Recitation

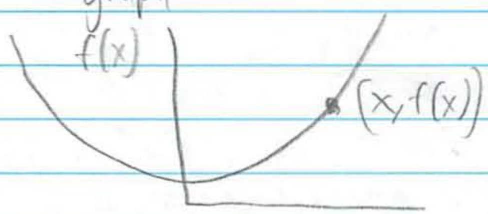
2/24

I got 66 / ~~100~~ 90 ~~some rounding stuff~~ ~~I did not do as well?~~
average 66 / 90
makeup < 55

Function of 1 variable x

$f(x) = x^2 + 1$

graph



Functions of several variables

$f(x, y) = xy + x$ ← function of variables x, y

$f(x, y, z) = xe^{x+y} + x^2 z$ ← x, y, z

Today: Domain of definition + how to represent

always tired on wed

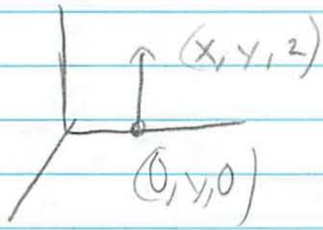
Ex 1

Find the function x, y, z giving

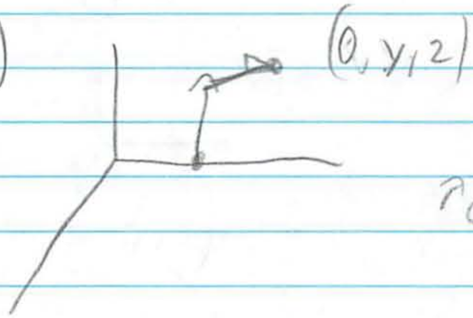
a) the distance of (x, y, z) from origin
b) from y axis
c) from y, z plane

→

a) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

b) 

$= f(x,y,z) = \sqrt{x^2+y^2}$

c) 

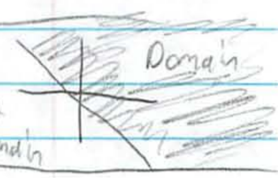
$= f(x,y,z) = |x|$

absolute value

Domain of $f(x,y)$: set of (x,y) for which $f(x,y)$ makes sense

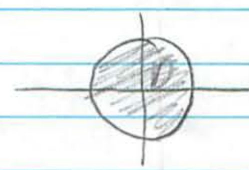
a) $f(x,y) = \frac{1}{x} + \frac{1}{y}$ $x \neq 0,\ y \neq 0$
everything except the axis

b) $f(x,y) = \sqrt{x+y}$ $x + y > 0$



c) $f(x,y) = \sqrt{1-x^2-y^2}$

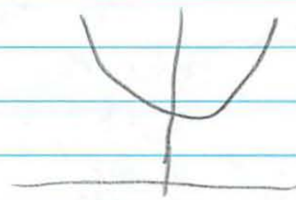
domain inside circle

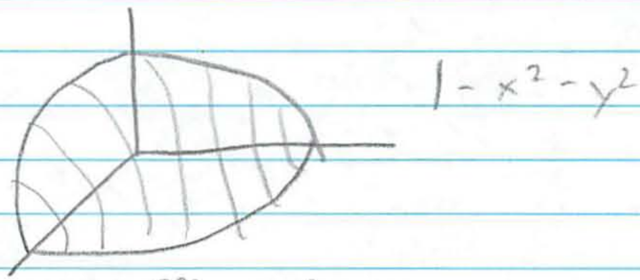


Representing $f(x, y)$

$f(x,y) = 1 - x^2 - y^2$

1. Try normal graph



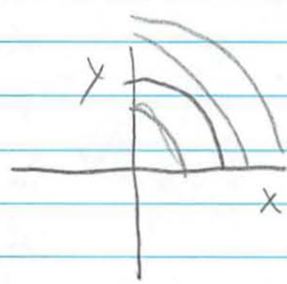


↑ 3D surface

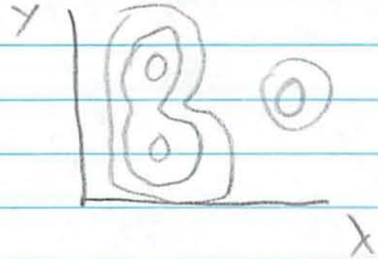
Difficult to represent

2. Level curves / equipotential lines / topo. map

- for volume c
- for set of x, y such that $f(x,y) = c$



like an architect looking at side view
but many curves as x changes



topo map
is 3 mountain peaks

Lecture 9

Partial Derivitives, Tangent Plane, Approx Formula 2/25

So apperently avg: 70/90
pass 60/90

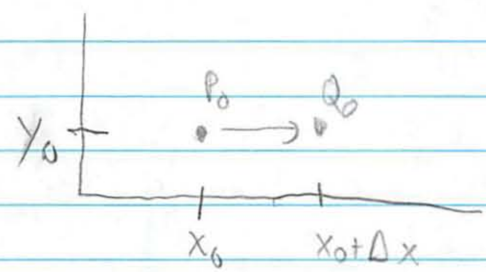
Main topic of term: functions of several variables

Today will be working in 2D (identical for nD)

$w = f(x, y)$

$w = f(x, y, z)$

temp °C at pt (x, y) in plane
(x, y, z) in space



$W_0 = W(x_0, y_0)$
$W_1 = W(x_0, y_0, z_0)$

$\Delta W = W_1 - W_0$

$$\left(\frac{\partial W}{\partial x}\right)_0 = \lim_{\Delta x \to 0} \frac{\Delta W}{\Delta X}$$

(at pt P_0)

(In this case its a 18.01 problem - only x changes)

$$= \frac{d}{dx} W(x, y_0)$$

fixed

partial function

"partial derivitive of $f(x, y)$ with respect to x at (x_0, y_0)

$$\frac{\partial w}{\partial x} = \frac{d}{dx} w(x, y_0) \quad \text{if } y \text{ fixed}$$

$$\frac{\partial w}{\partial y} = \frac{d}{dy} w(x_0, y) \quad \text{if } x \text{ fixed}$$

$$\frac{\partial}{\partial x}\left(x^4 y^2\right) = 4x^3 y^2$$

- hold y constant
- differentiate x like 18.01
- hold x constant
- differentiate y

$$\frac{\partial}{\partial y} \cos \frac{x}{y} = -\sin\left(\frac{x}{y}\right) \cdot \frac{-x}{y^2}$$

$$= \sin\left(\frac{x}{y}\right) \cdot \frac{x}{y^2}$$

2D Geometric

← level curves
- like topo map

$w = f(x, y)$

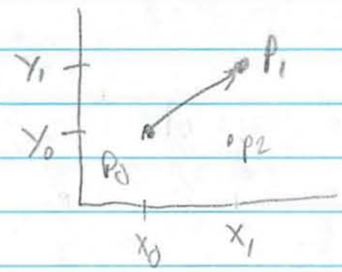
$\left(\frac{\partial w}{\partial x}\right)_0 \quad \left(\frac{\partial w}{\partial y}\right)_0$ approximate

need to define ⟵ = unit 1

$$\left(\frac{\partial W}{\partial x}\right)_0 \approx \frac{\Delta W}{\Delta X} = \frac{2-1}{1} = \frac{-1}{1}$$

$$\left(\frac{\partial W}{\partial y}\right)_0 \approx \frac{\Delta W}{\Delta Y} = \frac{3-2}{3/4} = \frac{1}{3/4} = \frac{4}{3}$$

*** Most important formula ***



By how much does temp change (ΔW)

Approx formula for ΔW from P_0 to P_1

Decompose the diagonal into horizontal + vertical line
- change in temp $\approx$ same

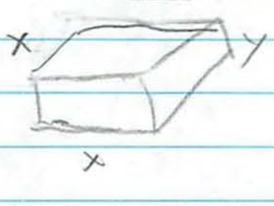
$\Delta W = \Delta W \uparrow + \Delta W \uparrow$ ↳ at P_2

$$\boxed{\Delta W \approx \left(\frac{\partial W}{\partial x}\right)_0 \Delta x + \left(\frac{\partial W}{\partial y}\right)_0 \Delta y}$$

Approx formula

Note $\left(\frac{\partial W}{\partial y}\right)_{P_2} \approx \left(\frac{\partial W}{\partial y}\right)_0$ if continuous

Now 3D



$V = x^2 y$

$x = 3m \pm 1cm$
$y = 1m \pm 1cm$

What is the possible error in volume?

