18.02 Exam 3 Solns Spring 2010

1 a) $\vec{F} = -\dfrac{\langle x, y\rangle}{\sqrt{x^2+y^2}}$

b) $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \underset{\text{on } C_1}{1} + \underset{\text{on } C_2}{0} = 1$

flux over $C = \underset{\text{on } C_1}{0} + \underset{\text{on } C_2}{\frac{\pi}{2}\cdot 1} = \frac{\pi}{2}$

2 $\displaystyle\oint_C (y^3+2y)\,dx + x(3y^2-1)\,dy$

$= \displaystyle\iint_R (N_x - M_y)\,dA$

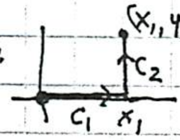
$= \displaystyle\iint_R (3y^2-1)-(3y^2+2)\,dA$

$= \displaystyle\iint_R -3\,dA = -3\cdot\underset{\text{area}}{(\sqrt{2})^2} = -6$

3 $\dfrac{\partial M}{\partial y} = a(x-2y)$ $\dfrac{\partial N}{\partial x} = 2x - by$

a) $a(x-2y) = 2x - by \Rightarrow a = 2$, $b = 2a = 4$

b) Method 1:



$f(x_1,y_1) = \displaystyle\int_{C_1+C_2} 2y(x-y)\,dx + (x^2-4xy+3y^2)\,dy$

$\displaystyle\int_{C_1} = 0 + 0$ (since $y=0$, $dy=0$)

$\displaystyle\int_{C_2} = \underset{(dx=0)}{0} + \int_0^{y_1}(x_1^2 - 4x_1y + 3y^2)\,dy$

$= x_1^2y_1 - 2x_1y_1^2 + y_1^3$

Method 2:

$f_x = 2y(x-y)$ $f_y = x^2 - 4yx + g'(y)$

$f = yx^2 - 2y^2x + g(y)$ $= x^2 - 4xy + 3y^2$

$\therefore g' = 3y^2$, $g = y^3$

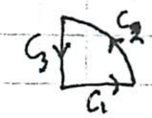
$f = yx^2 - 2y^2x + y^3$ (+c unnecessary)

c) $t = 0 \Rightarrow (x,y) = (0,0)$ use

$t = \pi/4 \Rightarrow (x,y) = (1,1)$ F.T. Calc for line integrals

$yx^2 - 2y^2x + y^3\Big]_{(0,0)}^{(1,1)} = 0 - 0 = 0$

4 a) $\displaystyle\oint_C -y^3\,dx + x^3\,dy = \iint_R 3x^2 + 3y^2\,dA$



b) On C_1: $y=0$, $dy=0$ $\displaystyle\int_{C_1} = 0$

C_3: $x=0$, $dx=0$ $\displaystyle\int_{C_3} = 0$

C_2: $x = a\cos\theta$ $y = a\sin\theta$

$dx = -a\sin\theta$ $dy = a\cos\theta$

$\displaystyle\int_{C_2} = \int_0^{\pi/2} -a^3\sin^3\theta(-a\sin\theta) + a^3\cos^3\theta\cdot(a\cos\theta)\,d\theta$

$= a^4\displaystyle\int_0^{\pi/2}(\sin^4\theta + \cos^4\theta)\,d\theta =$

$= a^4\left(\dfrac{3\cdot1}{4\cdot2}\cdot\dfrac{\pi}{2} + \dfrac{3\cdot1}{4\cdot2}\cdot\dfrac{\pi}{2}\right) = \dfrac{3\pi a^4}{8}$

c) $\displaystyle\iint_R 3x^2 + 3y^2\,dA = \int_0^{\pi/2}\int_0^a 3r^2\cdot r\,dr\,d\theta$

(use polar coords) $= \dfrac{3r^4}{4}\Big]_0^a\cdot\pi/2$

$= \dfrac{3\pi a^4}{8}$

5 $\vec\nabla f = \langle f_x, f_y\rangle$

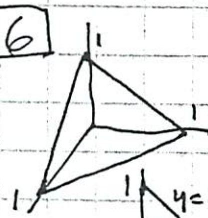
Flux of $\vec\nabla f$ over C (any C) $= \displaystyle\int_C M\,dy - N\,dx$

$M = f_x$, $N = f_y$

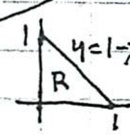
$\displaystyle\oint_{C_1'+C_2} M\,dy - N\,dx = \iint_R \underset{\operatorname{div}(\vec\nabla f)}{f_{xx} + f_{yy}}\,dA = 0$

$\displaystyle\int_{C_1'+C_2} = -\int_{C_1} + \int_{C_2} = 0$, $\therefore \displaystyle\int_{C_1} = \int_{C_2}$

6



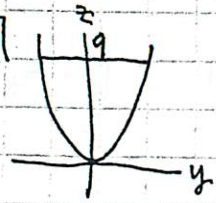
$\displaystyle\int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz\,dy\,dx$



Inner: $\dfrac{1}{2}z^2\Big]_0^{1-x-y}$

$\therefore \displaystyle\int_0^1\int_0^{1-x}\frac{1}{2}(1-x-y)^2\,dy\,dx$

$\boxed{7}$



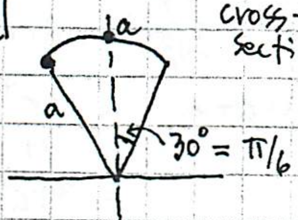
$z = x^2 + y^2$

$z = r^2$

(cross-section)

$$\int_0^{2\pi}\int_0^{3}\int_{r^2}^{9} \underbrace{r^2}_{\substack{\text{moment of inertia}\\ \text{about z-axis}}} \cdot \ dz \cdot \underbrace{r\,dr\,d\theta}_{\substack{dA \text{ in polar}\\ \text{coords}}}$$

$\boxed{8}$



cross-section

eqn cone: $\varphi = \pi/6$

eqn cap: $\rho = a$

$$\text{Vol} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{a} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

integrand = dV in spherical coords