u\,du\right)$

$-2u\cos u + 2\sin u$

look + decide plan of attach

(2)

$x = \csc t$ $\qquad 0 < t < \pi$

$y = \cot t$

- to graph
- need a trig idenity

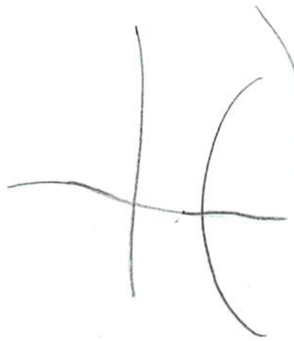
$$\frac{\sin^2 t}{\sin^2 t} + \frac{\cos^2 t}{\sin^2 t} = \frac{1}{\sin^2 t}$$

$$1 + \cot^2 t = \csc^2 t$$

$$1 + y^2 = x^2$$

or $x^2 - y^2 = 1$

↑ hyperbole



when $y = 0$, what are values for x

$-1, 1$

are they actually on curve ?

draw

x

y

③ Heavy sides

$$\int \frac{x^3+2x^2-x+1}{(x^2+1)(x-1)^2}\,dx$$

degree denom > degree numerator

$$\frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

can't = 0

Only works for highest factors

~~last~~ heavysides only

$$\frac{Ax+B\,(x-1)^2}{x^2+1} + \frac{C\,(x-1)^2}{x-1} + D = \frac{x^3+2x^2-x+1}{(x^2+1)}$$

plug in 1 so these = 0

$$D = \frac{3}{2}$$

Now multiply all terms by denom

$$(Ax+B)(x^2-1)^2 + C(x-1)(x^2+1) + \frac{3}{2}(x^2+1) = x^3+2x^2-x+1$$

multiply out

$$Ax^3 - 2Ax^2 + Ax + Bx^2 - 2Bx + B + Cx^3 + Cx - Cx^2 - C + \frac{3}{2}x^2 + \frac{3}{2} =$$

now systems

3) $A + C = 1$

2) $-2A - C + B\frac{3}{2} = 2 \rightarrow -2A + B - C = \frac{1}{2}$

1) $A + \;\; - 2B + C = -1$

(4) c) $\frac{3}{2} + B - C = 1$

~~use~~ solve

$1 - 2B = -1$

$-2B = -2$

$B = 1$

$-C + \frac{3}{2} + 1 = 1$

$C = \frac{3}{2}$

$A + \frac{3}{2} = 1$

$A = -\frac{1}{2}$

$$\int \frac{-\frac{1}{2}x + 1}{x^2+1} + \int \frac{\frac{3}{2}}{x-1} + \int \frac{\frac{3}{2}}{(x-1)^2} + C$$

$$\int \frac{-\frac{1}{2}x}{x^2+1}\,dx \qquad \int \frac{1}{x^2+1} +$$

~~factoring / u sub~~

u-sub now

$u = x^2+1$

$du = 2x\,dx$

$-\frac{1}{4}\int \frac{1}{u}$

$-\frac{1}{4}\ln u$

$u =$ ~~factor out~~ $\frac{3}{2}$

$\int x^{-1} - 1 + 1$

$\frac{3}{2}\ln(x-1)$

(know the tricks!)

$+ \frac{3}{2}\cancel{\int (x-1)^{-2}}$

factor out

$\frac{3}{2}\cancel{\int x^{-2} - 2x^{-1} + 1}$

$\frac{3}{2}\cancel{-x^{-1} - 2\ln x + x}$

$\cancel{-\frac{3}{2}(x-1)^{-1}}$

$\boxed{-\frac{3}{2(x-1)}}$ $\cancel{-\frac{3}{2x-2}}$

wolfram so can treat as a group (x-1)

(5) $\int \sec x = \int \frac{\sec x\,(\sec x + \tan x)}{(\sec x + \tan x)}$ ← du

↑ u

$\int \frac{1}{u}\,du = \ln(\sec x + \tan x)$

~~[illegible]~~

Night

I think I got concepts
- not so many this time

Its knowing which steps to do
- TA + Prof don't know instantly
- Try stuff

- Feeling good about this for some reason
  - perhaps over confident
  - Spent a lot of time doing Past [illegible]
- Just need to do practice
  - This class does not give good practice
  - Last years exam
- can do the easy qu

(6) $\int \frac{x-4}{(x+1)(x^2+4)}$

$\cancel{(x+2)}\cancel{(x-2)}$

can't

$\frac{A}{x+1} + \frac{Bx+C}{x^2+4}$

$\frac{x-4}{(x^2+4)} = A + \frac{(Bx+C)(x+1)}{(x^2+4)}$

what should x be so =0

can't do

$\frac{x-4}{(x+1)} = \frac{A(x^2+4)}{(x+1)} + Bx+C$

cant

-so multiply out

$\frac{x-4}{\cancel{(x+1)(x^2+4)}} = A(x^2+4) + (Bx+C)(x+1)$

$Ax^2+4A + Bx^2 + Cx + Bx + C$

2) $0 = A + B$

1) $1 = C + B \rightarrow B = 1 - C$

0) $-4 = C + 4A \rightarrow C = -4 - 4A$

$-C = 4 + 4A$

$0 = A + 1 - C$

$0 = A + 1 + 4 + 4A$

$0 = 5 + 5A$

$A = -1$

$B = 1$

$C = 0$ ← small math error here

⑦

$$\int -\frac{1}{x+1}\,dx + \int \frac{1x}{x^2+4}\,dx +$$

$-\cancel{1(x2)}+$

? u sub why can't I integrate this easily

$U = x+1$

$du = 1dx$

$-\int \frac{1}{u}\,du$

$-\int u^{-1}$

$U = x^2+4$

$du = 2x\,d\cancel{x}$

$\frac{1}{2}\int \frac{1}{u}$

↑ or 2 can't decide!

was right $\frac{1}{2}$

$\rightarrow \frac{1}{2}du = x\,dx$

? put in

↑ take out

$$-\ln|x+1| + \frac{1}{2}\ln|x^2+4| + C$$

- whoo - right exc abs vale

- took me a while - but got it

* power of do it * not just read solution

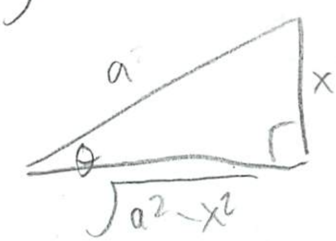
(8) Doing a triangle one

~~$\int \frac{\sqrt{a^2-x^2}}{x}\,dx$~~ ← what is a
- don't have sol

r can aal for a
- going to do it

~~Pset av~~

$\sin\theta = \frac{x}{a}$



$x = a\sin\theta$

now what?

$dx = a\cos\theta\, d\theta$

$$\int \frac{\sqrt{a^2-a^2\cos^2\theta}}{x}\, a\cos\theta\, d\theta$$

$$\int \frac{\sqrt{a^2(1-\cos^2\theta)}}{x}\, a\cos\theta\, d\theta$$

$$\int \frac{\sqrt{a^2\sin^2\theta}}{x}$$

$$\int \frac{a\sin\theta \cdot a\cos\theta\, d\theta}{x}$$ ← what now

← think not supposed to have

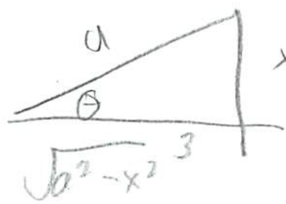
$\int a^2 \sin\theta\cos\theta$ ← ? double angle
- memorize

$\int a^2\, \frac{1}{2}\sin(2\theta)$

calc says ans not pretty
- prof did not test before posting

(9) Now going to do pset one w/o looking

$$\int \frac{dx}{(a^2-x^2)^{3/2}}$$



$\sin\theta = \frac{x}{a}$

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$$\int \frac{a\cos\theta\, d\theta}{\sqrt{a^2 - a^2\sin^2\theta}^{\,3}}$$

$$\int \frac{a\cos\theta\, d\theta}{\sqrt{a^2(1-\sin^2\theta)}^{\,3}}$$

$$\int \frac{\cancel{a}\,\cancel{\cos\theta}\, d\theta}{a^{\cancel{3}2}\cos\theta^{\cancel{3}2}}$$

$$\int \frac{1}{a^2\cos^2\theta}\, d\theta$$

? now what ↓ forgot this step before

$$\frac{1}{a^2}\int \sec^2\theta\, d\theta = \frac{1}{a^2}\tan^2\theta$$

↑ move out ↑ $\frac{1}{\cos^2}$ =

※ plug tan in.

$$\frac{1}{a^2}\,\frac{x}{\sqrt{a^2-y^2}^{\,3}} + C$$

ok think got this now.

Now a prob of first type

No distractions - good

(1) $\int \sin^3 x \sec^2 x$

$\frac{1}{\cos}$

$\boxed{\frac{\sin^3 x}{\cos^2 x}} = \sin x \tan^2 x$

? does that help

- want one think where deriv is there

$v = \cos$

$dv = -\sin$

$\frac{-dv^3}{v^2}$ ← getting nowhere

$\sin^3 x (\tan^2 + 1)$

$\sin^3 x \frac{\sin^2}{\cos^2} + 1$

$\int \frac{\sin^5 x}{\cos^2 x} + \sin^3 x$

$\sin^3 x + \sin$??

somehow $\sin^2 = 1 - \cos^2$

→ $\frac{1 - \cos^2 x}{\cos^2 x} \sin x \, dx$

use that

guessing its good that sin and cos

now v and dv

$v = \cos$

$dv = -\sin$

$-\int \frac{1 - v^2}{v^2} dv$

(11) $-\int \frac{1}{u^2} - \int \frac{u^2}{u^2}$

$+\frac{1}{u} - u$

$\frac{1}{\cos} - \cos \quad + C$

$\uparrow \sec x$

* Ok just know the steps for that

do another one like that

$\int \sin^3 x \cos^2 x \, dx$

$\int \sin x \sin^2 x \cos^2 x$

$\int \sin x (1 - \cos^2 x) \cos^2 x$

$u = \cos x$
$du = -\sin x \, dx$

$-\int du (1 - u^2) u^2$

$-\int u^2 - u^4$

$-\left(\frac{u^3}{3} - \frac{u^5}{\cancel{4}5}\right)$ ← stupid mistake forgot to +1

$-\frac{\cos^3 x}{3} + \frac{\cos^{\cancel{4}5} x}{\cancel{4}5} + C$ ✓ got it except for stupid mistake

(12)

- What do they mean reduction formula in pratice test?

- integrate - or perhaps specifically $\sin^x \cos^x$ type

Did all types Brubaker listed except by parts

$$uv - \int v\,du$$

and on arclenght

arc lenght $\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

? $\int \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$

$\int_A^B \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

? do we have to do actual area - don't think so

- If asks qu like pratic test would be far more confused
- heavy on parametrics

studdying went for better I think

13

More stuff to know

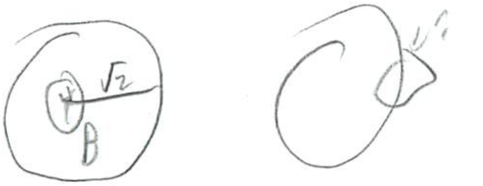
cycloid did not see any pratice q...

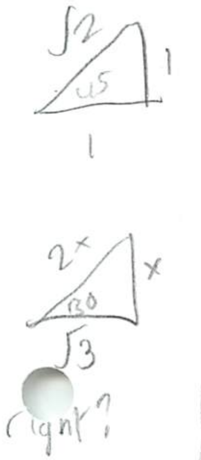
$x(\theta) = r(\theta - \sin\theta)$

$y(\theta) = r(1 - \cos\theta)$

tangent $\dfrac{y'(t)}{x'(t)}$

area $\displaystyle\int_{t_1}^{t_2} y(t)\, dx(t)$





$$\text{Area} = 2 \cdot \frac{1}{2} \int_0^{\pi/4} (2\cos\theta)^2 - (\sqrt{2})^2 \, d\theta$$

← time for 45°

← ? where is this from

Can also solve $r = 2\cos\theta$, $r = \sqrt{2}$ simultaneously for θ

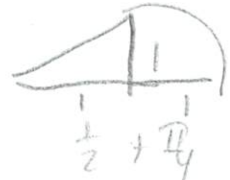
a) is polar ← oh were not supposed to do

b) eval integral

$$2\int_0^{\pi/4} (2\cos^2\theta - 1)\, d\theta$$

$$\frac{2\sin 2x}{2} \Big|_0^{\pi/4}$$

1



$= \frac{\pi(\sqrt{2})^2}{8} + A \quad A = \frac{1}{2}$

$2A = 1$

feel a lot more comfortable

— awake now

18.01 EXAM IV

Thursday, December 3, 2009

Name: Michael Plasmeier

E-mail: theplaz@mit.edu

Recitation Instructor (Circle One): C. Breiner / I. Losev / X. Ma / S. Ramakrishnan / B. Rhoades

Recitation Hour: ____________________

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 5 questions and a 50 minute time limit on this exam. Good luck.

