18.02 Practice Final 3 hrs.

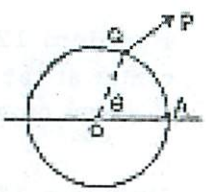
Problem 1. Given the points $P:(1,1,-1)$, $Q:(1,2,0)$, $R:(-2,2,2)$, find

a) $PQ \times PR$ b) a plane $ax+by+cz=d$ through P, Q, R.

Problem 2. Let $A = \begin{pmatrix} 1 & 0 & c \\ 2 & c & 1 \\ 1 & -1 & 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \times & \cdot \end{pmatrix}$.

a) For what value(s) of the constant c will $A\mathbf{x} = \mathbf{0}$ have a non-zero solution?

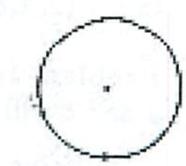
b) Take $c = 2$, and tell what entry the inverse matrix has in the position marked $\times$.



Problem 3. The roll of Scotch tape shown has outer radius a and is fixed in position (i.e., does not turn). Its end P is originally at the point A; the tape is then pulled from the roll so the free portion makes a 45-degree angle with the horizontal.

Write parametric equations $x = x(\theta), y = y(\theta)$ for the curve C traced out by the point P as it moves. (Use vector methods; θ is the angle shown).

Sketch the curve on the second picture, showing its behavior at its endpoints.



Problem 4. The position vector of a point P is $\mathbf{r} = \langle 3\cos t, 5\sin t, 4\cos t \rangle$.

a) Show its speed is constant.

b) At what point $A:(a,b,c)$ does P pass through the yz-plane?

Problem 5. Let $w = x^2y - xy^2$, and $P = (2,1)$.

a) Find the directional derivative $\dfrac{dw}{ds}$ at P in the direction of $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j}$.

b) If you start at P and go a distance .01 in the direction of $\mathbf{A}$, by approximately how much will w change? (Give a decimal with one significant digit.)

Problem 6. a) Find the tangent plane at $(1,1,1)$ to the surface $x^2 + 2y^2 + 2z^2 = 5$; give the equation in the form $ax + by + cz = d$ and simplify the coefficients.

b) What dihedral angle does the tangent plane make with the xy-plane? (Hint: consider the normal vectors of the two planes.)

Problem 7. Find the point on the plane $2x + y - z = 6$ which is closest to the origin, by using Lagrange multipliers. (Minimize the square of the distance. Only 10 points if you use some other method.)

Problem 8. Let $w = f(x,y,z)$ with the constraint $g(x,y,z) = 3$.

At the point $P:(0,0,0)$, we have $\nabla f = \langle 1,1,2 \rangle$ and $\nabla g = \langle 2,-1,-1 \rangle$. Find the value at P of the two quantities (show work): a) $\left(\dfrac{\partial z}{\partial x}\right)_y$ b) $\left(\dfrac{\partial w}{\partial x}\right)_y$

Problem 9. Evaluate by changing the order of integration: $\displaystyle\int_0^1 \int_{x^2}^1 x e^{-y^2}\, dy\, dx$.

Problem 10. A plane region R is bounded by four semicircles of radius 1, having ends at $(1,1), (1,-1), (-1,1), (-1,-1)$ and centerpoints at $(2,0), (-2,0), (0,2), (0,-2)$.

Set up an iterated integral in polar coordinates for the moment of inertia of R about the origin; take the density $\delta = 1$. Supply integrand and limits, but *do not evaluate* the integral. Use symmetry to simplify the limits of integration.

Problem 11. a) In the xy-plane, let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$. Give in terms of P and Q the line integral representing the flux of $\mathbf{F}$ across a simple closed curve C, with outward-pointing normal.

b) Let $\mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j}$. How should the constants a and b be related if the flux of $\mathbf{F}$ over any simple closed curve C is equal to the area inside C?

Problem 12. A solid hemisphere of radius 1 has its lower flat base on the xy-plane and center at the origin. Its density function is $\delta = z$. Find the force of gravitational attraction it exerts on a unit point mass at the origin.