total possible error

$$\Delta V \approx (2xy)_{(3,1)}\,\Delta x + (x^2)_{(3,1)}\,\Delta y$$

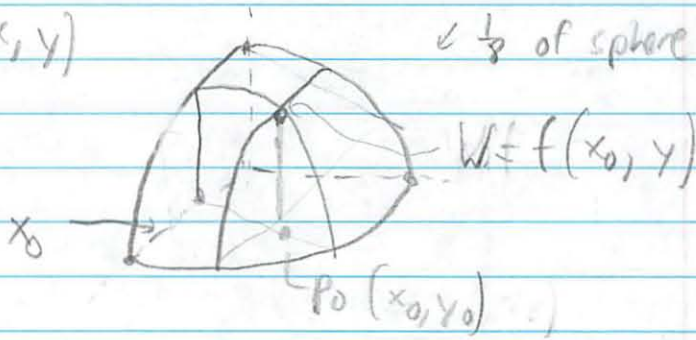
$$\approx (2(3)(1))\,\Delta x + (3^2)\,\Delta y$$

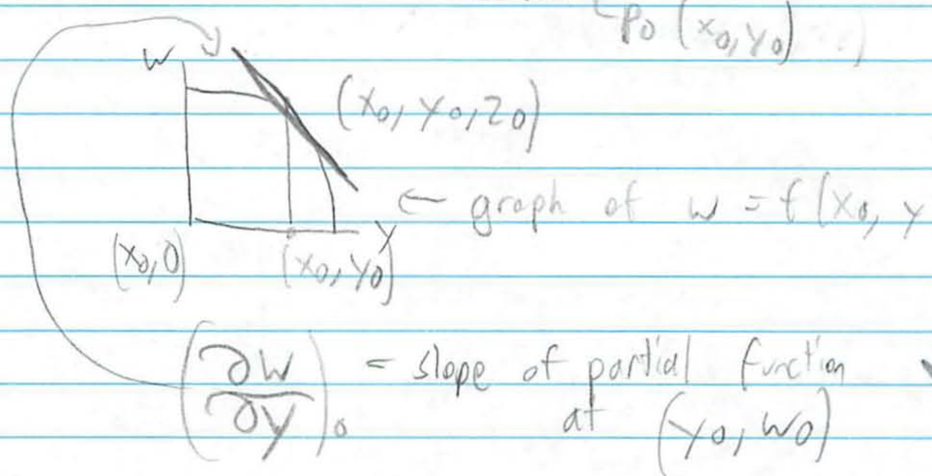
$$\approx 15 \text{ cubic cm}$$

Volume is more affected by Δy
- so pay more attention to that
- b/c it has biggest coefficent

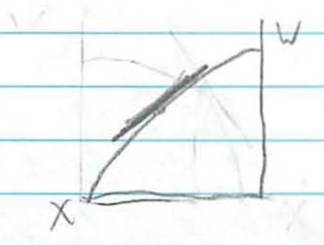
Geometric picture

$w = f(x, y)$





$\left(\frac{\partial w}{\partial y}\right)_0$ = slope of partial function at (y_0, w_0)



$\left(\frac{\partial w}{\partial x}\right)_0$ = slope

Alt method

tangent plane to graph $w = f(x,y)$ at (x_0, y_0, z_0)

= plane containing two tangents to the partial functions

$$w - w_0 = A(x - x_0) + B(y - y_0)$$

$$\qquad\quad \uparrow \qquad\qquad\qquad \uparrow$$

$$\qquad\quad \left(\frac{\partial w}{\partial x}\right)_x \qquad\quad \left(\frac{\partial w}{\partial y}\right)_0$$

$$\Delta w \approx \Delta w = \left(\frac{\partial w}{\partial x}\right)_0 \Delta x + \left(\frac{\partial w}{\partial y}\right)_0 \Delta y$$

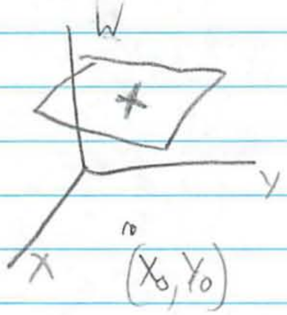
↑change in $f(x,y)$

↑change in tan plane function from $p_0 - p_1$

Lecture 10
Directional derivative, gradient

2/26

Review



plane through (x_0, y_0, w_0)

$$a(x-x_0)+b(y-y_0)+c(w-w_0)=0$$

$c \neq 0 \Rightarrow$ plane not vertical

$$\boxed{w-w_0 = A(x-x_0)+B(y-y_0)}$$

A = slope in $\hat{i}$ dir
$w-w_0=A(x-x_0)$ $\qquad y=y_0$

B = slope in $\hat{j}$ dir
$w-w_0=B(y-y_0)$ $\qquad x=x_0$

$$\boxed{w-w_0=\left(\frac{\partial w}{\partial x}\right)_0(x-x_0)+\left(\frac{\partial w}{\partial y}\right)_0(y-y_0)}$$

tangent to plane $w=f(x,y)$

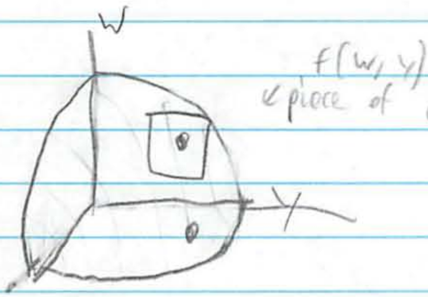
$$\boxed{\Delta w=\left(\frac{\partial w}{\partial x}\right)_0\Delta x+\left(\frac{\partial w}{\partial y}\right)_0\Delta y}$$ Approx formula

sphere $x\ y\ w$ space (radius = 3)

$\sqrt{x^2+y^2+w^2} = 3$
distance from 0

$(1,2,2)$

$w = \sqrt{9-x^2-y^2}$



tan plane at $(1,2,2)$

$$\frac{\partial w}{\partial x} = \frac{-x}{w}$$ ← same as

← other stuff is constant drops out

$$\frac{\partial w}{\partial y} = \frac{-y}{w}$$

$$w - 2 = \frac{-1}{2}(x - x_0) - \frac{2}{2}(y - y_0)$$

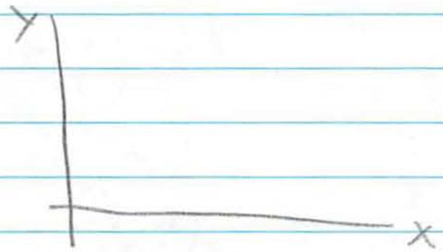
$$\Delta w = \frac{-1}{2}\Delta x - \Delta y$$

$x = 1.1$
$y = 2.1$

$$\Delta w = -\frac{1}{2}(.1) - (.1)$$

$= .15$
$w \approx 1.85$

$w = f(x, y)$

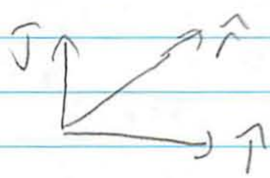


temp at (x, y) in plane at P_0

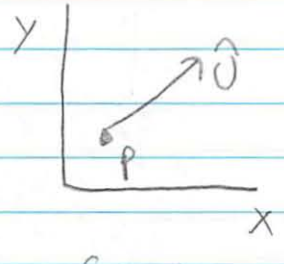
$\left(\frac{\partial w}{\partial x}\right)_0$ = rate of change of w with respect to x $\hat{\imath}$ dir

$\left(\frac{\partial w}{\partial y}\right)_0$ = rate of change of w w/ respect to y $\hat{\jmath}$ dir

But who picks $\hat{\imath}$ and $\hat{\jmath}$ direction



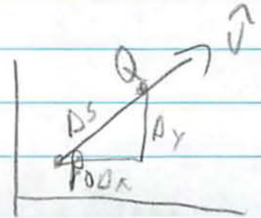
directional derivitive of w in dir $\hat{u}$



$w = w(x, y)$

$$\left(\frac{dw}{ds}\right)_{P_0, \hat{u}} = \lim_{\Delta s \to 0} \frac{\Delta w}{\Delta s}$$

$w = w(x, y)$



← approx formula

$$= \lim_{\Delta s \to 0} \left(\frac{\partial w}{\partial x}\right)_0 \frac{\Delta x}{\Delta s} + \left(\frac{\partial w}{\partial y}\right)_0 \frac{\Delta y}{\Delta s}$$

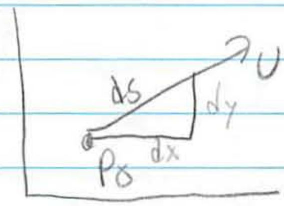
constants

$$= \left(\frac{\partial w}{\partial x}\right)_0 \cdot \frac{dx}{ds} + \left(\frac{\partial w}{\partial y}\right)_0 \cdot \frac{dy}{ds}$$

$$\boxed{= \left(\frac{dw}{ds}\right)_{0,\hat{u}} = \left(\frac{\partial w}{\partial x}\right)_0 \frac{dx}{ds} + \left(\frac{\partial w}{\partial y}\right)_0 \cdot \frac{dy}{ds}}$$