Question	Score	Maximum
1	3.5	5
2	1	5
3	3.5	5
4	0	7
5	2+1.5	6
Total	12	28

felt confident on this but did just like normal
– same plot on graph

 Name: ______________________

1. Compute the following integral:

$\frac{1}{t}$

$$\int_1^4 \sqrt{t}\,\ln t\,dt$$

by parts

$uv - \int v\,du$

$\int_1^4 t^{1/2} \ln t\,dt$

$u = \ln t$ $dv = t^{1/2}$

$du = \frac{1}{t}\,dt$ $v = \frac{2}{3}t^{3/2}$

$uv - \int v\,du$

$\ln t \cdot \frac{2}{3}t^{3/2} - \int \frac{2}{3}t^{3/2} \cdot t^{-1}\,dt$

practice more by parts

$\frac{2}{3}t^{3/2}\ln t - \frac{2}{3}\int t^{3/2} \cancel{\times}\ dt$ -1

← they add not multiply

$\frac{2}{3}t^{3/2}\ln t - \frac{2}{3}\frac{t^{3/2}}{\frac{3}{2}}$

← integrated wrong also

$\frac{2}{3}t^{3/2}\ln t - \frac{4}{9}\sqrt{t}^{3/2}$

$\frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9}$ ← multiply across!

$\frac{2}{3}(4)^{3/2} \cdot \ln(4) - \frac{4}{9}\sqrt{4}^{3/2} - \left[\frac{2}{3}(1)^{3/2} \cdot \ln(1) - \frac{4}{9}\sqrt{1}^{3/2}\right]$

$\frac{2}{3} \cdot \cancel{16}$, $4^{3/2} = 8$, $\ln 4 - \frac{8}{3} - \frac{2}{3}\ln(1) + \frac{4}{3}$

-5

$\frac{\cancel{32}}{3}$ 16 $\ln 4 - \frac{28}{9} - \frac{2}{3}\ln 1$

? prob know $\boxed{\ln(1) = 0}$ ← how am I supposed to know

opps $4 \cdot 4^{3/4} \to \sqrt{\ }$

I took $\frac{1}{2}$ grr

almost had it
key was putting u^2
in and expanding

 Name: ____________________

2. Compute the following integral:

$$\int_0^{\pi/4} \tan^4\theta \sec^6\theta \, d\theta$$

$\frac{d}{dx}\tan = \sec^2$

$\frac{d}{dx}\sec = \sec\tan$

$\sin^2 + \cos^2 = 1$

$\tan^2 + 1 = \sec^2$

oh what did we say for evens - keep reducing till easy - cut in half somehow

no evens

$$\int_0^{\pi/4} \tan^4\theta \sec^4\theta \sec^2\theta \, d\theta$$

do this one

which method is it?

(1) $u = \sec\theta$, $du = \sec\theta\tan\theta\, d\theta$ $\quad$ $u = \tan\theta$, $du = \sec^2\theta\, d\theta$

$u \to \tan\theta$, $du = \sec^2\theta$

(2) $\int_0^{\pi/4} \tan^4\theta \sec^4\theta (\tan^2\theta + 1)$

$\int_0^{\pi/4} \tan^6\theta \sec^4\theta + \tan^4\theta \sec^4\theta$ - got more complex

have to change bounds

$\tan\frac{\pi}{4} = 1$

$\tan 0 = 0$

(3) $\int_0^{\pi/4} \frac{\sin^4\theta}{\cos^4\theta} \cdot \frac{1}{\cos^6\theta}$

(1b) $\int_0^{\pi/4} u^2 \, du \sec^3\theta \tan^3\theta$

$u = \sec\theta$

$du = \sec\theta\tan\theta$

↑ now what

$\int_0^{\pi/4} u^2 \, du \sec^2\theta \sec\theta \tan^3\theta$

$w = \tan\theta$

$dw = \sec^2\theta$

$\int_0^{\pi/4} u^2 \, du \, w^3 \, dw \sec\theta$

(4) $\int_0^{\pi/4} u^4 \sec^4 du$

$\int_0^{\pi/4} u^4 (\tan^2 + 1) \, du$ key

bounds

almost

change to u^2

$$\int_0^1 u^4 (1+u^2)^2 \, du$$

simple algebra

$$\int_0^1 u^4(1 + 2u^2 + u^4)\, du$$

$$\int_0^1 u^4 + 2u^6 + u^8 \, du$$

now integrate

$$\frac{u^5}{5} + \frac{2u^7}{7} + \frac{u^9}{9} \Big|_0^1$$

$$\frac{1}{5} + \frac{2}{7} + \frac{1}{9} = \frac{188}{315}$$

 Name: ____________________

3. Compute the following integral:

$$\int \frac{10}{(x-1)(x^2+9)}\,dx$$

$$\frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

$$\frac{10}{(x^2+9)} = A + \frac{(Bx+C)(x-1)}{x^2+9}$$

$x=1$ $\quad$ $x=1$

$$\frac{10}{10} = 1 = A$$

$$\frac{10}{(x-1)} = \frac{A(x^2+9)}{(x-1)} + Bx+C$$

$x \neq$ anything

$$10 = 1(x^2+9) + (Bx+C)(x-1)$$

$$10 = x^2+9 + Bx^2 - Bx + Cx - C$$

2) $0 = 1 + B \Leftarrow B = -1$

1) $0 = -B + C \Leftarrow 0 = --1 + C \Leftarrow C = -1$

0) $10 = 9 - C \Leftarrow$ ✓ $10 = 9 - -1$ ✓

→ back

$$\int \frac{1}{x-1}\,dx + \int \frac{-1x}{x^2+9}\,dx + \int \frac{-1}{x^2+9}\,dx$$

$\int (x-1)^{-1}$

$u = x^2+9$

$du = 2x\,dx$

$-\int \cancel{x^{-2}+9^{-1}}$

C may be illegal

$\ln(x-1)$ + $-\frac{1}{2}du = -x\,dx$

$\cancel{+\frac{x^{-1}}{-1} + \ln(9)}$

$-\frac{1}{2}\int \frac{du}{u}$

$-\frac{1}{2}\ln(x^2+9)$

$\cancel{+\frac{1}{x} + \ln 9}$

$-\frac{1}{3}\tan^{-1}(x/3)$

$$\ln(x-1) - \frac{1}{2}\ln(x^2+9) \underbrace{\cancel{+\frac{1}{x} + \ln 9}}_{\text{wrong}} + C$$

I forgot $\frac{1}{x^2+1} = \tan^{-1}(x)$

but how to $\frac{1}{3}$?

divide 3

$\frac{1}{\frac{x^2}{3}+3} \quad \frac{1}{x^2+1} = \tan^{-1}(x)$

$\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$

How would you think about that

think got it

35

? should have been more

 Name: ______________________

Should have practiced more

Worth most pts

4. Compute the following integral:

Completely wrong

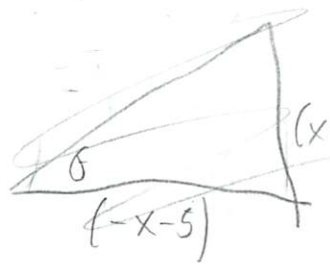
$$\int \frac{1}{(5-4x-x^2)^{5/2}}\,dx$$

$$\int \frac{1}{\sqrt{-x^2-4x+5}^{\,5}}$$

← ? triangle
no trig stuff, no a
but was no triangle qv
factor

$$\int \frac{1}{\sqrt{(-x-5)(x-1)}^{\,5}}$$

?, no what
- what other methods
U-sub
- by parts ← he said any function can do by
- compound method



$$\cancel{\frac{A}{-x-5}} + \cancel{\frac{B}{x-1}}$$

$U = -x^2-4x+5$

$du = -2x-4$

$\frac{dV}{V} =$ what goes here?

$\frac{1}{U^{5/2}}$ ← where does du go

$$\int \frac{1}{(9-(2+x)^2)^{5/2}}\,dx$$

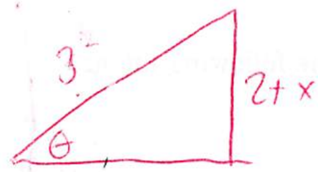
← Factor like this

? how we supposed to know

$x+2 = 3\sin\theta$

$x = 3\sin\theta - 2$

$dx = 3\cos\theta\,d\theta$



$\sin\theta = \frac{x+2}{3}$

$$\int \frac{3\cos\theta\,d\theta}{9^{5/2}(1-\sin^2\theta)^{5/2}}$$

? factor out 9 (2 step) and triangle

← $\frac{3}{9^{5/2}} = \frac{3}{9\cdot 9\cdot 9\cdot 9\cdot 9\cdot\sqrt{2}} = \frac{3}{243} = \frac{1}{81}$

$\sqrt{59049}$

? how did they do this, came out same but they have $\frac{1}{34}$

$$\frac{1}{81}\int \frac{\cos\theta\,d\theta}{(\cos^2)^{5/2}}$$

← $(\cos^2\theta)^{5/2} = |\cos^5\theta|$

$$\frac{1}{81}\int \frac{\cos\theta\,d\theta}{|\cos^5\theta|}$$

$$\frac{1}{81}\int \cos^{-4}\theta$$

realize

$$\frac{1}{81}\int \sec^4\theta$$

now must split again

$$\frac{1}{81}\int (1+\tan^2\theta)(\sec^2\theta)\,d\theta$$

$u = \tan\theta$

$du = \sec^2\theta$

$$\frac{1}{81}\int 1+u^2\,du$$

$$\frac{1}{81}\left(u+\frac{u^3}{3}\right)+C$$

← don't forget to plug back in

$$\frac{1}{81}\left(\tan\theta + \frac{\tan^3\theta}{3}\right)+C$$

→ and again use triangle

find 3th leg

$\sqrt{9-(x+2)^2}$

$\sqrt{5-4x-x^2}$

$\tan\theta = \frac{x+2}{\sqrt{5-4x-x^2}}$

$$\frac{1}{81}\left(\frac{x+2}{\sqrt{5-4x-x^2}} + \frac{(x+2)^3}{(5-4x-x^2)^{3/2}}\right) + C$$

Name: ______________________

5. (a) Set up (but do not solve) the integral for the arc length along the curve $x = y + y^3$ from $y = 1$ to $y = 4$.

$$\int_1^4 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad dx = 1 + 3y^2$$

$y = t$ << how they did this

$x = t + t^3$

$$\int_1^4 \sqrt{1 + (1+3y^2)^2}\, dy \quad \text{same} \qquad \int_1^4 \sqrt{(x'(t))^2 + (y'(t))^2}$$

not

$$\int_1^4 \sqrt{(1+3t^2) + 1}$$

$(1+3y^2)(1+3y^2)$

$$\cancel{\int_1^4 \sqrt{1 + (1 + 6y^2 + 9y^4)}\, dy}$$

$$\boxed{\cancel{\int_1^4 \sqrt{9y^4 + 6y^2 + 2}\, dy} \quad \cancel{\int_1^4 9y^2 + 6y + \sqrt{2}\, dy}}$$

don't do

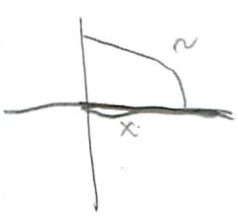
2/3

(b) Set up (but do not solve) the integral for the surface area of the surface obtained by rotating the curve given by

$$x = a\cos^3 t, \quad y = a\sin^3 t, \quad 0 \leqslant t \leqslant \pi/2$$

about the x-axis. Here a is an arbitrary constant.

Forgot diff $\cos^3$



$dx = a3\cos^2 t \sin t$

$dy = a\,3\sin^2 t \cos t$

$$\int_0^{\pi/2} 2\pi\, x\, y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$= \sin 3t$

radius duh

$$2\pi \int_0^{\pi/2} \cancel{2\pi x}\; |a| \sin 3t \sqrt{(a3\cos^2 t \sin t)^2 + (a3\sin^2 t \cos t)^2}\, dt$$

$$2\pi \int_0^{\pi/2} 3a^2 \sin^3 t \sqrt{\sin^2 t \cos^2 t\, (\cos^2 t + \sin^2 t)}$$

$\sin t \cos t$ 1.5/3   1

$$6a^2\pi \int_0^{\pi/2} \sin^4 t \cos t\, dt$$

<< simplify more

18.01 EXAM IV

Thursday, December 3, 2009

Name: SOLUTIONS.