Problem 13. Evaluate $\int_C (y-x)dx + (y-z)dz$ over the line segment C from $P:(1,1,1)$ to $Q:(2,4,8)$.

Problem 14. a) Let $\mathbf{F} = ay^2\,\mathbf{i} + 2y(x+z)\,\mathbf{j} + (by^2+z^2)\,\mathbf{k}$. For what values of the constants a and b will $\mathbf{F}$ be conservative? Show work.

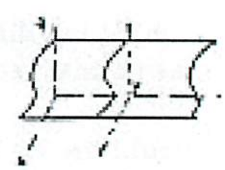
b) Using these values, find a function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$.

c) Using these values, give the equation of a surface S having the property: $\int_P^Q \mathbf{F}\cdot d\mathbf{r} = 0$ for any two points P and Q on the surface S.

Problem 15. Let S be the closed surface whose bottom face B is the unit disc in the xy-plane and whose upper surface is the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$. Find the flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across U by using the divergence theorem.

Problem 16. Using the data of the preceding problem, calculate the flux of $\mathbf{F}$ across U directly, by setting up the surface integral for the flux and evaluating the resulting double integral in the xy-plane.

Problem 17. An xz-cylinder in 3-space is a surface given by an equation $f(x,z) = 0$ in x and z alone; its section by any plane $y = c$ perpendicular to the y-axis is always the same xz-curve. (See picture.)



Show that if $\mathbf{F} = z^2\,\mathbf{i} + y^2\,\mathbf{j} + xz\,\mathbf{k}$, then $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for any simple closed curve C lying on an xz-cylinder. (Use Stokes' theorem.)

Problem 18. $\int e^{-x^2}dx$ is not elementary but $I = \int_0^\infty e^{-x^2}dx$ can still be evaluated.

a) Evaluate the iterated integral $\int_0^\infty \int_0^\infty e^{-x^2}e^{-y^2}\,dy\,dx$, in terms of I.

b) Then evaluate the integral in (a) by switching to polar coordinates. Comparing the two evaluations, what value do you get for I?

1. $P: (1,1,-1)$ $\quad \vec{PQ} = \langle 0,1,1\rangle$
$Q: (1,2,0)$ $\quad \vec{PR} = \langle -3,1,3\rangle$
$R: (-2,2,2)$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ -3 & 1 & 3 \end{vmatrix} = \langle 2,-3,3\rangle$$

Plane: $\boxed{2x-3y+3z=-4}$

(substitute any of the pts. into $2x-3y+3z=d$)

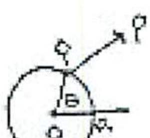
2. $\begin{vmatrix} 1 & 0 & c \\ 2 & c & 1 \\ 1 & -1 & 2 \end{vmatrix} = (2c-2c)-(c^2-1) = 1-c^2$

$\therefore |\;| = 0 \iff \boxed{c=\pm 1}$

$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$ cofactor $= -\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1$

det $= 1-2^2 = -3$

$\therefore \boxed{-1/3}$

3. 

$\vec{OP} = \vec{OQ} + \vec{QP}$
$\vec{OQ} = a\langle \cos\theta, \sin\theta\rangle$
$\vec{QP} = a\theta\left\langle \frac{\sqrt2}{2}, \frac{\sqrt2}{2}\right\rangle$

$\therefore x = a\left(\cos\theta + \frac{\theta\sqrt2}{2}\right)$
$y = a\left(\sin\theta + \frac{\theta\sqrt2}{2}\right)$

4. $\vec{r} = \langle 3\cos t, 5\sin t, 4\cos t\rangle$
$\vec{v} = \langle -3\sin t, 5\cos t, -4\sin t\rangle$
$|\vec{v}| = \sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t}$
$= \boxed{5}$

Passes through yz plane when $x=0$,
$\therefore$ when $\cos t = 0$: $t = \pi/2, 3\pi/2$
$\therefore$ at $\langle 0, \pm 5, 0\rangle$