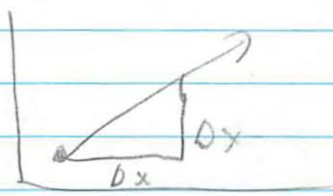
↑ took dot product of 2 vectors

$$\left(\frac{dw}{ds}\right)_{0,\hat{u}} = \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right\rangle_0 \cdot \left\langle \frac{dx}{ds}, \frac{dy}{ds} \right\rangle$$



$$= \left\langle \frac{dx}{ds}, \frac{dy}{ds} \right\rangle$$

$$= 1$$



$$= \left\langle \frac{\Delta x}{\Delta s}, \frac{\Delta y}{\Delta s} \right\rangle \quad \text{as } \Delta s \to 0$$

$$= 1$$

← the unit vector you started with

$$\left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right\rangle_0 \cdot \left\langle \frac{dx}{ds}, \frac{dy}{ds} \right\rangle$$

gradiant of w $\hat{u}$

$\vec{\nabla} w$ $\quad$ $\vec{\text{grad}}\, w$

del

nabla

- does not use a coord system
- just dir ant is traveling
- but left side in terms of $\hat{\imath}$ and $\hat{\jmath}$?
- what is the invariant vector

Meaning of gradiant $\vec{\nabla} w$

direction?

magnitude?

$w = w(x, y)$

<u>direction</u> $(\vec{\nabla} w)_0$

$\perp$ level curve through P_0

unit tangent vector to curve at P_0

$$= \boxed{\vec{\nabla} w \cdot \hat{u} = \left(\frac{dw}{ds}\right)_{0, \vec{A}}}$$

where $\hat{u} = \text{dir}\, \vec{A} = \frac{\vec{A}}{|\vec{A}|}$

because its a level curve

walking along curve - no change

<u>magnitude</u> of $\vec{\nabla} w$

$$\left(\frac{dw}{ds}\right)_{0, \vec{\nabla} w} = \vec{\nabla} w \cdot \frac{\vec{\nabla} w}{|\vec{\nabla} w|}$$

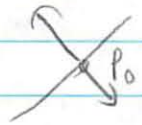
$\vec{A} \cdot \vec{A}$ = square of lenght

$$= |\vec{\nabla} w|$$

read backwards

$$|\vec{\nabla} w| = \left(\frac{dw}{ds}\right)_{0, \vec{\nabla} w}$$

both ≥ 0

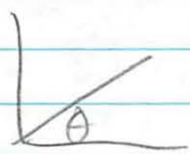


the direction you choose is the $\oplus$ one

So $\boxed{(\vec{\nabla} w)_0 \perp \text{ level curve through } P_0 \text{ in } \oplus \text{ dir}}$

Skeleton of calculation

$\underset{\text{polar angle}}{\theta} = \tan^{-1}\left(\frac{y}{x}\right)$



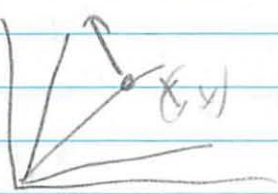
$\vec{\nabla} w = ?$

- be patient

$\vec{\nabla} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

Check that compatable

level curves are the rays
$\theta = \text{Constant}$



which way is θ increasing

vector perpendicular

$\langle x, y \rangle \perp \langle -y, x \rangle$ So it works out

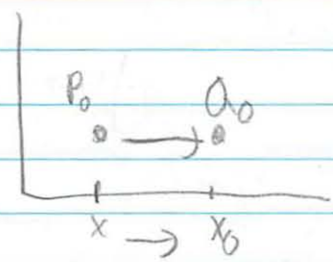
magnitude $= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$

Lectures 9 + 10
Review

2/28

working in 2D
- same in ND

have a temp at a pt in space



$$\left(\frac{\partial w}{\partial x}\right)_0 = \lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x}$$

at that pt

derivitive

- So basically keep the rest constant
while taking deriv of oter 1

pratically very simple
- math describing it goofy as always
(don't be math major)

ask yourself - how much does temp change
topograph map

$\Delta w = \Delta w \uparrow + \Delta w \uparrow$

at P_0

what does that mean?
but = if continueas
So no pratical implications??

Then imagine it in 3D
- slope of each section

Lecture 10

$$W - W_0 = A(x - x_0) + B(y - y_0)$$

- so basically same thing??

$$\boxed{\text{Approx formula} \quad \Delta W = \left(\frac{\partial W}{\partial x}\right)_0 \Delta x + \left(\frac{\partial W}{\partial y}\right)_0 \Delta y}$$

but another way to write it

$$\sqrt{x^2 + y^2 + w^2} = 3$$

↑ radius

$$w = \sqrt{9 - x^2 - y^2}$$

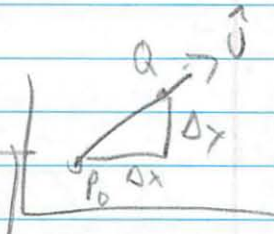
$$\frac{\partial w}{\partial x} = \frac{-x}{w} \qquad \frac{\partial w}{\partial y} = \frac{-y}{w}$$

$$w - 2 = -\frac{1}{2}(x - x_0) - \frac{2}{2}(y - y_0)$$

but problem with $\hat{\imath}$ $\hat{\jmath}$ is that who picks what is what

So directional derivitive of w in dir $\hat{u}$

$$\left(\frac{dw}{ds}\right)_{P_0, \hat{u}} = \lim_{\Delta s \to 0} \frac{\Delta w}{\Delta s}$$



$$\boxed{\left(\frac{dw}{ds}\right)_{P_0 \hat{u}} = \left(\frac{\partial w}{\partial x}\right)_0 \frac{dx}{ds} + \left(\frac{\partial w}{\partial y}\right)_0 \frac{dy}{ds}}$$

$$\left(\frac{dw}{ds}\right)_{0,\hat{u}} = \underbrace{\left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right\rangle}_{\vec{\nabla} w} \cdot \underbrace{\left\langle \frac{dx}{ds}, \frac{dy}{ds} \right\rangle}_{\hat{u}}$$

$$= \left\langle \frac{\Delta x}{\Delta s}, \frac{\Delta x}{\Delta s} \right\rangle = 1$$

↑ same vector started with

$$= \vec{\nabla} w \cdot \hat{u}$$

$$= \vec{\nabla} w \cdot \frac{\vec{\nabla} w}{|\nabla w|}$$

$\vec{\nabla} w \perp$ level curve through P_0

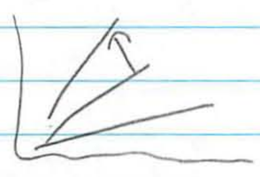
↑ topo map

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\vec{\nabla} = -\left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

- level curves are the rays

θ constant



magnitude $= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$

TA Course Notes
Reading

2/28

$w = f(x, y)$

↳ fix $y = y_0$ get $w = f(x, y_0)$
slope at pt P $x = x_0$

$$\frac{\partial f}{\partial x}_{(x_0, y_0)}$$

Can calc w/ tangent plane or approx formula

- Section not helpful in showing what f_{xy} or f_{ww} means.

Recitation

3/1

Ex 0 $\quad f(x,y) \qquad P_0 = (x_0, y_0)$

	numbers	vectors
$f(P_0)$	✓	
$\left(\frac{\partial f}{\partial x}\right)_{P_0}$	✓	
$\nabla f(P_0)$		✓
$\left(\frac{\partial f}{\partial x}\right)_{P_0, \vec{v}}$	✓	

2 coords the 2 partial derivs

ex1 Calculate partial derivitive

a. $f(x,y) = \frac{y}{x}$

b $g(x,y) = \sin(3x+2y)$

a $\frac{\partial f}{\partial x} = y \cdot -1 x^{-2} = -\frac{y}{x^2}$ ✓

$\frac{\partial f}{\partial y} = \frac{1}{x}$ ✓

b $\frac{\partial f}{\partial x} = \cancel{\cos(3x+2y) + (3+\underset{\text{constant}}{2y})}$ chain rule $3\cos(3x+2y)$

$\frac{\partial f}{\partial y} = \cancel{\cos(3x+2y) + (\underset{\text{constant}}{3x}+2)}$ chain rule $2\cos(3x+2y)$

* do chain rule correctly

ex 2
2B-7

Assume $x = 3$ with certain error 0.01
$y = 4$

a With what accuracy can the polar coordinate (r, θ) be calculated? Is it more sensitive on x or y edge