E-mail: ____________________

Recitation Instructor (Circle One): C. Breiner / I. Losev / X. Ma / S. Ramakrishnan / B. Rhoades

Recitation Hour: ____________________

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 5 questions and a 50 minute time limit on this exam. Good luck.

Question	Score	Maximum
1		5
2		5
3		5
4		7
5		6
Total		28

 Name: ______________________

1. Compute the following integral:

$$\int_1^4 \sqrt{t}\ln t\,dt$$

$u = \ln t$ $\qquad v = \frac{2}{3}t^{3/2}$

$du = \frac{1}{t}\,dt$ $\qquad dv = \sqrt{t}\,dt$

$$\int_1^4 \sqrt{t}\ln t\,dt = \frac{2}{3}t^{3/2}\ln t\Big|_1^4 - \int_1^4 \frac{2}{3}\underbrace{t^{3/2}\cdot t^{-1}}_{t^{1/2}}\,dt$$

$$= \left(\frac{2}{3}t^{3/2}\ln t - \frac{4}{9}t^{3/2}\right)\Big|_1^4 =$$

$$= \frac{16}{3}\ln 4 - \frac{28}{9}$$

$$\frac{28}{9} = \frac{4}{9}\left(4^{3/2}-1\right)$$

 Name: ____________________

2. Compute the following integral:

$$\int_0^{\pi/4} \tan^4\theta \sec^6\theta \, d\theta$$

$\sec^2\theta = 1 + \tan^2\theta$

$$\int_0^{\pi/4} \tan^4\theta \sec^6\theta \, d\theta = \int_0^{\pi/4} \tan^4\theta \, (1+\tan^2\theta)^2 \sec^2\theta \, d\theta =$$

$$\left(\begin{array}{ll} u = \tan\theta & du = \sec^2\theta \, d\theta. \\ 0 \le \theta \le \pi/4 \Rightarrow & 0 \le u \le 1. \end{array} \right)$$

$$= \int_0^1 u^4 \, (1+u^2)^2 \, du = \int_0^1 u^4 \, (1 + 2u^2 + u^4) \, du =$$

$$= \int_0^1 u^4 + 2u^6 + u^8 \, du = \frac{1}{5}u^5 + \frac{2}{7}u^7 + \frac{1}{9}u^9 \Big|_0^1 =$$

$$= \frac{1}{5} + \frac{2}{7} + \frac{1}{9}$$

$$\left(= \frac{188}{315} \right).$$

 Name: ____________________

3. Compute the following integral:

$$\int \frac{10}{(x-1)(x^2+9)}\,dx$$

$$\int \frac{10}{(x-1)(x^2+9)}\,dx = \int \frac{1}{x-1} - \frac{x+1}{x^2+9}\,dx$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$$

$$= \ln(x-1) - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}(x/3) + C.$$

Name: ______________________

4. Compute the following integral:

$$\int \frac{1}{(5-4x-x^2)^{5/2}}\,dx$$

$$\int \frac{1}{(9-(2+x)^2)^{5/2}}\,dx$$

$x+2 = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$$= \int \frac{3\cos\theta\, d\theta}{(9)^{5/2}(1-\sin^2\theta)^{5/2}}$$

$$= \frac{1}{3^4}\int \frac{\cos\theta\, d\theta}{\cos^5\theta} = \frac{1}{3^4}\int \sec^4\theta\, d\theta$$

$$= \frac{1}{3^4}\int (1+\tan^2\theta)(\sec^2\theta)\, d\theta$$

$u = \tan\theta$

$du = \sec^2\theta\, d\theta$

$$= \frac{1}{3^4}\int (1+u^2)\,du = \frac{1}{3^4}\left(u + \frac{u^3}{3}\right) + C.$$

$$= \frac{1}{3^4}\left(\tan\theta + \frac{\tan^3\theta}{3}\right) + C$$

$\sin\theta = \dfrac{x+2}{3}$

$\sqrt{9-(x+2)^2} = \sqrt{5-4x-x^2}$

$$\boxed{= \frac{1}{3^4}\left(\frac{x+2}{\sqrt{5-4x-x^2}} + \frac{(x+2)^3}{3(5-4x-x^2)^{3/2}} + C\right)}$$

$\Rightarrow \tan\theta = \dfrac{x+2}{\sqrt{5-4x-x^2}}$

 Name: ____________________

5. (a) Set up (but do not solve) the integral for the arc length along the curve $x = y + y^3$ from $y = 1$ to $y = 4$.

$y = t$

$x = t + t^3$

$$AL = \int ds = \int_{t=1}^{4} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt =$$

$$= \int_1^4 \sqrt{(1+3t^2)^2 + 1}\, dt$$

(b) Set up (but do not solve) the integral for the surface area of the surface obtained by rotating the curve given by

$$x = a\cos^3 t, \quad y = a\sin^3 t, \quad 0 \leqslant t \leqslant \pi/2$$

about the x-axis. Here a is an arbitrary constant.

$$SA = 2\pi \int_{t=0}^{\pi/2} |a| \sin^3 t \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= 2\pi \int_0^{\pi/2} |a| \sin^3 t \sqrt{(3a\cos^2 t \sin t)^2 + (3a \sin^2 t \cos t)^2}\, dt =$$

$$= 2\pi \int_0^{\pi/2} 3a^2 \sin^3 t \sqrt{\sin^2 t \cos^2 t\,(\cos^2 t + \sin^2 t)}\, dt$$

$$= 6a^2\pi \int_0^{\pi/2} \sin^4 t \cos t\, dt.$$

Recitation / Review

12/2

No make up

office hrs 3-5

Last assignment before final

If I do half right - should pass - concentrate on just integral stuff not parametric - since 1/5 and did most recently

Trig Functions

- tools
  - $\sin^2\theta + \cos^2\theta = 1$
  - double angles

    $\cos 2\theta = \cos^2\theta - \sin^2\theta$
  - know common derivitives + intergrals

    $2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

    ? could $\int \cos^2\theta$

    $\int \frac{1}{2} \ldots$

    $\tan^2\theta + 1 = \sec^2\theta$

    $1 + \cot^2\theta = \csc^2\theta$

Also know when to stop - when they don't want you to go further

$\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta$

$\frac{d}{d\theta}\tan\theta = \sec^2\theta$

$\frac{d}{d\theta}\csc\theta = -\cot\theta\csc\theta$ ← can get back somehow if don't know

$\frac{d}{d\theta}\cot\theta = -\csc^2\theta$

$\int \tan^5\theta \sec^3\theta$

- which to get rid of + how many
- which one is better ?

- want 1 tan and a bunch of secants

$\int \tan\theta \, (\sec^2\theta - 1)^2 \sec^3\theta \, d\theta$

$\sec\theta = u$

$\sec\theta\tan\theta = du$

$$\int \underbrace{\tan\theta}_{u} \underbrace{\sec^2\theta \, d\theta}_{du}$$

want tan odd or secant even

replace all tan but 1 | replace secants except 2

so $\int \sec \text{ stuff } \tan\theta \, d\theta$ — $u = \sec\theta$, $du = \sec\theta \tan\theta$

so $\int \tan \text{ stuff } \sec^2\theta$ — $u = \tan\theta$, $du = \sec^2\theta$

$$\int \cos^2\theta \sin^2\theta \, d\theta$$

$\leftarrow 2\sin\theta\cos\theta = \sin 2\theta$

$$\int \frac{\sin(2\theta)^2}{4} d\theta$$

$\leftarrow$ trig sub

$$\frac{1}{4}\int \frac{\cos(4\theta) - 1}{-2} d\theta$$

can integrate that

$$\int \tan^5\theta \sec^2\theta \, d\theta$$

$$\int \tan^5\theta \,(1 - \cos^2\theta)\, d\theta$$

$$\int \tan^5\theta - \int \frac{\sin^5\theta}{\cos^3\theta} d\theta \rightarrow \underline{\underline{or}} \int \frac{\sec\theta(1-\cos^2\theta)^2}{\cos^3\theta}$$

radicles

$$\int \tan^5\theta \, d\theta - \int \tan^3\theta \, d\theta + \int \frac{\sin^3\theta}{\cos\theta} d\theta$$

$\uparrow$ can do 1 more time

How to do fast

- rule out stuff quickly
- not magic, just feels like it

Estimate Integrals
- compare to function bounded in between

$\int_a^b f(x)$

^ if $g(x) \le f(x) \le h(x)$

easy to integrate

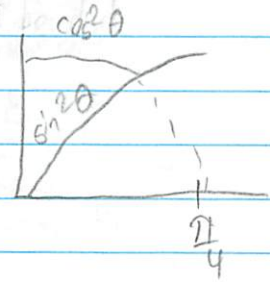
$$\int_0^{\pi/4} \underbrace{\sqrt{1+4\sin^2 x\cos^2 x}}_{} \, dx \quad < 1.2$$

$< \sqrt{5}$ since sin + cos bounded by 1

$\frac{\sqrt{5}\pi}{4} > 1.2$

$\sin^2 x + \cos^2 x = 1 \rightarrow \sin^2 x \cos^2 x = \cos^2 x - \cos^4 x$

↑ bounded < 1



Could use Simpson's rule
- how many terms??
- want bounded above
- not really a good choice

key is find right function bounded by

$$\sqrt{1+(\sin 2x)^2} < \sqrt{2}$$

$$\frac{\sqrt{2}\,\pi}{4} < 1.2$$ ✓

She keeps trying things quickly — have always hated that

Integration by Parts

- logs + polynomials
- exponentials + polynomials
- exponential + trig

$$\int \frac{\ln x}{x^7}\,dx$$

which should be u?

← want to derive always

← polynomial goes easy each way

$u = \ln x$ $\quad dv = x^{-7}$

$du = \frac{dx}{x}$ $\quad v = \frac{x^{-8}}{-8}$

$$uv - \int du\, v\, dx$$

$$\ln x \frac{x^{-8}}{-8} - \int \frac{dx}{x} \cdot \frac{x^{-8}}{-8}$$

might have to do again if lots powers left

- exponential + trig
- esp w/ e^x

$\sin x$ $\cos x$

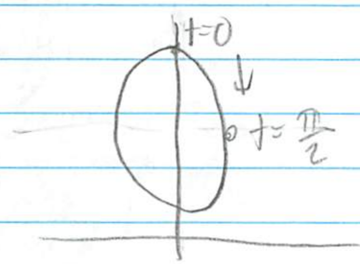
Parametrics

- be able to find tangent line

$x = \sin t$ $\qquad 0 \le t \le 2\pi$
$y = \cos t + 2$

circle
radius = 1 $\qquad \sin^2 t + \cos^2 t = 1$
from (0, 2)

$$x^2 + (y-2)^2 = 1$$



$\frac{dx}{dt} = \cos t \qquad \frac{dy}{dt} = -\sin t$

when t is small and + cos is +
- so $\frac{dx}{dt}$ is + getting bigger in time →

when t is small and + sin is −
- so going ↓

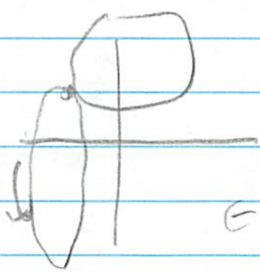
Find tangent at $t = \frac{\pi}{4}$

still $\frac{\Delta y}{\Delta x} = \frac{dy}{dx}\Big|_t = \frac{dy/dt}{dx/dt} = -\tan t \Big|_{\frac{\pi}{4}} = -1$

To find equation, plug in point (x, y)

- Some have slope $= 0$, some have ∞ slope

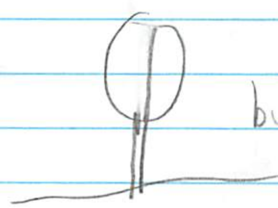
Rotate curve about x - axis



← spin the curve around

$\int 2\pi \text{ radius } ds$

cos t + 2 ← y value in this case



but if rotating t...

end w/ donut / torus

write in terms of 2

$\int_0^{2\pi} 2\pi(\cos t + 2)\,dt$

in this case $ds = dt$

Lecture
Infinite Series

12/4

#4 was hardest on exam 4

Was a completing the square

$$\int \frac{dx}{\sqrt{5-4x-x^2}^5}$$

Could factor - but not right path

Complete the square
- supposed to pick 9

$$9-(4+4x+x^2)$$
$$9-(x+2)^2$$
$$a^2-x^2 \qquad x+2=3\sin\theta$$
$$dx=3\cos\theta\, d\theta$$

No passing score since no makeup
- about 11 or 12

Enough points to pass
- final is worth a lot
- if get A or B prob pass
- everything was warm up for final
- final is comprehensive
- 1/3 straightforward
- 2/3 thinking
- from past tests

Math is not an algorithm - it's thinking

Infinite Series

Last time, saying best degree k polynomial approx to a function $f(x)$ at $x=a$

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$\leftarrow$ nth deriv of f eval at a

Consider $T_\infty(x) = \lim_{k\to\infty} \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$

notation $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

reminds of notation for improper integrals

Questions 1. Does $T_\infty(x)$ make sense?
↳ for what values of x does the limit exist?

today →

tmo → 2. How does $T_\infty(x)$ relate to original function $f(x)$?