5. $w = x^2y - xy^3$, $\quad P = (2,1)$

a) $\vec{\nabla w} = (2xy-y^3)\hat{\imath} + (x^2-3xy^2)\hat{\jmath}$
$(\vec{\nabla w})_P = 3\hat{\imath} - 2\hat{\jmath}$
$\left(\frac{dw}{ds}\right)_P = (3\hat{\imath}-2\hat{\jmath})\cdot\frac{(3\hat{\imath}+4\hat{\jmath})}{5} = \boxed{\frac{1}{5}}$

b) $\frac{\Delta w}{\Delta s} \approx \frac15$, $\therefore \Delta w \approx \frac15(.01) = \boxed{.002}$

6. $x^2+2y^2+2z^2=5$
$\vec{\nabla w} = \langle 2x, 4y, 4z\rangle = \langle 2,4,4\rangle$ at $(1,1,1)$
tan plane: $\boxed{x+2y+2z=5}$

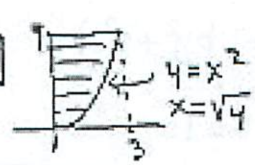
dihedral angle θ: $\cos\theta = \frac{\langle 1,2,2\rangle}{3}\cdot\hat{k} = \frac{2}{3}$
($\angle$ between normals)
$\boxed{\theta = \cos^{-1}(2/3)}$

7. Minimize $x^2+y^2+z^2$, with $2x+y-z-6=0$ $\circledast$
Lagrange equations:
$2x = 2\lambda$ $\quad$ substituting into $\circledast$:
$2y = \lambda$ $\quad 2\lambda + \frac{\lambda}{2} - \left(\frac{-\lambda}{2}\right) = 6$
$2z = -\lambda$ $\quad \therefore \lambda = 2$

Ans: $(2,1,-1)$

8. $g(x,y,z)=3$ $\quad (\vec{\nabla g})_P = \langle 2,-1,-1\rangle$
$\therefore g_x + g_z\cdot\frac{\partial z}{\partial x} = 0$; at P, $\frac{\partial z}{\partial x} = -\frac{g_x}{g_z} = \frac{-2}{-1} = \boxed{2}$ Ans.

$$\left(\frac{\partial w}{\partial x}\right)_y = \underset{1}{(f_x)}\underset{1}{\left(\frac{\partial x}{\partial x}\right)_y} + \underset{1}{(f_y)}\underset{0}{\left(\frac{\partial y}{\partial x}\right)_y} + \underset{2}{(f_z)}\underset{2}{\left(\frac{\partial z}{\partial x}\right)_y}$$
$= \boxed{5}$

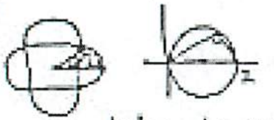
9. 

$$\int_0^3\int_{x^2}^9 xe^{-y^2}\,dy\,dx = \int_0^9\int_0^{\sqrt y} xe^{-y^2}\,dx\,dy$$

Inner: $\frac12 x^2e^{-y^2}\Big]_0^{\sqrt y} = \frac12 ye^{-y^2}$

Outer: $-\frac{e^{-y^2}}{4}\Big]_0^9 = \frac14\left[1-e^{-81}\right]$

[10]



Circle is $r = 2\cos\theta$

Integrate over 1/8 region:

$$8\int_0^{\pi/2}\int_0^{2\cos\theta} r^2 \cdot r\,dr\,d\theta \qquad \left[\text{or: } 4\int_{-\pi/4}^{\pi/4}\int \cdots\right]$$

[11] $\oint P\,dy - Q\,dx \quad [\text{or: } \oint -Q\,dx + P\,dy]$

b) By Green's Thm: above

$$= \iint_R (P_x + Q_y)\,dx\,dy = \iint_R (a+b)\,dx\,dy$$

$$= \text{area of } R \iff \boxed{a+b=1}$$

[12]

$$F = G\iiint \frac{\cos\varphi}{\rho^2}\cdot\delta\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$\delta = z = \rho\cos\varphi$$