$r = \sqrt{x^2 + y^2}$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$x = 3 + \Delta x$ with $|\Delta x| \le 0.01$

$$r(x,y) \approx r(3,4) + \underbrace{\frac{\partial r}{\partial x}(3,4)\,\Delta x + \frac{\partial r}{\partial y}(3,4)\,\Delta y}_{\Delta r}$$

$$\sqrt{x^2+y^2} = \sqrt{3^2+4^2} + \frac{x}{\sqrt{x^2+y^2}}\Delta x + \frac{y}{\sqrt{x^2+y^2}}\Delta y$$

$$\frac{d}{dx}\ (x^2+y^2)^{1/2}$$
$$\tfrac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x$$

I actually pretty much solved it

$$\sqrt{x^2+y^2} = 5 + \frac{3}{\sqrt{3^2+4^2}}\Delta x + \frac{4}{\sqrt{3^2+4^2}}\Delta y$$

$$\sqrt{x^2+y^2} = 5 + \frac{3}{5}\Delta x + \frac{4}{5}\Delta y$$

Know $|\Delta r| \le \left|\frac{3}{5}\Delta x + \frac{4}{5}\Delta y\right| \le \frac{3}{5}|\Delta x| + \frac{4}{5}|\Delta y|$
is at most this value

$\le .006 \cdot .001 + .08 \cdot .001 = .014$

haven book
ML pay
ML printing
laminate

$$\theta(x,y) = \theta(3,4) + \frac{\partial\theta}{\partial x}(3,4)\Delta x + \frac{\partial\theta}{\partial y}(3,4)\Delta y$$

Recall

$(\tan(x))' = \frac{1}{1+x^2}$

keep the y constant

$$\frac{d}{dx}\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot \frac{-y}{x^2} = \frac{-y}{\left(1+\left(\frac{y}{x}\right)^2\right)x^2} = \frac{-y}{x^2+y^2}$$

distribute / simplify

$$\frac{d}{dy}\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{1}{\left(1+\left(\frac{y}{x}\right)^2\right)x} = \frac{x}{x^2+y^2}$$

keep the constant

multiply by x^2

$$\frac{1}{x}\cdot\frac{x^2}{x^2+y^2}$$

$$\frac{x}{x^2+y^2}$$

$$\frac{\partial\theta}{\partial x}(3,4) = \frac{-4}{3^2+4^2} = \frac{-4}{25}$$

$$\frac{\partial\theta}{\partial y}(3,4) = \frac{3}{3^2+4^2} = \frac{3}{25}$$

$$\Delta\theta = \frac{-4}{25}\Delta x + \frac{3}{25}\Delta y$$

$$\Delta\theta \le \tfrac{4}{25}|\Delta x| + \tfrac{3}{25}|\Delta x| \le \tfrac{4}{25}\cdot .001 + \tfrac{3}{25}\cdot .001$$
$$= .0028$$

Error is more sensitive on r to error on y
θ to error on x

$$\frac{\partial r}{\partial y} > \frac{\partial r}{\partial x} \qquad \frac{\partial\theta}{\partial y} < \frac{\partial\theta}{\partial x}$$

Ex 3 $\quad w = \sqrt{x^2 + y^2}$

a) Find equation of tangent plane at $(0, 1, 1)$

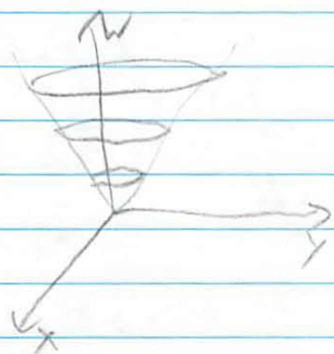
$\sqrt{0^2 + 1^2}$ is on plane

This day is easier since did P-set

b) More generally, find the tangent plane at $A = (a, b, c)$

$c = \sqrt{a^2 + b^2}$

c) Show that the line OA is included in the tangent plane.



Formula for tangent plane at $P_0 = (x_0, y_0, z_0)$

$$w - w_0 = \frac{\partial w}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial w}{\partial y}(x_0, y_0)(y - y_0)$$

$$\frac{\partial w}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x_0}{\sqrt{x_0^2 + y_0^2}} = \frac{0}{\sqrt{0^2 + 1^2}}$$

$$\frac{\partial w}{\partial y} = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2y = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y_0}{\sqrt{x_0^2 + y_0^2}} = \frac{1}{\sqrt{0^2 + 1^2}}$$

Equation of plane

$$w - 1 = 0(x - 0) + 1(y - 1)$$

$$w = y - 1 + 1$$

simplify

$$\boxed{w = y}$$

b Now more generally at (a,b,c)

$$\frac{a}{\sqrt{a^2+b^2}} \qquad \frac{b}{\sqrt{a^2+b^2}}$$

$$w-\sqrt{a^2+b^2} = \frac{a}{\sqrt{a^2+b^2}}(x-a) + \frac{b}{\sqrt{a^2+b^2}}(y-b)$$

$$w-c = \frac{a}{c}(x-a) + \frac{b}{c}(y-b)$$

simplify

$$cw - c^2 = ax - a^2 + by - b^2$$

$$\boxed{cw = ax + by}$$

can check $1w = 0x + 1y$

$w = y$ ✓

c We know O is in the plane

$0w = 0x + 0y$

$0 = 0$

We know A is in the plane

So we know OA is in the plane

Lecture 11

3D gradient, level surfaces, tan planes 3/2

$w = f(x,y)$

$$\boxed{\vec{\nabla} w = \langle w_x, w_y \rangle}$$

gradiant vector $= \langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \rangle$

$$\boxed{\left.\frac{dw}{ds}\right|_{P_0,\hat{u}} = \vec{\nabla} w \cdot \hat{u}}$$

directional derivitive

$$\left.\frac{dw}{ds}\right|_{P_0\vec{A}} = \vec{\nabla} w \cdot \frac{\vec{A}}{|\vec{A}|}$$

Review

temp in a place

dir $\vec{\nabla} w \perp$ to level curve direction

increasing w

$$|\vec{\nabla} w| = \left.\frac{dw}{ds}\right|_{P_0,\vec{\nabla} w}$$

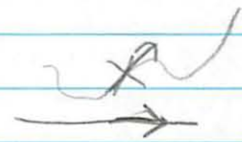
← lenght i.e. directional derivitive

$$\approx \left.\frac{\Delta w}{\Delta s}\right|_{\vec{\nabla} w} \approx \frac{1}{1/2} = 2$$

$|\vec{\nabla} w|$ big ~~if level curves are~~ when level curves are close

small when level curves are far apart

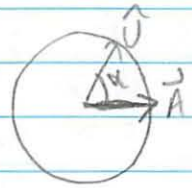
* gradient is on plane *
not on the mountain



→ represents all possible directions at that point

$\vec{A} \cdot \hat{u}$ when is it at max?
^varries   when $\hat{u}$ is parrallel and same direction as $\vec{A}$

max $\hat{u} = \text{dir}\,\vec{A}$
0 $\vec{u} \perp \vec{A}$
min $\hat{u} = -\text{dir}\,\vec{A}$



$\cos\alpha$ max when $\alpha = 0$
min $\alpha = \pi$
0 $\pi/2$

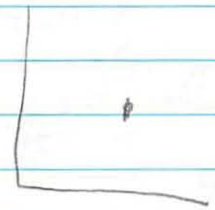
$\vec{A} \cdot \vec{u}$ = scalar component of $\vec{A}$ in the direction $\hat{u}$

So how does this matter for gradient?

$\left.\frac{dw}{ds}\right|_{\hat{u}}$

max $\hat{u} = \text{dir}(\vec{\nabla} w)$
0 $\hat{u} \perp \vec{\nabla} w$
min $\hat{u} = -\text{dir}(\vec{\nabla} w)$

$W = x^2 y$ at $(1,1)$



Which way and how far does it go to raise temp 2°

- So walk in dir $\vec{\nabla} r$ - but how far to go?