Given some numbers $\{b_n\}_n$ → non neg integers

When does $\lim_{k\to\infty} \sum_{n=0}^{k} b_n$ exist?

Here's one example r = real # $\lim_{k\to\infty} \sum_{n=0}^{k} r^n$

$$= 1 + r + r^2 + r^3 + \dots + r^k$$

$$= \frac{r^{k+1} - 1}{r-1}$$

Need to know $\lim_{k\to\infty}\left(\frac{r^{k+1}-1}{r-1}\right)$

If $|r|>1$ then $|r^{k+1}|\to\infty$ as $k\to\infty$
then the limit does not exist.

If $|r|<1$ then $r^{k+1}\to 0$ as $k\to\infty$

$$=\left(\frac{0-1}{r-1}\right)=\boxed{\frac{1}{1-r}}$$

If $r=1$ then $\lim_{k\to\infty}(k+1)\to$ "∞"

If $r=-1$ then $\lim_{k\to\infty}\sum_{n=0}^{k}-1^n \begin{cases}0 & \text{if } k=\text{odd}\\ 1 & \text{if } k=\text{even}\end{cases}$

osilating so limit does not exist

$$\sum_{n=0}^{\infty} r^n=\frac{1}{1-r} \quad \text{if } |r|<1$$

eg if $r=\frac{1}{2}\to \left(\frac{1}{2}\right)^0+\left(\frac{1}{2}\right)^1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\dots$

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots$

$\to$ gets closer to 2

v! $\frac{1}{1-\frac{1}{2}}=2$ ✓

Many Test to determine when a series converges
- ie has a limit

Today Ratio test
- series we just did
- motivated by geometric series $\sum_{n=0}^{\infty} r^n$

Given $\sum_{n=0}^{\infty} b_n$ consider $\lim_{n\to\infty} \left|\frac{b_{n+1}}{b}\right|$

as n gets bigger + bigger

$= \begin{cases} L<1, & \text{series converges, limit exists} \\ L>1, & \text{" diverges, limit \underline{not} exist} \\ L=1, & \text{test inconclusive} \end{cases}$

Example $\sum_{n=0}^{\infty} \frac{n^3}{3^n}$ ← grows slower / ← grows very fast ☆ ker gets small fast

do ratio test $\left|\frac{b_{n+1}}{b_n}\right|$

$b_{n+1} = \frac{(n+1)^3}{3^{n+1}}$ $b_n = \frac{n^3}{3^n}$

$\frac{(n+1)^3/3^{n+1}}{n^3/3^n} = \frac{1}{3}\frac{(n+1)^3}{n^3}$

$\lim_{n\to\infty} \frac{1}{3}\frac{(n+1)^3}{n^3} = \frac{1}{3}$

$\lim = 1$

$\frac{n^3+3n^2+3n+1}{n^3} \cdot \frac{1/n^3}{1/n^3} = \frac{1+3/n+3/n^2+1/n^3}{1}$

now $\lim_{n\to\infty}$

$\frac{1+0+0+0}{1} = 1$

So series converges at some #
But not actually finding it
Use a calculator

Why does this test work?

-in geometric series what is $\left|\frac{b_{n+1}}{b_n}\right| = |r|$

if $|r| < 1$ converges
$|r| > 1$ diverges

Not going to build a proof for

Series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$

power series in $(x-a)$

constants that depend on n

think of it as $\frac{f^{(n)}(a)}{n!}$

For which x does the power series converge?
Use the ratio test

Example 1 $\sum_{n=0}^{\infty} n!\, x^n$

Use ratio test, solve for x so that $\lim L < 1$

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{(n+1)! \cdot x^{n+1}}{n! \cdot x^n} = (n+1)|x|$$

When is $\lim_{n\to\infty} (x+1)|x| < 1$

- any non 0 $= \infty$
- so only works for $x=0$

Example 2 $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \frac{|x|}{n+1}$$

When does $\lim_{n\to\infty} \frac{|x|}{n+1} < 1$

- any value for x, including 0
- Converges for all real ~~#s~~ x

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

Example 3 $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{(x-3)^{n+1}/n+1}{(x-3)^n/n}\right| = |x-3| \frac{n}{n+1}$$

$$\lim_{n\to\infty} |x-3| \frac{n}{n+1} = |x-3| \lim_{n\to\infty} \frac{n}{n+1} = |x-3|$$

When is $|x-3| < 1$? $\to x \in (2,4)$

Next Tue: Explain series always converge in interval centered at a.

Lectures are out of order   Skipping a lot

Lecture 36: Infinite Series and Convergence Tests

Infinite Series

Geometric Series

A geometric series looks like

$$1 + a + a^2 + a^3 + ... = S$$

There's a trick to evaluate this: multiply both sides by a:

$$a + a^2 + a^3 + ... = aS$$

Subtracting,

$$(1 + a + a^2 + a^3 + \cdots) - (a + a^2 + a^3 + \cdots) = S - aS$$

In other words,

$$1 = S - aS \implies 1 = (1-a)S \implies S = \frac{1}{1-a}$$

This only works when $|a| < 1$, i.e. $-1 < a < 1$.

$a = 1$ can't work:

$$1 + 1 + 1 + ... = \infty$$

$a = -1$ can't work, either:

$$1 - 1 + 1 - 1 + ... \neq \frac{1}{1-(-1)} = \frac{1}{2}$$

Notation

Here is some notation that's useful for dealing with series or sums. An infinite sum is written:

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + ...$$

The finite sum

$$S_n = \sum_{k=0}^{n} a_k = a_0 + ... + a_n$$

is called the "n^{th} partial sum" of the infinite series.

Definition

$$\sum_{k=0}^{\infty} a_k = s$$

means the same thing as

$$\lim_{n\to\infty} S_n = s, \text{ where } S_n = \sum_{k=0}^{n} a_k$$

We say the series *converges* to s, if the limit exists and is finite. The importance of convergence is illustrated here by the example of the geometric series. If $a = 1, S = 1 + 1 + 1 + ... = \infty$. But

$$S - aS = 1 \quad \text{or} \quad \infty - \infty = 1$$

does not make sense and is not usable!

Another type of series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

We can use integrals to decide if this type of series converges. First, turn the sum into an integral:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \sim \int_1^{\infty} \frac{dx}{x^p}$$

If that improper integral evaluates to a finite number, the series converges.

Note: This approach only tells us whether or not a series converges. It does **not** tell us what number the series converges to. That is a much harder problem. For example, it takes a lot of work to determine

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Mathematicians have only recently been able to determine that

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges to an irrational number!

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sim \int_1^{\infty} \frac{dx}{x}$$

We can evaluate the improper integral via Riemann sums.

We'll use the upper Riemann sum (see Figure 1) to get an upper bound on the value of the integral.

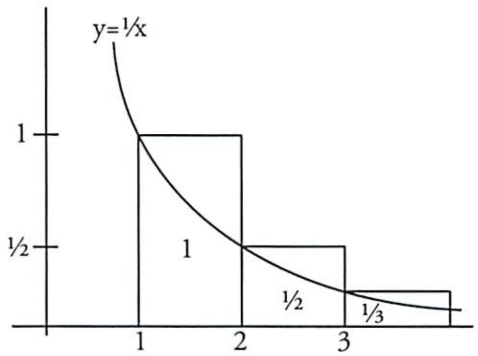


Figure 1: Upper Riemann Sum.

$$\int_1^N \frac{dx}{x} \le 1 + \frac{1}{2} + \ldots + \frac{1}{N-1} = s_{N-1} \le s_N$$

We know that

$$\int_1^N \frac{dx}{x} = \ln N$$

As $N \to \infty$, $\ln N \to \infty$, so $s_N \to \infty$ as well. In other words,

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Actually, s_N approaches ∞ rather slowly. Let's take the lower Riemann sum (see Figure 2).

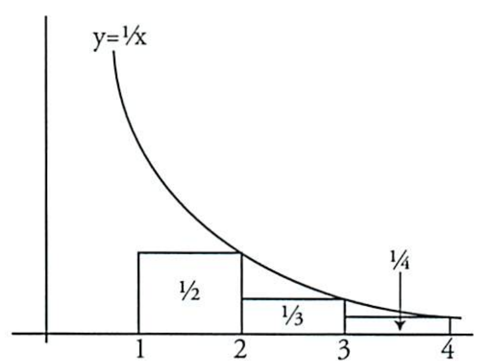


Figure 2: Lower Riemann Sum.

$$s_N = 1 + \frac{1}{2} + \ldots + \frac{1}{N} = 1 + \sum_{n=2}^{N} \frac{1}{n} \le 1 + \int_1^N \frac{dx}{x} = 1 + \ln N$$

Therefore,

$$\ln N < s_N < 1 + \ln N$$

Integral Comparison

Consider a positive, decreasing function $f(x) > 0$. (For example, $f(x) = \frac{1}{x^p}$)

$$\left| \sum_{n=1}^{\infty} f(n) - \int_1^{\infty} f(x)dx \right| < f(1)$$

So, either both of the terms converge, or they both diverge. This is what we mean when we say

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \sim \int_1^{\infty} \frac{dx}{x^p}$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \leq 1$ and converges for $p > 1$.

Lots of fudge room: in comparison.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+10}}$$

diverges, because

$$\frac{1}{\sqrt{n^2+10}} \sim \frac{1}{(n^2)^{1/2}} = \frac{1}{n}$$

Limit comparison:
If $f(x) \sim g(x)$ as $x \to \infty$, then $\sum f(n)$ and $\sum g(n)$ either both converge or both diverge.

What, exactly, does $f(x) \sim g(x)$ mean? It means that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = c$$

where $0 < c < \infty$.

Let's check: does the following series converge?

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5-10}}$$

$$\frac{n}{\sqrt{n^5-10}} \sim \frac{n}{n^{5/2}} = n^{-3/2} = \frac{1}{n^{3/2}}$$

Since $\frac{3}{2} > 1$, this series does converge.

Playing with blocks

At this point in the lecture, the professor brings out several long, identical building blocks.

Do you think it's possible to stack the blocks like this?

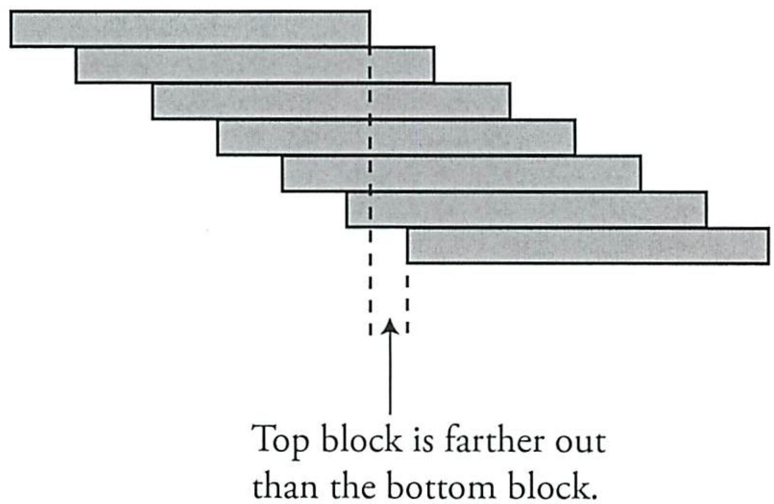


Figure 3: Collective center of mass of upper blocks is always over the base block.

In order for this to work, you want the collective center of mass of the upper blocks always to be over the base block.

The professor successfully builds the stack.

Is it possible to extend this stack clear across the room?

The best strategy is to build from the *top* block down.
Let C_0 be the left end of the first (top) block.
Let C_1 = the center of mass of the first block (top block).
Put the second block as far to the right as possible, namely, so that it's left end is at C_1 (Figure 4).
Let C_2 = the center of mass of the top two blocks.
Strategy: put the *left end* of the next block underneath the center of mass of all the previous ones combined. (See Figure 5).