$$\therefore F = G\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho\cos^2\varphi\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= G\cdot 2\pi\cdot\int_0^{\pi/2}\cos^2\varphi\sin\varphi\,d\varphi\cdot\int_0^1\rho\,d\rho$$

$$= G\cdot 2\pi\cdot \left.\frac{-\cos^3\varphi}{3}\right]_0^{\pi/2}\cdot\left.\tfrac{1}{2}\rho^2\right]_0^1$$

$$= 2\pi G\cdot\frac{1}{3}\cdot\frac{1}{2} = \boxed{\frac{\pi G}{3}}$$

[13] Line from $P:(1,1,1)$ to $Q:(2,4,8)$

is: $x = 1+t,\quad y = 1+3t,\quad z = 1+7t$

(since $\vec{PQ} = \langle 1,3,7\rangle$. $\quad 0\le t\le 1$

$$\therefore \int_C (y-x)\,dx + (y-z)\,dz = \int_0^1 2t\,dt - 4\cdot 7t\,dt$$

$$= \int_0^1 -26t\,dt = \left.-13t^2\right]_0^1 = \boxed{-13}$$

[14] a) $\vec F = \langle ay^2,\ 2yx+2yz,\ by^2+z^2\rangle$

Test: $2ay = 2y \quad\therefore a=1$
$2y = 2by \quad\therefore b=1$
$0=0$ ✓

b) By any method, $f(x,y,z) = \boxed{xy^2 + y^2z + \frac{z^3}{3}}$

c) Any surface S: (contour surface of f) $\boxed{xy^2+y^2z+\frac{z^3}{3} = C}$

[15]

$$\iint_S \vec F\cdot d\vec S = \iiint_V \operatorname{div}\vec F\,dV$$

$$\therefore \iint_D \vec F\cdot d\vec S + \iint_U \vec F\cdot d\vec S = \iiint_V 3\,dV = 3\cdot\text{vol } V.$$

$$\text{Volume } V = \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta = 2\pi\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 = \pi/2$$

$\iint_D = 0$ since $\vec F\cdot\hat n = z = 0$ on xy-plane

$$\therefore \iint_U \vec F\cdot d\vec S = \boxed{3\pi/2}$$

[16] $\vec F = \langle x,y,z\rangle \qquad z = 1-x^2-y^2$

$$\hat n\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy = \langle 2x,2y,1\rangle\,dx\,dy$$

$$\vec F\cdot\hat n\,dS = (2x^2+2y^2+z)\,dx\,dy$$

$$= (x^2+y^2+1)\,dx\,dy$$

$\therefore$ flux over U is:

$$\iint (x^2+y^2+1)\,dx\,dy = \int_0^{2\pi}\int_0^1 (r^2+1)\,r\,dr\,d\theta$$

$$= 2\pi\left[\frac{r^4}{4}+\frac{r^2}{2}\right]_0^1 = 2\pi\cdot\frac{3}{4} = \boxed{\frac{3\pi}{2}}$$

[17] $\oint_C \vec F\cdot d\vec r = \iint_S \nabla\times\vec F\cdot d\vec S$

$$\vec\nabla\times\vec F = \begin{vmatrix} i & j & k\\ \partial_x & \partial_y & \partial_z\\ x^2 & y^2 & xz\end{vmatrix} = -z\,\hat j$$

The normal vector to $f(x,z)=0$ is

$$\hat n = \frac{\nabla f}{|\nabla f|} = \frac{f_x\hat\imath + f_z\hat k}{|\nabla f|}$$

$\therefore \vec\nabla\times\vec F\cdot\hat n = 0$, so $\oint_C \vec F\cdot d\vec r = 0$

[18] $\int_0^\infty\int_0^\infty e^{-x^2}e^{-y^2}\,dy\,dx$

a) $= \int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy = I\cdot I$

b) $= \int_0^{\pi/2}\int_0^\infty e^{-r^2}\cdot r\,dr\,d\theta$

$$= \pi/2\cdot\left.\frac{e^{-r^2}}{-2}\right]_0^\infty = \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4}$$

$I^2 = \pi/4, \quad\therefore I = \boxed{\sqrt{\pi}/2}$