$\vec{\nabla W} = \langle 2xy, x^2 \rangle_{(1,1)} = \langle 2, 1 \rangle$

↑ direction

$|\vec{\nabla W}| = \sqrt{5}$

$\left|\frac{dw}{ds}\right|_{\nabla W} = \sqrt{5}$

$\wr\wr \qquad \wr\wr$

diff pieces to put together

$\left|\frac{\Delta w}{\Delta s}\right|_{\vec{\nabla W}} = \frac{2}{\Delta s}$

$\frac{2}{\Delta s} \approx \sqrt{5}$

$\Delta s \approx \frac{2}{2.2} = .9$

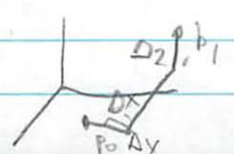
In 3-Space

$w = f(x, y, z)$

can still use temp instead of ant → flea

$\Delta w \approx w_x \Delta x + \dots + w_z \Delta z$

- same reasoning as 2D



$$\frac{dw}{ds} = w_x \frac{dx}{ds} + \dots + w_z \frac{dz}{ds} = \vec{\nabla} w \cdot \hat{u}$$

$$\vec{\nabla w} = \langle w_x, w_y, w_z \rangle$$

$$\boxed{\left.\frac{dw}{ds}\right|_{\hat{u}} = \vec{\nabla} w \cdot \hat{u}}$$ still works

All of the rest of it is the same too

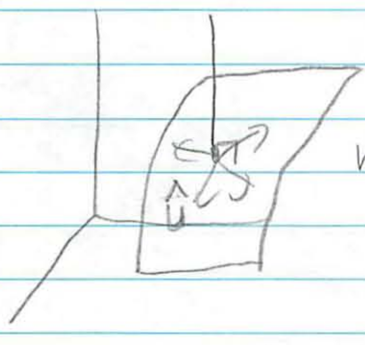
$\text{dir}(\vec{\nabla} w)_0 \perp$ level surface through P_0 ← (not curves but surfaces)

↑ means P_0

$$|\vec{\nabla w}| = \left.\frac{dw}{ds}\right|_{P_0, \vec{\nabla w}}$$

$$\vec{\nabla} w \perp \hat{u} \Rightarrow \vec{\nabla} w \cdot \hat{u} = 0$$

└ the unit vectors that lie along level surface



$w = c$ level surface for $w = f(x, y, z)$

must be perpendicular to the little tangent vectors at that point

↓

which $\hat{u}$ is the dir derivative 0?

↗ the ones tangent to level surface

The $\vec{v}$ which are tangent to level surface

Alternative calculation of tangent plane to a surface at $(1,2,1)$

$x^2+y^2+2z^2$ — ellipsoid
$= 1^2+2^2+2(1)^2$
$= 7$

Find tangent plane at $(1,2,1)$

This is a level surface of $w = x^2+y^2+z^2$
→ need the normal vector to that surface

↳ $(\vec{\nabla} w)_{(1,2,1)}$ is the normal vector

$\vec{\nabla} w = \langle 2x, 2y, 4z \rangle$

$= \langle 2, 4, 4 \rangle \perp$ to surface
$= 2\langle 1, 2, 2 \rangle$
normal vector

So tan plane

$1(x-1) + 2(y-2) + 2(z-2) = 0$

expand

Recitation

ex 1 $\quad f(x,y,z) = \ln(x^2+y^2+z^2)$

$$\left\langle \frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2} \right\rangle$$

take der right
simplify
$$\frac{2\langle x,y,z\rangle}{x^2+y^2+z^2}$$

a. Find value + gradiant at $(1,2,-2)$

value $= \ln(1^2+2^2+(-2)^2)$

$$\vec{\nabla w} = \left\langle \frac{2}{9}, \frac{4}{9}, \frac{-4}{9} \right\rangle \text{ or } \frac{2}{9}\langle 1,2,-2\rangle$$

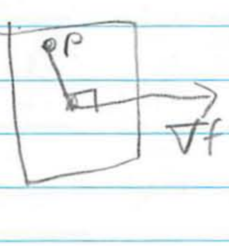
b. Find tangent plane equation of level curve $f(x,y,z) = \ln(4)$

level curve value - extraneous

∇f is $\perp$ to the level curve
- So it is the tangent plane

So want $\perp$ $\frac{2}{9}\langle 1,2,-2\rangle$ and goes through $\langle 1,2,-2\rangle$

can drop



$$1(x-1)+2(x-2)-2(x+2)=0$$

$$\frac{2}{9}\langle 1,2,-2\rangle \cdot \langle x-1, y-2, z+2\rangle = 0$$

right we are doing dot product to find normal vector

ah ha!

c Find the directional derivitive of f at P in dir $\hat{i} + 3\hat{j} - \hat{k}$

$$\left(\frac{df}{ds}\right)_{P,\hat{u}} = \nabla f \cdot \hat{u}$$

rate of change of f if one goes at speed 1 in the direction u

$$\hat{u} = dir(A) = \frac{\vec{A}}{|\vec{A}|} = \frac{\langle 1, 3, -1 \rangle}{\sqrt{1^2+3^2+1^2}} = \frac{\langle 1, 3, -1 \rangle}{\sqrt{11}}$$

$$\left(\frac{df}{ds}\right)_{P,\vec{u}} = \nabla f \cdot \vec{u} = \frac{2}{9}\langle 1, 2, -2 \rangle \cdot \frac{1}{\sqrt{11}}\langle 1, 3, -1 \rangle =$$

$$\frac{2}{9\sqrt{11}}(1+6+2) = \frac{2}{\sqrt{11}}$$

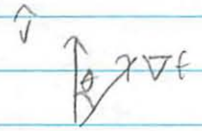
now do dot product

get this better now

d What are the min/max of directional deriv at P, What direction are they?

want to max $\frac{df}{ds}$?

max $\nabla f \cdot \vec{u}$ over all unit vector choices of $\hat{u}$



maximize when $\hat{u}$ in dir (∇f) $\quad |\nabla f||\vec{u}|\cos\theta \quad [-1,1]$

$$\hat{u} = \frac{\nabla f}{|\nabla f|}$$

has value $\nabla f \cdot \frac{\nabla f}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|$

min if $\hat{u} = -\frac{\Delta f}{|\Delta f|}$

has value $-|\nabla f|$

$-\frac{2}{9}\sqrt{9}$

So $|\nabla f| = \frac{2}{9}|\langle 1, 2, -2 \rangle|$

$= \frac{2}{9}\sqrt{9}$

do manually

on calc will be weird decimal

e In which direction should one travel to get largest increase of f at p
What is the rate of increase if one goes in this dir at speed = 7,

To increase f, one should go in dir $\frac{\nabla f}{|\nabla f|}$

The rate of change of f for speed 7 in that dir is

$$7\left(\frac{\partial f}{\partial s}\right)_{P,\hat{v}} = 7 \cdot \frac{2}{9}\sqrt{A}$$

$\hat{v} = \frac{\nabla f}{|\nabla f|}$

we found dir in last problem

ex 2 Let S be a surface given by $x^2y^2 + y + z = 0$

Let C be a parametric curve $\vec{r}(t) = \langle 2t, \ldots \rangle$

Is C included in S (on surface)

- plug coords into equation
- must satisfy for all t

18.02 Problem Set 3B DUE, STAPLED WITH 3A, ON THU. MAR. 4, 10:45 2-106

Part I (15 points)

Recit. Wed. Feb. 24 Functions of several variables: examples, graphs, contour curves and level curves.
Read 19.1 Work: ~~2A-1be~~

Lecture 9. Thurs. Feb. 25 Partial derivatives; tangent plane, approximation formula
Read: 19.2, Notes TA Work: ~~2A-2ae, 3b~~, 5b; ~~2B-1b~~, ~~4, 6, 9~~

Lecture 10. Fri. Feb. 26 Directional derivative; gradient.
Read: 19.5 pp. 681-top 683 (lecture and exercises will use only two dimensions)
Work: ~~2D-1aed, 2a, 4, 7, 9bcde~~ (describe in words where B and C are)

Lecture 11. Tues. Mar.2 3D gradient: level surfaces; tan. planes.
Read: finish 19.5. Work: ~~2D-1b~~, 2b, ~~3a, 5bc, 8;~~

Lecture 12. Thurs. Mar. 4 Max-min problems. Least squares approximation.
Read 19.7 to bottom p.693; Notes LS

Part II (25 points)

Directions. Try each problem alone for 15 minutes. If you subsequently collaborate, solutions must be written up independently. It is illegal to consult problem sets from previous semesters. With each problem is the day it can be done.

Problem 1. (Wed., 9 pts: 4,3,2) This problem introduces you to MatLab's 3D plots of a surface, and the corresponding plot of its level curves. Do it all at the computer, using the sheet of general MatLab directions given out with this problem set. You will use the sections: Directions for 3D Graphs and Special MatLab Operators.

a) Plot the surface $z = x^2 - y^2$; adjust the viewpoint and the scale so that all the interesting features are best displayed. Include also a reasonable number of contour curves on the graph. Then print it out. (See Note at the end of this problem.)

b) Plot $f(x, y) = xy\, e^{x^2-y^2}$; follow the same instructions as in (a).

c) For the function in part (b), also make a 2D contour plot, using the same number of level curves on the 2D plot as you used for the contour curves on the 3D graph, and print that out also.

Note: In entering the functions, use parentheses freely, use * for multiplication, and use exp for the exponential function; don't forget the . for the array operation (see the directions for 3D Graphs).