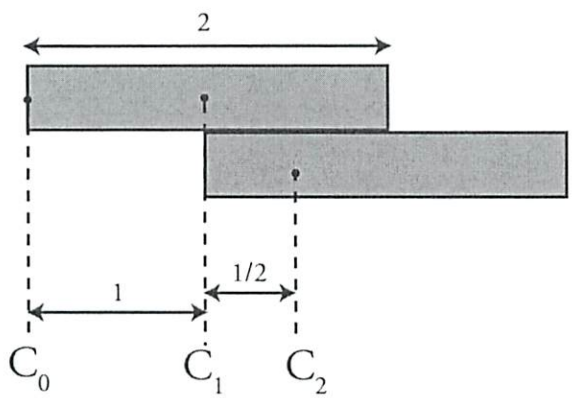


Figure 4: Stack of 2 Blocks.

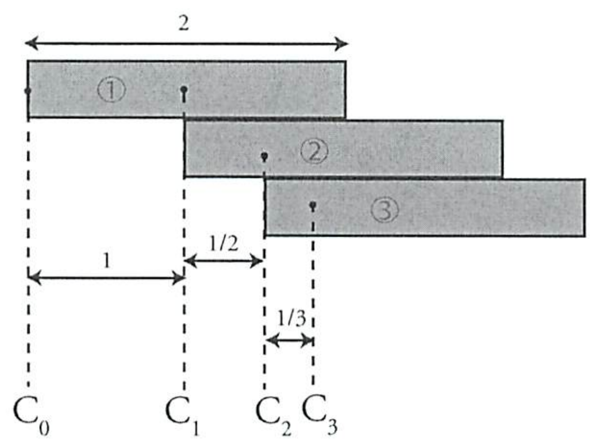


Figure 5: Stack of 3 Blocks. Left end of block 3 is C_2 = center of mass of blocks 1 and 2.

$$C_0 = 0$$
$$C_1 = 1$$
$$C_2 = 1 + \frac{1}{2}$$

$$C_{n+1} = \frac{nC_n + 1(C_n + 1)}{n+1} = \frac{(n+1)C_n + 1}{n+1} = C_n + \frac{1}{n+1}$$
$$C_3 = 1 + \frac{1}{2} + \frac{1}{3}$$
$$C_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$
$$C_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} > 2$$

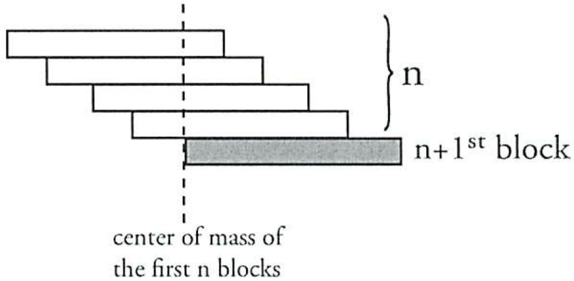


Figure 6: Stack of $n + 1$ Blocks.

So yes, you can extend this stack as far (horizontally) as you want — provided that you have enough blocks. Another way of looking at this problem is to say

$$\sum_{n=1}^{N} \frac{1}{n} = S_N$$

Recall the Riemann Sum estimation from the beginning of this lecture:

$$\ln N < S_N < (\ln N) + 1$$

as $N \to \infty$, $S_N \to \infty$.

How high would this stack of blocks be if we extended it across the two lab tables here at the front of the lecture hall? The blocks are 30 cm by 3 cm (see Figure 7). One lab table is 6.5 blocks, or 13 units, long. Two tables are 26 units long. There will be $26 - 2 = 24$ units of overhang in the stack.

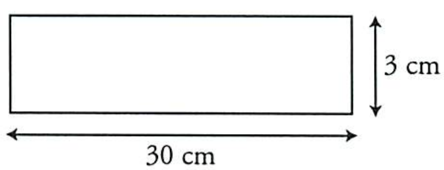


Figure 7: Side view of one block.

If $\ln N = 24$, then $N = e^{24}$.

$$\text{Height} = 3\,\text{cm} \cdot e^{24} \approx 8 \times 10^8\,\text{m}$$

That height is roughly twice the distance to the moon.

If you want the stack to span this room ($\sim$ 30 ft.), it would have to be 10^{26} meters high. That's about the diameter of the observable universe.

Recitation

Infinite Series / Power Series

12/7

$$\lim_{n \to \infty} \sum_{k=0}^{n} a_k$$

↑ sum of $n+1$ #
keep adding more things to sum
does it have a limit?
aka does it converge?

$S_0 = a_0$
$S_1 = a_0 + a_1$
$S_2 = a_0 + a_1 + a_2$
↑ n

partial sums
– sumation expanded

Something must happen for s to converge
Nessassry conditions

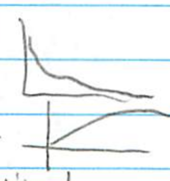
For the sum to converge $a_k \xrightarrow{k \to \infty} 0$
↑ necessary (has to happen for sum to converge)
but <u>not sufficent</u> (even if $a_k \to 0$ sum might diverge)

Could go to 0 but not fast enough
– Harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} \leftarrow \text{diverges}$$

↑ Integral is log

$a_k \to 0$
$\sum \to \infty$ ←



like log

not asymptotic to anything

Have done 1 test
- ratio test - series convergence
- check if series converges

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = L$$

$L < 1 \Rightarrow$ series converges
$L > 1 \Rightarrow$ series diverges
$L = 1 \to$ do more to find out

ratio test for harmonic series

$$\lim_{k\to\infty} \left| \frac{1}{k+1} \cdot \frac{k}{1} \right| = 1$$ so can't determine

coefficents of highest degrees =

Try one

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k} \qquad \frac{(k+1)^2}{2^{(k+1)}} \cdot \frac{2^k}{k^2}$$

$$\boxed{\frac{2^k}{2^{k+1}} = \frac{1}{2^1}} \qquad \frac{1}{2} \left| \frac{(k+1)^2}{k^2} \right|$$

$\to 1$

$\frac{1}{2}$ converges

Ex 2

$$\sum_{n=0}^{\infty} \frac{n!}{e^n}$$

$$\left| \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} \right|$$

$$\frac{1}{e} \left| \frac{(n+1)!}{n!} \right|$$

$\frac{1}{e} |n+1|$ as $\underline{n \to \infty}$

$\boxed{\text{diverges}}$

Ex3

$$a_{k+1} = \frac{1+\sin k}{k}, \ a_k$$

does $\sum a_k$ converge?

$$\left| \frac{\frac{1+\sin k}{k} \cancel{a_k}}{\cancel{a_k}} \right|$$

don't split abs value and take limits →

$\left| \frac{1+\sin k}{k} \right|$ ← max sin can go to $= 1$
min $= -1$

$0 \leq |1 + \sin k| \leq 2$

$\cancel{\left| \frac{2}{k} \right|}$

squeeze theorm

Compare to 2 things can deal with

$$\frac{0}{k} \leq \frac{|1+\sin k|}{k} \leq \frac{2}{k} \leftarrow \text{as } k \to \infty$$

$\downarrow$ $\downarrow$

0 0

↑ so main thing $\to 0$

$0 < 1 \to$ converges

$$\boxed{|a+b| \leq |a| + |b| \quad \text{triangle inequality} \\ \text{not equal to}}$$

Power Series

will be functions where they converge

General Formula

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

each a_i constant depends on n

c is a constant

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$\leftarrow 0$ for x

$\frac{0^0}{0!} = 1$

1 for x

? finite

75 for x

???

converges? depends on x

Run a ratio test

$$\left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$\left| \frac{x}{n+1} \right| \xrightarrow{n \to \infty} 0$$

look! for any x

$0 < 1$ so converges for any x

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$f(0) = 1$

$f(5) =$ plug it in $\sum_{n=0}^{\infty} \frac{5^n}{n!}$

$$\frac{x^0}{0!} + \frac{x^1}{1} + \frac{x^2}{2 \cdot 1} + \frac{x^3}{3 \cdot 2 \cdot 1} + \frac{x^4}{4 \cdot 3 \cdot 2 \cdot 1} + \ldots$$

1

quadratic approx for e^x at 0

$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots$$

$$f'(x) = 0 + 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$$

get same thing

$$\boxed{= e^x}$$

so $f(5) = e^5$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n}$$

For what values of x does it converge?

$\boxed{-1^n \leftarrow \text{does not matter abs value}}$

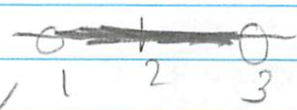
$$\left| \frac{(x-2)^{(n+1)}}{n+1} \cdot \frac{n}{(x-2)^n} \right|$$

$$\left| \frac{x-2}{1} \right|$$

$$|x-2|$$

The sum converges when $|x-2| < 1$

converges



↑ interval of covergence

test pts

At $x=1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n} = \frac{1}{n}$$ ← diverges

* harmonic series (know/remember)

At $x=3$

$\frac{(-1)^n (1)^n}{n}$ = alternating harmonic series - converges

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$ ✓ justification

↑ now everyother is a minus sign

converges since goes to 0 - fast enough

Lecture

More on Power series / Taylor Series 12/8

Today - More on Power series

Thur - Finish series / Review for exam

Pratice Problems on web insted of P-set

Office Hrs Thur 2:30-4
Tue 10-12, 1-3

This is last exam
- have Tue evening + Wed

Need to work hard on this

Last Time: Ratio test
- determines convergence / divergance

Applied to power series in $(x-a)$

$$\underbrace{\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}}_{b_n} \leftarrow \text{power series } (x-3)$$

Determine the x's for which this is <1
$x \in (2,4)$ had $\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| < 1$

Compounding when you concentrate + do well

Theorm Given a power series

$$\sum_{n=0}^{\infty} C_n (x-a)^n$$ there are 3 possibilities for convergence

1) Converge at $x=a$
2) Converge every where (all real x)
3) Converge → There is a # R (real #) so that the power →

series converges in $x \in (a-R, a+R)$
diverges in $|x-a| > R$

← symetric about a

Sometimes we call a power series in $(x-a)$ a power series "centered at a"

More formalized way of recitation yesterday →

__Remark__ Still have not answered about end points $(a-R)$ $(a+R)$

Can't use ratio test → Inconclusive

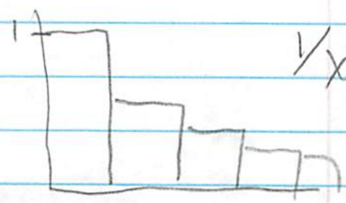
__Need new tests__

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} \quad \text{at } x=2, \ x=4$$

$\underline{x=2}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ $\underline{x=4}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

converge or diverge?

↑ showed before (w/ Riemann sum) that diverges with $\Delta n = 1$



Integral Test

$\sum_{n=1}^{\infty} f(n)$ converges/diverges if $\int_1^{\infty} f(t)\,dt$ converges/diverges

$= \lim_{b\to\infty} \int_1^b \ln(b) - \cancel{\ln(1)}$

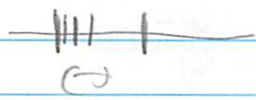
$\lim_{b\to\infty} \ln(b)$

∞ slowly diverges

$\ln e^{100} = 100$

$\ln e^{1000} = 1000$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$



ping pong in increasingly thin region

Alternating Series Test

If your series alternates in sign $(+ \to - \to + \to -)$ say $\sum_{n=1}^{\infty} (-1)^n b_n$ and $b_n > b_{n+1}$ for all n

and $\lim_{n\to\infty} b_n = 0$

$\boxed{\text{know it will converge}}$

- alternating to 0 shrinking

*interested in series - what the sum of the terms is

converges $x \in [2, 4)$ diverges

Call $\sum_{n=0}^{\infty} C_n (x-a)^n$, for $x \in (a-R, a+R)$ where series converges

$\overset{def}{=} g(x)$

theorm * $\quad g'(x) \overset{?}{=} \sum_{n=1}^{\infty} C_n \cdot n (x-a)^{n-1}$ ✓

↓ derivite

↑ killed constant

not proving

$\int g(x)\,dx = \sum_{n=0}^{\infty} \frac{C_n (x-a)^{n+1}}{n+1} + C$ for $x \in (a-R, a+R)$

We know $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $x \in (-1,1)$

$\rightarrow \frac{1}{(1-x^2)} = \sum_{n=1}^{\infty} n x^{n-1}$, $\xrightarrow[\text{+ theorm}]{\text{integrate}}$ $-\ln|1-x| = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} + C$

↑ differentiate + theorm

set $x=0$ $\quad -\ln 1 = 0 + c$

$0 \qquad$ ↑ 0

One more application of therom * $\quad$ say $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$

- Suppose $f(x)$ has a power series expansion in $(x-a)$
- Let's use theorm to solve C_n in terms of f

Use differenitation

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$

$x = a \rightarrow f(a) = c_0$

differenitdate $\rightarrow f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$

$x = a \rightarrow f'(a) = c_1$

differentiate again $\rightarrow f''(x) = 2c_2 + 6c_3(x-a) + \dots$

$x = a \rightarrow \frac{f''(a)}{2!} = c_2$

So

$x = a \rightarrow \frac{f^n(a)}{n!} = c_n$ If it has a power series expansion

* If $f(x)$ has a power series expansion at $(x-a)$ then power series must look like

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Pick $f(x) = \sin x$

find a power series like

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

How to prove that $f(x)$ has a power series expansion?

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!} x^{n}$$

$$= \lim_{n \to \infty} \underbrace{\sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!} (x-a)^{n}}_{T_k(x) \text{ at } a}$$

kth degree aprox

Dumb idea

$$f(x) - T_k(x) = R_k(x)$$

Show $\lim\limits_{k \to 0} R_k(x) = 0$ remainder

Taylor's Inequality

Suppose $|f^{(k+1)}(x)| \leq M$ some constant M for $|x-a| \leq d$

then $|R_k(x)| \leq \frac{M}{(k+1)!} |x-a|^{k+1}$

Example e^{x} has Taylor series at 0: $\sum\limits_{n=0}^{\infty} \frac{x^{n}}{n!}$

$f^{n}(0) = 1$ if $f(x) = e^{x}$

Showed Radius R of convergence is ∞
- ie convergence for all x

Want M so that $|f^{(k+1)}| \leq M$

for $|x-0| \leq d$

$|e^x| \leq M$ for $|x| \leq d$

Choose $M = e^d$

$$|R_k(x)| \leq \frac{e^d |x|^{k+1}}{(k+1)!}$$

For any $x \in (-d, d]$, analyze $\lim_{k \to \infty} \frac{e^d |x|^{k+1}}{(k+1)!} \leq$

$$\lim_{k \to \infty} \frac{e^d d^{k+1}}{(k+1)!} = 0$$

Proved remainder goes to 0 ... no matter ...
So proved e^x ...

Felt like longer than hr

Lecture 37: Taylor Series

General Power Series

What *is* $\cos x$ anyway?

Recall: geometric series

$$1 + a + a^2 + \cdots = \frac{1}{1-a} \quad \text{for } |a| < 1$$

General power series is an infinite sum:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

represents f when $|x| < R$ where $R =$ radius of convergence. This means that for $|x| < R$, $|a_n x^n| \to 0$ as $n \to \infty$ ("geometrically"). On the other hand, if $|x| > R$, then $|a_n x^n|$ does not tend to 0. For example, in the case of the geometric series, if $|a| = \frac{1}{2}$, then $|a^n| = \frac{1}{2^n}$. Since the higher-order terms get increasingly small if $|a| < 1$, the "tail" of the series is negligible.

Example 1. If $a = -1$, $|a^n| = 1$ does not tend to 0.

$$1 - 1 + 1 - 1 + \cdots$$

The sum bounces back and forth between 0 and 1. Therefore it does not approach 0. Outside the interval $-1 < a < 1$, the series diverges.

Basic Tools

Rules of polynomials apply to series within the radius of convergence.

Substitution/Algebra

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

Example 2. x = -u.

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots$$

Example 3. $x = -v^2$.

$$\frac{1}{1+v^2} = 1 - v^2 + v^4 - v^6 + \cdots$$

Example 4.

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right) = (1+x+x^2+\cdots)(1+x+x^2+\cdots)$$

Term-by-term multiplication gives:

$$1 + 2x + 3x^2 + \cdots$$

Remember, here x is some number like $\frac{1}{2}$. As you take higher and higher powers of x, the result gets smaller and smaller.

Differentiation (term by term)

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[1+x+x^2+x^3+\cdots\right]$$

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \cdots \quad \text{where 1 is } a_0, 2 \text{ is } a_1 \text{ and 3 is } a_2$$

Same answer as Example 4, but using a new method.

Integration (term by term)

$$\int f(x)\,dx = c + \left(a_0 + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots\right)$$

where

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

Example 5. $\displaystyle\int \frac{du}{1+u}$

$$\left(\frac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots\right)$$

$$\int \frac{du}{1+u} = c + u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots$$

$$\ln(1+x) = \int_0^x \frac{du}{1+u} = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$$

So now we know the series expansion of $\ln(1+x)$.

Example 6. Integrate Example 3.

$$\frac{1}{1+v^2} = 1 - v^2 + v^4 - v^6 + \cdots$$

$$\int \frac{dv}{1+v^2} = c + \left(v - \frac{v^3}{3} + \frac{v^5}{5} - \frac{v^7}{7} + \cdots\right)$$

$$\tan^{-1} x = \int_0^x \frac{dv}{1+v^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Taylor's Series and Taylor's Formula

If $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, we want to figure out what all these coefficients are. Differentiating,

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$
$$f''(x) = (2)(1)a_2 + (3)(2)a_3 x + (4)(3)a_4 x^2 + \cdots$$
$$f'''(x) = (3)(2)(1)a_3 + (4)(3)(2)a_4 x + \cdots$$

Let's plug in $x = 0$ to all of these equations.

$$f(0) = a_0; \ f'(0) = a_1; \ f''(0) = 2a_2; \ f'''(0) = (3!)a_3$$

Taylor's Formula tells us what the coefficients are:

$$\boxed{f^{(n)}(0) = (n!)a_n}$$

Remember, $n! = n(n-1)(n-2)\cdots(2)(1)$ and $0! = 1$. Coefficients a_n are given by:

$$\boxed{a_n = \left(\frac{1}{n!}\right) f^{(n)}(0)}$$

Example 7. $f(x) = e^x$.

$$f'(x) = e^x$$
$$f''(x) = e^x$$
$$f^{(n)}(x) = e^x$$
$$f^{(n)}(0) = e^0 = 1$$

Therefore, by Taylor's Formula $a_n = \dfrac{1}{n!}$ and

$$e^x = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

Or in compact form,

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

Now, we can calculate e to any accuracy:

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

Example 7. $f(x) = \cos x$.

$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$
$$f^{(4)}(x) = \cos x$$
$$f(0) = \cos(0) = 1$$
$$f'(0) = -\sin(0) = 0$$
$$f''(0) = -\cos(0) = -1$$
$$f'''(0) = \sin(0) = 0$$

Only *even* coefficients are non-zero, and their signs alternate. Therefore,

$$\boxed{\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots}$$

Note: $\cos(x)$ is an even function. So is this power series — as it contains only even powers of x.

There are two ways of finding the Taylor Series for $\sin x$. Take derivative of $\cos x$, or use Taylor's formula. We will take the derivative:

$$-\sin x = \frac{d}{dx}\cos x = 0 - 2\left(\frac{1}{2}\right)x + \frac{4}{4!}x^3 - \frac{6}{6!}x^5 + \frac{8}{8!}x^7 + \cdots$$

$$= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\boxed{\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

Compare with quadratic approximation from earlier in the term:

$$\boxed{\cos x \approx 1 - \frac{1}{2}x^2 \qquad \sin x \approx x}$$

We can also write:

$$\cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}(-1)^k = (-1)^0\frac{x^0}{0!} + (-1)^2\frac{x^2}{2!} + \cdots = 1 - \frac{1}{2}x^2 + \cdots$$

$$\sin x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}(-1)^k \leftarrow n = 2k+1$$

Example 8: Binomial Expansion. $f(x) = (1+x)^a$

$$(1+x)^a = 1 + \frac{a}{1}x + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots$$

Taylor Series with Another Base Point

A Taylor series with its base point at a (instead of at 0) looks like:

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2 + \frac{f^{(3)}(b)}{3!}(x-b)^3 + ...$$

Taylor series for $\sqrt{x}$. It's a bad idea to expand using $b = 0$ because $\sqrt{x}$ is not differentiable at $x = 0$. Instead use $b = 1$.

$$x^{1/2} = 1 + \frac{1}{2}(x-1) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(x-1)^2 + \cdots$$

Last
Recitation

12/9

Review Wed 2-4

infinite series · $(x-a)^n$
$\sum^{\infty} (x-a)^n$

Power Series + Taylor Series

- Classic example

$1 + x + x^2 + x^3 + \dots \quad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad |x|<1$

Example of geometric series w/ ratio x

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad |r|<1$$

equal between $-1 < x < 1$

$\sum_{n=0}^{\infty} .2^n = \frac{1}{1-.2}$ ← can eval like this

$\sum_{n=1}^{\infty} .2^n = \frac{1}{1-.2} - 1$

Play some games

Find power series for $\frac{1}{1-x^2}$

$\boxed{f(x) = \frac{1}{1-x}} \to f(x^2)$

So eval series at x^2 $\to \sum_{n=0}^{\infty} x^{2n}, \quad |x|<1$

Endgame
Get better and better approximations
as # terms $\to \infty$
get it to = function

$$\frac{1}{1+x} = f(-x) = \sum_{n=0}^{\infty} -x^n, \quad |x| < 1$$

$\downarrow$ same thing just cleaner

$$(-1)^n (x)^n$$

General Form for Power Series

$$\sum_{n=0}^{\infty} C_n (x-a)^n$$

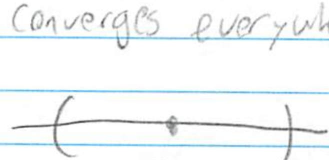
$\uparrow$ centered at a
for what x is this sum finite?

Radius of convergence

- Run ratio test

$$|x-a| < R$$

- Can be
  - $< 0 \to$ could go up or down R units $(a+R)(a-R)$
  - $= 0 \to$ only finite at $x = a$
  - $\infty \to$ converges everywhere



must check boundries

can integrate power series

$$\ln(1+x) = \int \frac{dx}{1+x}$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx$$

$$= \sum_{n=0}^{\infty} \int (-1)^n x^n \, dx$$

$$(-1)^n \frac{x^{n+1}}{n+1} + C$$

leave alone
constant
since multiply
not adding

<u>To get C Plug in 0</u>

$$\ln(1) = (-1)^n \frac{0^{n+1}}{n+1}$$

$$\ln(1) = 0 + C$$

$C = 0$

to find
estimates
of functions

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1} x^m}{m}$$

$\boxed{m = 1+n}$

Taylor Series

- centered at a
- explicit definition of c_n in terms of a function

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

← eval at a

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots$$

↑ the approx we have been doing

- need a general form

Taylor Series for $\sin x$ at $a=0$

$f(x) = \sin x$	0
$f'(x) = \cos x$	1
$f''(x) = -\sin x$	-0
$f'''(x) = -\cos x$	-1
$f^{(4)}(x) = \sin x$	0

$x=a=0$ ← eval at center

$$\sin(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$0 + 1x + \frac{-0}{2}x^2 + \frac{-1}{3!}x^2 + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \dots$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \leftarrow \sin x$$

should be able to do ln

Remainder

Define

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k \quad \leftarrow \text{finite \# of terms}$$

↑ lim as $n \to \infty$ is taylor series

$$\lim_{n\to\infty} T_n \sum_{k=0}^{\infty} \ldots$$

Does that really = the function?

Define $R_n(x) = f(x) - T_n(x)$ ← chop off those terms get what's left

- look at just what going on at end

Want $R_n(x) \to 0$ as $n \to \infty$

- should get arbitraly small
- to make sure it actually converges

Do

$$|R_n(x)| \leq \frac{M}{(n+1)!} (x-a)^{n+1}$$

What is M? M bounds $|f^{(n+1)}|$ on the interval

For $f(x) = \sin(x)$, consider $|x| < 100$

$$|R_n| \leq \frac{M}{(n+1)!} \cdot \overset{x}{100}^{n+1}$$

Find M that it bounds $|f^{(n+1)}|$ on interval
- can be sin or cos
- either way evals to 1

$$|R_n| \leq \frac{1}{(n+1)!} 100^{n+1}$$

↑ grows to 0 faster

$$|R_n| \leq 0$$

so remainder → 0, so converges

Brendon Office Hrs

12/10

Series P-Set
- will be on there

15 qu

- intervals
- finding Taylor series
- remainder

only one there
- great
- did not write much down

$\frac{1}{(3+4x^2)}$ about $x=0$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$\sum (-1)^n x^n$$

knowing this

plug in for x

split

$$\frac{1}{3+4x^2} = \frac{1}{3}\,\frac{1}{1+\left(\frac{4x^2}{3}\right)} = \frac{1}{3}\left(1 - \frac{4x^2}{3} + \left(\frac{4x^2}{3}\right)^2\right.$$

$\frac{16x^4}{9}$ ← multiply everything

$$\frac{-1}{(1+x)^2}$$ ← is the derivative of $\frac{1}{1+x}$ ← $(1+x)^{-1}$

$-1(1+x)^{-2}\cdot 1$

$-\frac{1}{(1+x)^2}$

taylor series
- have a function
- form a power series
- around that function

Redo old tests
- Riemann sums
- differentiation
- quotient ~~test~~ rule

$$\frac{f}{g} = \frac{gf' - fg'}{g^2}$$

pass line set after finals graded

~ 80% will pass
- broadly

Need to do well

Last Lecture
Review

12/10

Talked to prof about past exams
- Has to talk to math office
OCW not to helpful
30 qu on past exams
He will pick 15 from them randomly

Series example will be on final last qu

$\boxed{}$ = hidden

- Given the function $f(x) = \boxed{}$

a) Find the Taylor series for $f(x)$ in $x-a$ $a = \boxed{}$

b) Find the radius of convergence for Taylor Series

c) Prove that $f(x) = T(x)$ ← taylor series
in the interval of convergence

Example $f(x) = \sin x$
$a = 0$

a) $$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$ ← 1 pt, general

When $a = 0$

← compute this explicitly

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$$

All questions on

1. Definition of derivitive ← limits of secants
2. Its applications - related rates, curve sketching
3. Definition of integrals ← riemann sum
4. Its applications
5. Series

First 3-4 qu → take derivitves + integrals

Don't think as math as magic
Based in logic
Derive applications
Why things are true
Work as hard as possible to become a super genius
Is a week left!