Problem 2. (Thurs. 4 pts: 2,2). The surface $z = x^2 - y^2$ is a saddle-shaped surface, but even though it is curved everywhere, it contains two families of lines, each of which fills out the surface completely. (This is not so evident from the MatLab plot.)

To put it another way, two lines – one from each family – go through every $P_0 : (x_0, y_0, z_0)$ on the surface. They span the tangent plane at P_0, and are the intersection of the surface with the tangent plane.

Verify these facts for the point $P:(3,2,5)$ on the surface, as follows.

a) Write the parametric equations for a general non-horizontal line through P: i.e., one passing through P and parallel to the vector $a\,\mathbf{i} + b\,\mathbf{j} + \mathbf{k}$.

Then show that there are exactly two sets of values for the pair (a,b) for which the resulting line lies entirely in the surface. (The line will lie in the surface if every point on the line satisfies the equation of the surface.)

b) Find the equation of the tangent plane to the surface at P.

Then show that both of the lines you found in (a) also lie in this tangent plane. (Use the same method as in part (a).)

Problem 3. (Thurs. 3 pts.) Work 2B-8. (Use the law of cosines.)

Problem 4. (Fri., 5 pts.: 2,1,1,1)

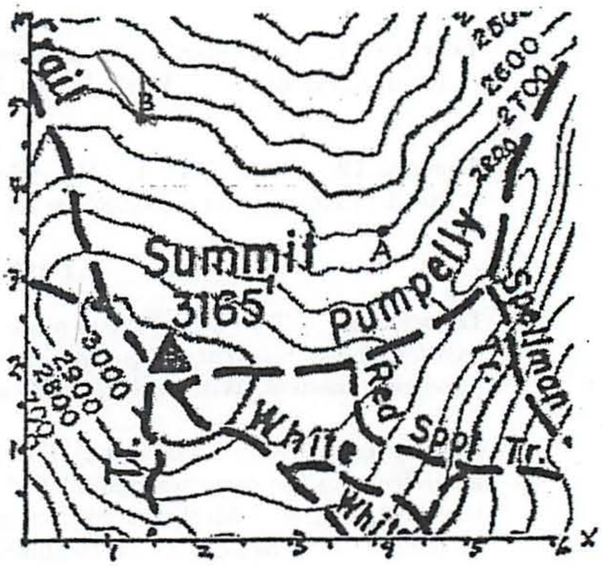
The map shows a portion of the region around the summit of Mt. Monadnock, in southern N.H.. The level curves indicate height h above sea-level, in feet. Use as the horizontal scale: $1/4'' = 500$ feet, so the x and y scales are marked in kilofeet. In your answers, identify points by their approximate coordinates.

a) Estimate h_x and h_y at the point B.

b) Estimate dh/ds at B in the direction of the vector $\mathbf{j} - \mathbf{i}$.

c) Find the points P and Q having the smallest y-coordinates such that $h(P) = h(Q) = 3000$, and $h_x(P) = 0$, $h_y(Q) = 0$.

d) Find the R with the largest x-coordinate such that $h(R) = 2900$, and $dh/ds = 0$ at R in the direction $\mathbf{i} + \mathbf{j}$.



Problem 5. (Tues. 4 pts) In the first octant of 3-space, the equation $xyz = a$, for a fixed constant $a > 0$, has as its graph a surface called a *hyperboloid*; this surface has the three coordinate planes as its asymptotes, i.e., it gets closer and closer to a coordinate plane as any of the variables goes to infinity.

At any point $P_0:(x_0,y_0,z_0)$ on this surface, the tangent plane cuts off a tetrahedron from the first octant, having the origin as one vertex, and triangular portions of the three coordinate planes and the tangent plane as its four faces. As P_0 varies, the shape of this tetrahedron varies, but its volume remains constant.

Prove this, and find the volume.

(Think of $xyz = a$ as a level surface of $w = xyz$; use this to write the equation of its tangent plane at P_0, find where this plane intersects the coordinate axes, then find the volume of the resulting tetrahedron.

Volume of a tetrahedron $= \frac{1}{3}bh$, where b is the area of a face and h the length of the corresponding altitude – the line segment with one end perpendicular to the face and the other end at the opposite vertex.)

18.02 pset

3B

Michael Plasmeier 2/28

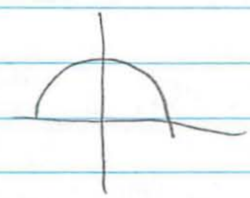
Recitation + Function of several variables

examples, graphs, contour + level curves

2A - 1b Sketch level curves for functions

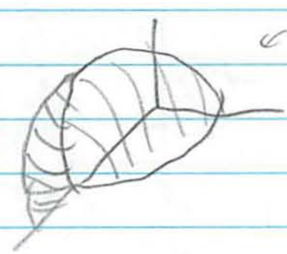
$\sqrt{x^2+y^2}$

Step 1 try normal graph



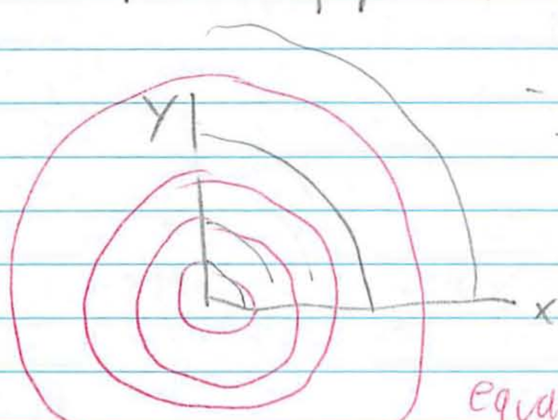
? I think

Step 2 Try 3D graph



← half sphere drawn very bad

Step 3 Equipotential curves

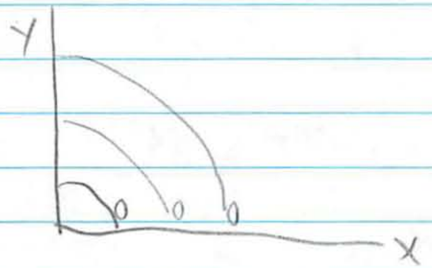


- can try points
- is there any thing eaise?

equal spaced
all 4 quadrants

2A - 1e

$x^2 - y^2$

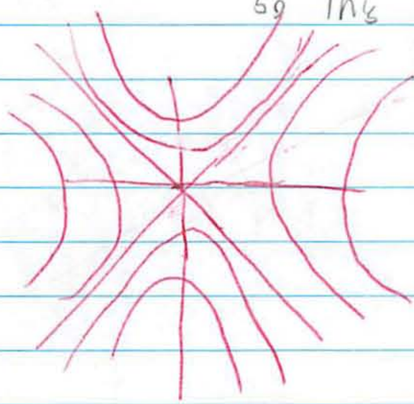


$1^2 - 1^2 = 0$

$4^2 - 4^2$

$6^2 - 2^2 = 14$

So this alternates? How to draw?



I am bad at visualizing this - need more examples

Lecture 9 Partial derivitives, tangent plane, approx formula

2A-2a Calculate partial deriv

$w = x^3 y - 3xy^2 + 2y^2$

$\frac{\partial}{\partial x}$ hold y constant differenitate x $f_x = 3x^2 y - 3 \cdot 1 y^2 + 0$

$\frac{\partial}{\partial y}$ hold x constant differenitate y $f_y \; x^3 \cdot 1 - 3x \cdot 2y + 2 \cdot 2y$

~~? now add~~

$\cancel{3x^2 y - 3y^2 + x^3 - 6xy + 4y}$

no just keep as 2 equations

e $z = x \ln(2x+y)$

$\frac{\partial}{\partial x}$ $f_x = 1 \ln(2x+y) + x \frac{1}{2x+y} \cdot 2$

← product rule ↓ Chain rule

$\frac{\partial}{\partial y} = f_y = x \frac{1}{2x+y} \cdot 1$

(4)

Cross derivitive $f_x = \frac{\partial f}{\partial x} \rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

2nd do 1st

3b Verify $f_{xy} = f_{yx}$ for each

$$\frac{x}{x+y} = x(x+y)^{-1}$$

what is $f_{xy} = (f_x)_y$

$$\frac{\partial}{\partial x} = f_x = 1 \cdot \frac{1}{x+y} + x \cdot -1(x+y)^{-2} \cdot 1$$

$$\frac{1}{x+y} - \frac{x}{(x+y)^2}$$

$$\frac{x+y}{(x+y)^2} - \frac{x}{(x+y)^2}$$

$$\frac{y}{(x+y)^2}$$

$$\frac{\partial}{\partial y} = f_y = x - (x+y)^{-2} \cdot 1$$

$$= -\frac{x}{(x+y)^2}$$

$$(f_x)_y = \cancel{\frac{y}{(x+y)^2}} \cdot \cancel{\frac{-x}{(x+y)^2}}$$ → multiply across

over

why? see back

did not do this right

$$\cancel{\frac{-xy}{(x+y)^4}}$$

They say $\frac{x-y}{(x+y)^3}$

$$(f_y)_x = \cancel{\frac{-x}{(x+y)^2}} \cdot \cancel{\frac{y}{(x+y)^2}}$$

$$\frac{-(y-x)}{(x+y)^3}$$

$$\cancel{\frac{-xy}{(x+y)^4}}$$

but why?