Skipping the rest due to lack of time

Lecture 34: Indeterminate Forms - L'Hôpital's Rule

Feel free to use on exam if know it

L'Hôpital's Rule

(Two correct spellings: "L'Hôpital" and "L'Hospital")

Sometimes, we run into indeterminate forms. These are things like

$$\frac{0}{0}$$

and

$$\frac{\infty}{\infty}$$

For instance, how do you deal with the following?

$$\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \frac{0}{0}??$$

Example 0. One way of dealing with this is to use algebra to simplify things:

$$\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x^2+x+1}{x+1} = \frac{3}{2}$$

In general, when $f(a) = g(a) = 0$,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}} = \frac{\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a} \frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}$$

This is the easy version of L'Hôpital's rule:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Note: this only works when $g'(a) \neq 0$!

In example 0,

$$f(x) = x^3 = 1; \; g(x) = x^2 - 1$$

$$f'(x) = 3x^2; \; g'(x) = 2x \implies f'(1) = 3; \; g'(1) = 2$$

The limit is $f'(1)/g'(1) = 3/2$. Now, let's go on to the full L'Hôpital rule.

Example 1. Apply L'Hôpital's rule (a.k.a. "L'Hop") to

$$\lim_{x\to 1} \frac{x^{15}-1}{x^3-1}$$

to get

$$\lim_{x\to 1} \frac{x^{15}-1}{x^3-1} = \lim_{x\to 1} \frac{15x^{14}}{3x^2} = \frac{15}{3} = 5$$

Let's compare this with the answer we'd get if we used linear approximation techniques, instead of L'Hôpital's rule:

$$x^{15} - 1 \approx 15(x-1)$$

(Here, $f(x) = x^{15} - 1, a = 1, f(a) = b = 0, m = f'(1) = 15$, and $f(x) \approx m(x-a) + b$.)
Similarly,

$$x^3 - 1 \approx 3(x-1)$$

Therefore,

$$\frac{x^{15}-1}{x^3-1} \approx \frac{15(x-1)}{3(x-1)} = 5$$

Example 2. Apply L'Hop to

$$\lim_{x\to 0} \frac{\sin 3x}{x}$$

to get

$$\lim_{x\to 0} \frac{3\cos 3x}{1} = 3$$

This is the same as

$$\frac{d}{dx}\sin(3x)\Big|_{x=0} = 3\cos(3x)\Big|_{x=0} = 3$$

Example 3.

$$\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}} \frac{\cos x + \sin x}{1} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

$$f(x) = \sin x - \cos x, \; f'(x) = \cos x + \sin x$$

$$f'\left(\frac{\pi}{4}\right) = \sqrt{2}$$

Remark: Derivatives $\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$ are always a $\frac{0}{0}$ type of limit.

Example 4. $\lim_{x\to 0} \frac{\cos x - 1}{x}$.
Use L'Hôpital's rule to evaluate the limit:

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{-\sin x}{x} = 0$$

Example 5. $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$.

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x} = \lim_{x\to 0} \frac{-\cos x}{2} = -\frac{1}{2}$$

Just to check, let's compare that answer to the one we would get if we used quadratic approximation techniques. Remember that:

$$\cos x \approx 1 - \frac{1}{2}x^2 \quad (x \approx 0)$$

$$\frac{\cos x - 1}{x^2} \approx \frac{1 - \frac{1}{2}x^2 - 1}{x^2} = \frac{(-\frac{1}{2})x^2}{x^2} = -\frac{1}{2}$$

Example 6. $\lim_{x\to 0} \frac{\sin x}{x^2}$.

$$\lim_{x\to 0} \frac{\sin x}{x^2} = \lim_{x\to 0} \frac{\cos x}{2x} \quad \text{By L'Hôpital's rule}$$

If we apply L'Hôpital again, we get

$$\lim_{x\to 0} -\frac{\sin x}{2} = 0$$

But this doesn't agree with what we get from taking the linear approximation:

$$\frac{\sin x}{x^2} \approx \frac{x}{x^2} = \frac{1}{x} \to \infty \quad \text{as} \quad x \to 0^+$$

We can clear up this seeming paradox by noting that

$$\lim_{x\to 0} \frac{\cos x}{2x} = \frac{1}{0}$$

The limit is not of the form $\frac{0}{0}$, which means L'Hôpital's rule cannot be used. *The point is: look before you L'Hôp!*

More "interesting" cases that work.

It is also okay to use L'Hôpital's rule on limits of the form $\frac{\infty}{\infty}$, or if $x \to \infty$, or $x \to -\infty$. Let's apply this to rates of growth. Which function goes to ∞ faster: x, e^{ax}, or $\ln x$?

Example 7. For $a > 0$,

$$\lim_{x\to\infty} \frac{e^{ax}}{x} = \lim_{x\to\infty} \frac{ae^{ax}}{1} = +\infty$$

So e^{ax} grows faster than x (for $a > 0$).

Example 8.

$$\lim_{x\to\infty} \frac{e^{ax}}{x^{10}} = \text{by L'Hôpital} = \lim_{x\to\infty} \frac{ae^{ax}}{10x^9} = \lim_{x\to\infty} \frac{c^2e^{ax}}{10\cdot 9x^8} = \cdots = \lim_{x\to\infty} \frac{a^{10}e^{ax}}{10!} = \infty$$

You can apply L'Hôpital's rule ten times. There's a better way, though:

$$\left(\frac{e^{ax}}{x^{10}}\right)^{1/10} = \frac{e^{ax/10}}{x}$$

$$\lim_{x\to\infty} \frac{e^{ax}}{x^{10}} = \lim_{x\to\infty} \left(\frac{e^{ax/10}}{x}\right)^{10} = \infty^{10} = \infty$$

Example 9.

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/3}} = \lim_{x\to\infty} \frac{1/x}{1/3x^{-2/3}} = \lim_{x\to\infty} 3x^{-1/3} = 0$$

Combining the preceding examples, $\ln x \ll x^{1/3} \ll x \ll x^{10} \ll e^{ax} \quad (x \to \infty, a > 0)$

L'Hôpital's rule applies to $\frac{0}{0}$ and $\frac{\infty}{\infty}$. But, we sometimes face other indeterminate limits, such as 1^{∞}, 0^0, and $0 \cdot \infty$. Use algebra, exponentials, and logarithms to put these in L'Hôpital form.

Example 10. $\lim\limits_{x\to 0} x^x$ for $x > 0$.
Because the exponent is a variable, use base e:

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x \ln x}$$

First, we need to evaluate the limit of the exponent

$$\lim_{x\to 0} x \ln x$$

This limit has the form $0 \cdot \infty$. We want to put it in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Let's try to put it into the $\frac{0}{0}$ form:

$$\frac{x}{1/\ln x}$$

We don't know how to find $\lim\limits_{x\to 0} \frac{1}{\ln x}$, though, so that approach isn't helpful.

Instead, let's try to put it into the $\frac{\infty}{\infty}$ form:

$$\frac{\ln x}{1/x}$$

Using L'Hôpital's rule, we find

$$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0}(-x) = 0$$

Therefore,

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x \ln x} = e^{\lim\limits_{x\to 0}(x \ln x)} = e^0 = 1$$

Lecture 35: Improper Integrals

Definition.

An *improper integral*, defined by

$$\int_a^\infty f(x)dx = \lim_{M\to\infty}\int_a^M f(x)dx$$

is said to converge if the limit exists (diverges if the limit does not exist).

Example 1. $\displaystyle\int_0^\infty e^{-kx}dx = 1/k \quad (k>0)$

$$\int_0^M e^{-kx}dx = (-1/k)e^{-kx}\Big|_0^M = (1/k)(1-e^{-kM})$$

Taking the limit as $M\to\infty$, we find $e^{-kM}\to 0$ and

$$\int_0^\infty e^{-kx}dx = 1/k$$

We rewrite this calculation more informally as follows,

$$\int_0^\infty e^{-kx}dx = (-1/k)e^{-kx}\Big|_0^\infty = (1/k)(1-e^{-k\infty}) = 1/k \qquad (\text{since } k>0)$$

Note that the integral over the infinite interval $\displaystyle\int_0^\infty e^{-kx}dx = 1/k$ has an easier formula than the corresponding finite integral $\displaystyle\int_0^M e^{-kx}dx = (1/k)(1-e^{-kM})$. As a practical matter, for large M, the term e^{-kM} is negligible, so even the simpler formula $1/k$ serves as a good approximation to the finite integral. Infinite integrals are often easier than finite ones, just as infinitesimals and derivatives are easier than difference quotients.

Application: Replace x by t = time in seconds in Example 1.
R = rate of decay = number of atoms that decay per second at time 0.
At later times $t>0$ the decay rate is Re^{-kt} (smaller by an exponential factor e^{-kt})

Eventually (over time $0\le t<\infty$) every atom decays. So the total number of atoms N is calculated using the formula we found in Example 1,

$$N = \int_0^\infty Re^{-kt}dt = R/k$$

The half life H of a radioactive element is the time H at which the decay rate is half what it was at the start. Thus

$$e^{-kH} = 1/2 \quad\Longrightarrow\quad -kH = \ln(1/2) \quad\Longrightarrow\quad k = (\ln 2)/H$$

Hence

$$R = Nk = N(\ln 2)/H$$

Let us illustrate with Polonium 210, which has been in the news lately. The half life is 138 days or

$$H = (138\text{days})(24\text{hr/day})(60^2\text{sec/hr}) = (138)(24)(60)^2\text{seconds}$$

Using this value of H, we find that one gram of Polonium 210 emits (1 gram)$(6 \times 10^{23}/210$ atoms/gram$)(ln2)/H = 1.6610^{14}$ decays/sec ≈ 4500 curies

At 5.3 MeV per decay, Polonium gives off 140 watts of radioactive energy per gram (white hot). Polonium emits alpha rays, which are blocked by skin but when ingested are 20 times more dangerous than gamma and X-rays. The lethal dose, when ingested, is about 10^{-7} grams.

Example 2. $\int_0^\infty dx/(1+x^2) = \pi/2$.

We calculate,

$$\int_0^M \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^M = \tan^{-1} M \to \pi/2$$

as $M \to \infty$. (If $\theta = \tan^{-1} M$ then $\theta \to \pi/2$ as $M \to \infty$. See Figures 1 and 2.)

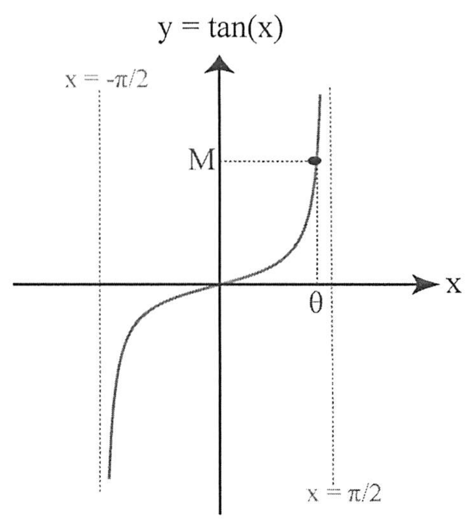


Figure 1: Graph of the tangent function, $M = \tan\theta$.

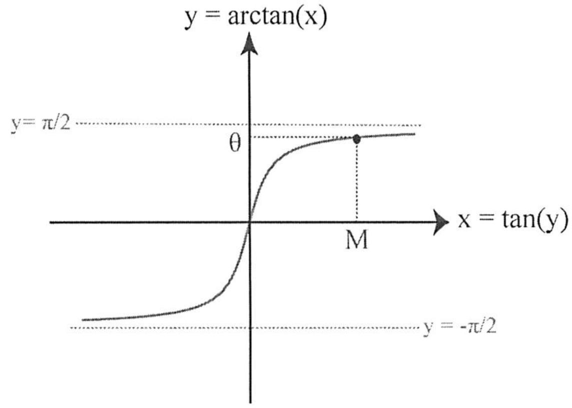


Figure 2: Graph of the arctangent function, $\theta = \tan^{-1} M$.

Example 3. $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$

Recall that we already computed this improper integral (by computing a volume in two ways, slices and the method of shells). This shows vividly that a finite integral can be harder to understand than its infinite counterpart:

$$\int_0^M e^{-x^2}dx$$

can only evaluated numerically. It has no elementary formula. By contrast, we found an explicit formula when $M = \infty$.

Example 4. $\int_1^\infty dx/x$

$$\int_1^M dx/x = \ln x\Big|_1^M = \ln M - \ln 1 = \ln M \to \infty$$

as $M \to \infty$. This improper integral is infinite (called divergent or not convergent).

Example 5. $\int_1^\infty dx/x^p \quad (p > 1)$

$$\int_1^M dx/x^p = (1/(1-p))x^{1-p}\Big|_1^M = (1/(1-p))(M^{1-p} - 1) \to 1/(p-1)$$

as $M \to \infty$ because $1 - p < 0$. Thus, this integral is convergent.

Example 6. $\int_1^\infty dx/x^p \quad (0 < p < 1)$

This is very similar to the previous example, but diverges

$$\int_1^M dx/x^p = (1/(1-p))x^{1-p}\Big|_1^M = (1/(1-p))(M^{1-p} - 1) \to \infty$$

as $M \to \infty$ because $1 - p > 0$.

Determining Divergence and Convergence

To decide whether an integral converges or diverges, don't need to evaluate. Instead one can compare it to a simpler integral that can be evaluated.

The General Story for powers: $\int_1^\infty \frac{dx}{x^p}$
From Examples 4, 5 and 6 we know that this diverges (is infinite) for $0 < p \leq 1$ and converges (is finite) for $p > 1$.

The comparison of integrals says that a larger function has a larger integral. If we restrict ourselves to nonnegative functions, then even when the region is unbounded, as in the case of an improper integral, the area under the graph of the larger function is more than the area under the graph of the smaller one. Consider $0 \leq f(x) \leq g(x)$ (as in Figure 3)

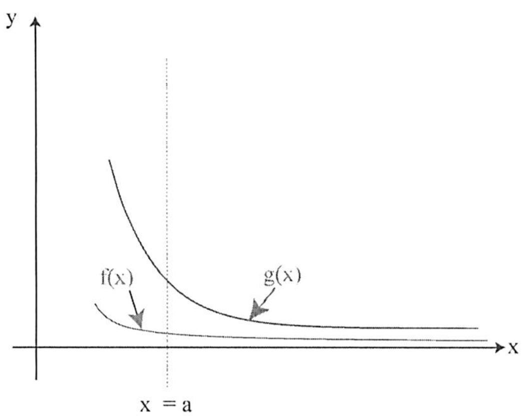


Figure 3: The area under $f(x)$ is less than the area under $g(x)$ for $a \leq x < \infty$.

If $\int_a^\infty g(x)\,dx$ converges, then so does $\int_a^\infty f(x)\,dx$. (In other words, if the area under g is finite, then the area under f, being smaller, must also be finite.)

If $\int_a^\infty f(x)\,dx$ diverges, then so does $\int_a^\infty g(x)\,dx$. (In other words, if the area under f is infinite, then the area under g, being larger, must also be infinite.)

The way comparison is used is by replacing functions by simpler ones whose integrals we can calculate. You will have to decide whether you want to trap the function from above or below. This will depend on whether you are demonstrating that the integral is finite or infinite.

Example 7. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x^3+1}}$ It is natural to try the comparison

$$\frac{1}{\sqrt{x^3+1}} \le \frac{1}{x^{3/2}}$$

But the area under $x^{-3/2}$ on the interval $0 < x < \infty$,

$$\int_0^\infty \frac{dx}{x^{3/2}}$$

turns out to be infinite because of the infinite behavior as $x \to 0$. We can rescue this comparison by excluding an interval near 0.

$$\int_0^\infty \frac{dx}{\sqrt{x^3+1}} = \int_0^1 \frac{dx}{\sqrt{x^3+1}} + \int_1^\infty \frac{dx}{\sqrt{x^3+1}}$$

The integral on $0 < x < 1$ is a finite integral and the second integral now works well with comparison,

$$\int_1^\infty \frac{dx}{\sqrt{x^3+1}} \le \int_1^\infty \frac{dx}{x^{3/2}} < \infty$$

because $3/2 > 1$.

Example 8. $\displaystyle\int_0^\infty e^{-x^3}\,dx$
For $x \ge 1$, $x^3 \ge x$, so

$$\int_1^\infty e^{-x^3}dx \le \int_1^\infty e^{-x}dx = 1 < \infty$$

Thus the full integral from $0 \le x < \infty$ of e^{-x^3} converges as well. We can ignore the interval $0 \le x \le 1$ because it has finite length and e^{-x^3} does not tend to infinity there.

Limit comparison:

Suppose that $0 \le f(x)$ and $\lim\limits_{x\to\infty} f(x)/g(x) \le 1$. Then $f(x) \le 2g(x)$ for $x \ge a$ (some large a).
Hence $\displaystyle\int_a^\infty f(x)\,dx \le 2\int_a^\infty g(x)\,dx$.

Example 9. $\displaystyle\int_0^\infty \frac{(x+10)\,dx}{x^2+1}$
The limiting behavior as $x \to \infty$ is

$$\frac{(x+10)dx}{x^2+1} \simeq \frac{x}{x^2} = \frac{1}{x}$$

Since $\displaystyle\int_1^\infty \frac{dx}{x} = \infty$, the integral $\displaystyle\int_0^\infty \frac{(x+10)\,dx}{x^2+1}$ also diverges.

Example 10 (from PS8). $\displaystyle\int_0^\infty x^n e^{-x} dx$
This converges. To carry out a convenient comparison requires some experience with growth rates of functions.

$x^n << e^x$ not enough. Instead use $x^n/e^{x/2} \to 0$ (true by L'Hop). It follows that

$$x^n << e^{x/2} \implies x^n e^{-x} << e^{x/2}e^{-x} = e^{-x/2}$$

Now by limit comparison, since $\displaystyle\int_0^\infty e^{-x/2}dx$ converges, so does our integral. You will deal with this integral on the problem set.

Improper Integrals of the Second Type

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

We know that $\dfrac{1}{\sqrt{x}} \to \infty$ as $x \to 0$.

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 x^{-1/2}dx$$

$$\int_a^1 x^{-1/2}dx = 2x^{1/2}\Big|_a^1 = 2 - 2a^{1/2}$$

As $a \to 0$, $2a^{1/2} \to 0$. So,

$$\int_0^1 x^{-1/2}\,dx = 2$$

Similarly,

$$\int_0^1 x^{-p}dx = \frac{1}{-p+1}$$

for all $p < 1$.
For $p = \dfrac{1}{2}$,

$$\frac{1}{\left(-\frac{1}{2}\right)+1} = 2$$

However, for $p \geq 1$, the integral diverges.

Lecture 38: Final Review

Review: Differentiating and Integrating Series.

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad \int f(x)dx = C + \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$$

Example 1: Normal (or Gaussian) Distribution.

$$\int_0^x e^{-t^2} dt = \int_0^x \left(1 - t^2 + \frac{(-t^2)^2}{2!} + \frac{(-t^2)^3}{3!} + \cdots\right) dt$$

$$= \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \ldots\right) dt$$

$$= x - \frac{x^3}{3} + \frac{1}{2!}\frac{x^5}{5} - \frac{1}{3!}\frac{x^7}{7} + \ldots$$

Even though $\int_0^x e^{-t^2} dt$ isn't an elementary function, we can still compute it. Elementary functions are still a little bit better, though. For example:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \implies \sin\frac{\pi}{2} = \frac{\pi}{2} - \frac{(\pi/2)^3}{3!} + \frac{(\pi/2)^5}{5!} - \cdots$$

But to compute $\sin(\pi/2)$ numerically is a waste of time. We know that the sum if something very simple, namely,

$$\sin\frac{\pi}{2} = 1$$

It's not obvious from the series expansion that $\sin x$ deals with angles. Series are sometimes complicated and unintuitive.

Nevertheless, we can read this formula backwards to find a formula for $\frac{\pi}{2}$. Start with $\sin\frac{\pi}{2} = 1$. Then,

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big|_0^1 = \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

We want to find the series expansion for $(1-x^2)^{-1/2}$, but let's tackle a simpler case first:

$$(1+u)^{-1/2} = 1 + \left(-\frac{1}{2}\right)u + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{1\cdot 2}u^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{1\cdot 2\cdot 3}u^3 + \cdots$$

$$= 1 - \frac{1}{2}u + \frac{1\cdot 3}{2\cdot 4}u^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}u^3 + \cdots$$

Notice the pattern: odd numbers go on the top, even numbers go on the bottom, and the signs alternate.

Now, let $u = -x^2$.

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots$$

$$\int (1-x^2)^{-1/2}dx = C + \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}\right) + \cdots$$

$$\frac{\pi}{2} = \int_0^1 (1-x^2)^{-1/2}dx = 1 + \frac{1}{2}\left(\frac{1}{3}\right) + \left(\frac{1\cdot 3}{2\cdot 4}\right)\left(\frac{1}{5}\right) + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\left(\frac{1}{7}\right) + \cdots$$

Here's a hard (optional) extra credit problem: why does this series converge? Hint: use L'Hôpital's rule to find out how quickly the terms decrease.

The Final Exam

Here's another attempt to clarify the concept of weighted averages.

Weighted Average

A weighted average of some function, f, is defined as:

$$\text{Average}(f) = \frac{\int_a^b w(x)f(x)\,dx}{\int_a^b w(x)\,dx}$$

Here, $\displaystyle\int_a^b w(x)\,dx$ is the total, and $w(x)$ is the weighting function.

Example: taken from a past problem set.
You get \$t if a certain particle decays in t seconds. How much should you pay to play? You were given that the likelihood that the particle has not decayed (the weighting function) is:

$$w(x) = e^{-kt}$$

Remember,

$$\int_0^\infty e^{-kt}\,dt = \frac{1}{k}$$

The payoff is

$$f(t) = t$$

The expected (or average) payoff is

$$\frac{\int_0^\infty f(t)w(t)\,dt}{\int_0^\infty w(t)\,dt} = \frac{\int_0^\infty te^{-kt}\,dt}{\int_0^\infty e^{-kt}\,dt}$$

$$= k\int_0^\infty te^{-kt}\,dt = \int_0^\infty (kt)e^{-kt}\,dt$$

Do the change of variable:

$$u = kt \quad \text{and} \quad du = k\,dt$$

$$\text{Average} = \int_0^\infty ue^{-u}\frac{du}{k}$$

On a previous problem set, you evaluated this using integration by parts: $\int_0^\infty ue^{-u}\,du = 1$.

$$\text{Average} = \int_0^\infty ue^{-u}\frac{du}{k} = \frac{1}{k}$$

On the problem set, we calculated the half-life (H) for Polonium^{120} was $(131)(24)(60)^2$ seconds. We also found that

$$k = \frac{\ln 2}{H}$$

Therefore, the expected payoff is

$$\frac{1}{k} = \frac{H}{\ln 2}$$

where H is the half-life of the particle in seconds.

Now, you're all probably wondering: who on earth bets on particle decays?

In truth, no one does. There is, however, a very similar problem that is useful in the real world. There is something called an annuity, which is basically a retirement pension. You can buy an annuity, and then get paid a certain amount every month once you retire. Once you die, the annuity payments stop.

You (and the people paying you) naturally care about how much money you can expect to get over the course of your retirement. In this case, $f(t) = t$ represents how much money you end up with, and $w(t) = e^{-kt}$ represents how likely your are to be alive after t years.

What if you want a 2-life annuity? Then, you need multiple integrals, which you will learn about in multivariable calculus (18.02).

Our first goal in this class was to be able to differentiate anything. In multivariable calculus, you will learn about another chain rule. That chain rule will unify the (single-variable) chain rule, the product rule, the quotient rule, and implicit differentiation.

You might say the multivariable chain rule is

One thing to rule them all
One thing to find them
One thing to bring them all
And in a matrix bind them.

(with apologies to JRR Tolkien).