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

Oliver
Ott

$\frac{\partial}{\partial y} \frac{y}{(x+y)^2} = \frac{\partial}{\partial y} y(x+y)^{-2} = 1(x+y)^{-2} + y \cdot -2(x+y)^{-3} \cdot 1$

$= \frac{1}{(x+y)^2} + \frac{2y}{(x+y)^3}$

what did wrong? just simplify

$\frac{-(x+y)+2x}{(x+y)^3} = \frac{x-y}{(x+y)^3}$

so it was simplify more

$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)$

$\frac{\partial}{\partial x} \frac{-x}{(x+y)^2} = \frac{\partial}{\partial x} -x(x+y)^{2}$

$-1(x+y)^{-2} + -x \cdot -2(x+y)^{-3} \cdot 1$

$\frac{-1}{(x+y)^2} + \frac{2x}{(x+y)^3}$

$\frac{-(x+y)+2x}{(x+y)^3} = \frac{x-y}{(x+y)^3}$ ✓ Does match

5b Show $f(x,y)$ satisfies $w_{xx} + w_{yy} = 0$
— the 2d Laplace equation

did not → know about WP; something w/ diff.eq

$$w = \ln(x^2+y^2)$$

$$\frac{\partial}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2}$$

$$\frac{\partial}{\partial y} = \frac{1}{x^2+y^2} \cdot 2y = \frac{2y}{x^2+y^2}$$

w_{xx} $\qquad \frac{2(y^2-x^2)}{(x^2+y^2)^2}$

↑ what does this mean?

if interchange x and y
$w = \ln(x^2+y^2)$ remains same
↑ yeah that makes sense
w_{xx} turns into w_{yy}
since interchange just changes right hand side sign

$$w_{yy} = -w_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)$$

?

$$\frac{\partial}{\partial y}\frac{2y}{x^2+y^2} = \frac{\partial}{\partial y} 2y(x^2+y^2)^{-1} = 2(x^2+y^2)^{-1} + 2y \cdot -1(x^2+y^2)^{-2} \cdot 2y$$

$$= \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} = \frac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

→

$$-\frac{\partial}{\partial x}\frac{2x}{x^2+y^2} = -\frac{\partial}{\partial x} 2x(x^2+y^2)^{-1}$$

$$= -2(x^2+y^2)^{-1} + 2x \cdot -1 (x^2+y^2)^{-2} \cdot 2x$$

$$= \frac{-2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2}$$

$$= \frac{-2(x^2+y^2) - 4x^2}{(x^2+y^2)^2}$$

$$= \frac{2y^2 - 2x^2}{(x^2+y^2)^2}$$

$$= \frac{2x^2 - 2y^2}{(x^2+y^2)^2}$$

2B-1b Give the equation of the tangent plane at each surface indicated

$$w = \frac{y^2}{x} \quad (1,2,4)$$

$$\frac{\partial}{\partial x} = \cancel{x^2 \ln x} \quad -\frac{y^2}{x^2}$$

remember rules - don't confuse $\frac{d}{}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ & $\int$

$\int \frac{1}{x} = \ln x$

$$\frac{\partial}{\partial y} = \frac{1}{x} \cdot 2y = \frac{2y}{x}$$ ✓

? what do we do with the points -? - plug them in?

$$w - w_0 = \underset{\left(\frac{\partial w}{\partial x}\right)_0}{A}(x - x_0) + \underset{\left(\frac{\partial w}{\partial y}\right)_0}{B}(y - x_0)$$

$$w - 4 = \left(\cancel{x^2 \ln x}\ \frac{y^2}{x^2}\right)(x-1) + \left(\frac{2y}{x}\right)(y-2)$$

Close

plug pts in for FF

$$w_x = -\frac{y^2}{x^2} = -\frac{2^2}{1^2} = -4$$

$$w_y = \frac{2x}{x} = \frac{2 \cdot 2}{1} = 4$$

tangent plane

$$w = 4 - 4(x-1) + 4(y-2)$$

$$w = -4x + 4y$$

+simplify

4. The combined resistance of 2 wires in parral

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If $R_1 = 1\ \Omega$ with error $\pm .1\ \Omega$
$R_2 = 2\ \Omega$

What is R and error window?

just another way to write

$$R = \frac{R_1 R_2}{(R_1 + R_2)}$$

-remember doing 1 like this w/ box in lecture

R first off is $\frac{1}{1} + \frac{1}{2} = \frac{3}{2}\ \Omega$

?? I thought $\Omega < 1$

total possible error

$$\Delta R = \frac{\partial}{\partial x} \Delta x + \frac{\partial}{\partial y} \Delta y$$

R_1 in this case, R_2

$$\left(\frac{R_1}{R_1+R_2}\right)^2$$

but why is this diff then? (where mistake was)

$$-\frac{1}{R_1^2} + \frac{1}{R_2} \cdot (.1) + -\frac{1}{R_2^2} + \frac{1}{R_1} \cdot (.1)$$

$$\left(\frac{R_2}{R_1+R_2}\right)^2$$

$$\left(-\frac{1}{1^2} + \frac{1}{2}\right) \cdot .1 + \left(-\frac{1}{2^2} + \frac{1}{1}\right) \cdot (.1)$$

$$-.05 + .075$$

$.025\ \Omega$ max error

$$\frac{4}{9} \Delta R_1 + \frac{1}{9} \Delta R_2$$

Hypothesis: $|\Delta R_i| \leq .1$ for $i = 1,2$ so $|\Delta R| \leq \frac{4}{9}(.1) + \frac{1}{9}(.1)$
$= .06$ possible error

6. To determine volume of a cylinder

How accurately should radius + height be measured so volume error < .1

$V = \pi r^2 h$

$V = \pi 2^2 \cdot 3$

$= 12\pi \pm .1$

So basically last problem in reverse

$.1 = \frac{\partial V}{\partial r} \Delta r + \frac{\partial V}{\partial h} \Delta h$

$.1 = \pi 2rh \Delta r + \pi r^2 \Delta h$

$.1 = \pi 2(2)(3) \Delta r + \pi (2)^2 \Delta h$

$.1 = 12\pi \Delta r + 4\pi \Delta h$

r hold 0

$\Delta r = \frac{.1}{\boxed{12\pi}} = .00265$

$\Delta h = \frac{.1}{\boxed{4\pi}} = .00795$

but here

Assume same accuracy $|\Delta r| \le \varepsilon$ $|\Delta h| \le \varepsilon$ for both measurements

? but why ?

$|\Delta V| \le 12\pi\varepsilon + 4\pi\varepsilon = 16\pi\varepsilon$

is $< .1$ if $\varepsilon < \frac{1}{160\pi} < .002$ ← So basically what I found

9.a. Around the point $(1,0)$ is $w = x^2(y+1)$ ~~which~~ is more sensitive to changes in x or in y?

- So this is just like example in class

$\Delta error = 2x(y+1)\,\Delta x + x^2(y+1)\cdot 1\,\Delta y$

plug in it

$= 2\cdot 1\cdot(0+1)\,\Delta x + 1^2(0+1)\,\Delta y$

$= 2\Delta x + 1\,\Delta y$ ← not addition

$= 3$

↑ biggest in x dir

b. What is the ratio $\frac{\Delta y}{\Delta x}$ so that small changes produce no changes in w?

(ie first order only, changes a little like $(\Delta x)^2$ not Δx)

↑ ??? what? very confusing

$\frac{\Delta y}{\Delta x} = \frac{1}{2}$ or will cancle out if $-\frac{1}{2}$?

$\Delta w = 2\Delta x + \Delta y$ ← what I found above

For $\Delta w = 0 \Rightarrow 2\Delta x + \Delta y = 0$

$\frac{\Delta y}{\Delta x} = -2$

was close - but thinking about it in wrong way

- So yeah have - to cancle

- and the $2\Delta x$ should make Δx smaller

Lecture 10 Directional derivitives, gradiants (only in 2D)

2D-1a

$f = x^3 + 2y^3$

$P = (1, 1)$

$\vec{A} = \hat{i} - \hat{j}$

Calc gradiant of f at a point

Directional deriv $\left.\frac{df}{ds}\right|_u$ at the pt in the dir u of given vector A

$\vec{\nabla} w = \langle \cancel{3x^2 + 2y^3}, \cancel{x^3 + 6y^2} \rangle$

$(\cancel{3(1)^2 + 2(1)^3}, \cancel{(1)^3} + 6(1)^2)$

$(\cancel{5, 7})$

remember vector $\rightarrow 3x^2\hat{i} + 6y^2\hat{j}$

at pt $3\hat{i} + 6\hat{j}$

① $\Delta w = \Delta w\,\hat{i} + \Delta w\,\hat{j}$

② approx formula

③ plane $w - w_0 = \frac{\partial}{\partial x}(x - x_0) + \frac{\partial}{\partial x}(y - y_0)$

② $\Delta w = \frac{\partial}{\partial x}\Delta x + \frac{\partial}{\partial x}\Delta y$

I guess I never actually found gradiant.

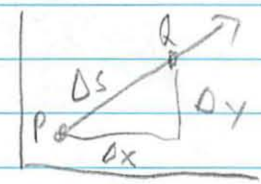
* No did right *

S constant = 0

↑

$\left.\frac{df}{ds}\right|_u$ ← what is that again. of w in dir $\hat{u}$ $= \lim_{\Delta s \to 0} \frac{\Delta w}{\Delta s}$

$w = w(x, y)$



$= \left(\frac{\partial w}{\partial x}\right)_0 \frac{dx}{ds} + \left(\frac{\partial w}{\partial y}\right)_0 \frac{dy}{ds}$

$= \underbrace{\left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right\rangle_0}_{\vec{\nabla} w} \cdot \underbrace{\left\langle \frac{dx}{ds}, \frac{dy}{ds} \right\rangle}_{\hat{u}}$

$= \langle 3, 6 \rangle \cdot \frac{\langle 1, -1 \rangle}{\sqrt{2}}$ ← is this right

? then what - dot product

$(5 \cdot 1) + (7 \cdot -1)$
-2

do dot product

$(3\hat{i} + 6\hat{j})$ $\frac{\hat{i} - \hat{j}}{\sqrt{2}}$

$\hookrightarrow -\frac{3\sqrt{2}}{2}$

Why the $\sqrt{2}$? 2

$x^2 + y^2$

~~I guess it is a~~

c $z = x \sin y + y \cos x$ $(0, \frac{\pi}{2})$ $-3\hat{i} + 4\hat{j}$

$\nabla \vec{w} = \langle 1 \sin y - y \sin x, x \cos y + \cos x \rangle$ ✓

now at the pt

$\langle \sin \frac{\pi}{2} - \frac{\pi}{2} \sin 0, 0 \cos \frac{\pi}{2} + \cos 0 \rangle$

$\langle 1 - \frac{\pi}{2} \cdot 0, 0 + 1 \rangle$

$\langle 1, 1 \rangle$ ✓ $\hat{i} + \hat{j}$ they say

now directional derivative

$\langle 1, 1 \rangle \cdot \frac{\langle -3\hat{i} + 4\hat{j} \rangle}{5}$

$\tan^{-1}\left(\frac{4}{3}\right)$

do dot product

where from? oh $\sqrt{x^2 + y^2}$

$\frac{1}{5}$

given vector its pointing in

1 $w = \ln(2t + 3v)$ $(-1, 1)$ $4\hat{\imath} - 3\hat{\jmath}$

what is $\frac{d}{dx}\ln(x)$ $= \frac{1}{x}$

$\frac{d}{dx}\ln(x^2) = \frac{1}{x}$

so how

ti $\vec{\nabla} w = \left\langle \frac{2}{2t+3v}, \frac{3}{2t+3v} \right\rangle$

at pt $\left(\frac{2}{2(-1)+3(1)}, \frac{3}{2(-1)+3(1)} \right) = (2, 3)$

$2\hat{\imath} + 3\hat{\jmath}$

$\frac{2\hat{\imath}+3\hat{\jmath}}{2t+3v}$

$2\hat{\imath} + 3\hat{\jmath} \cdot \frac{4\hat{\imath} - 3\hat{\jmath}}{5}$

$\left(2 \cdot \frac{4}{5}\right) + \left(3 \cdot -\frac{3}{5}\right)$ do dot product

$\frac{8}{5} - \frac{9}{5}$

$-\frac{1}{5}$

2a. Give max and min $\left.\frac{df}{ds}\right|_{\hat{u}}$ as $\hat{u}$ varries

so this is actually lecture 11

max $\hat{u}$ is when $\hat{u} = \mathrm{dir}(\vec{\nabla} w)$

0 $\hat{u}$ when $\perp$ $\vec{\nabla} w$

min $\hat{u}$ when $\hat{u} = -\mathrm{dir}(\vec{\nabla} w)$

$w = \ln(4x - 3y)$ at $(1,1)$

~~so we know from above~~

opps is different

$\left(\frac{4}{2(1) \cancel{+} 3(1)}, \frac{-3}{2(1) \cancel{+} 3(1)} \right) = \left(\cancel{\tfrac{2}{5}}, \cancel{\tfrac{3}{5}} \right)$ $4\hat{\imath} - 3\hat{\jmath}$

dir of a vector $= \dfrac{\vec{\nabla} w}{|\vec{\nabla} w|}$

$$\cancel{\dfrac{\langle \frac{2}{5}, \frac{3}{5} \rangle}{\sqrt{\frac{2}{5}^2 + \frac{3}{5}^2}}}$$

~~<.5547, .8320>~~

$$\frac{\langle 4, 3 \rangle}{\sqrt{4^2+3^2}} = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$$

just that

Or just

$(4\hat{\imath} - 3\hat{\jmath}) \cdot v$ has max 5

in dir $v = \dfrac{4\hat{\imath} - 3\hat{\jmath}}{5}$

4. The function $T = \ln(x^2+y^2)$ gives temp at each pt in plane except $(0,0)$

a) At $(1,2)$ what dir to most rapid ↑ in T

in dir $(\vec{\nabla} w)$

$$\vec{\nabla} w = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle = \left\langle \frac{2}{5}, \frac{4}{5} \right\rangle$$ ← mental math error

T increasing most rapidly in dir gradiant

$$\left\langle \frac{\frac{2}{5}, \frac{4}{5}}{\sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{4}{5}\right)^2}} \right.$$

Do the reduction manually

$$\sqrt{\frac{4}{25}+\frac{16}{25}} = \sqrt{\frac{20}{25}} = \sqrt{\frac{4}{5}} = \frac{2\sqrt{5}}{5}$$

$$\left\langle \frac{\frac{2}{5}}{\frac{2\sqrt{5}}{5}}, \frac{\frac{4}{5}}{\frac{2\sqrt{5}}{5}} \right\rangle$$

$$\left\langle \frac{2}{5} \cdot \frac{5}{2\sqrt{5}}, \frac{4}{5}, \frac{5}{2\sqrt{5}} \right\rangle$$

$$\left\langle \frac{10}{10\sqrt{5}}, \frac{20}{10\sqrt{5}} \right\rangle$$

$$\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

or $\frac{1\hat{\imath} + 2\hat{\jmath}}{\sqrt{5}}$

b At P how far should you go in above to get increase of .2 in T?

$$|\vec{\nabla w}| = .2$$

so just fill in Δw?? or where is this from?

$$\left|\frac{dw}{ds}\right|_{\vec{\nabla w}} = .2 = \left|\frac{\Delta w}{\Delta s}\right|_{\vec{\nabla w}} \quad \boxed{\frac{2}{\sqrt{5}}\Delta s = .20}$$

how got this?

See back

$$\Delta s = \frac{.2 \cdot \sqrt{5}}{2} = .1\sqrt{5} = \frac{\sqrt{5}}{10} \approx .22$$

simp

$\to$ other

Oliver

OH

have a pt and a dir

dir. $\hat{u}$ ← increase in that direction (rate of increase)

$$= \nabla f \cdot \hat{u} = |\nabla f|$$

$$\vec{\nabla} f = \hat{i} + 2\hat{j}$$

$$|\nabla f| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

↑ approx increase of temp if 1 unit in dir $\hat{i}+2\hat{j}$

But we want .2 increase (Δf)

$$\sqrt{5} = \frac{\Delta f}{\Delta s} \begin{matrix} \leftarrow .2 \\ \leftarrow ? \end{matrix}$$

$$\Delta s = \frac{.2}{\sqrt{5}} \approx .22$$

↑ much clearer, since not reduced

c) At (1,2) ~~how far~~ what dir ? to go in $\hat{i}+\hat{j}$ to get increase of 1.2 ?

$\vec{\nabla} w = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

$|\Delta w| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2} = 1$ What is $|\Delta s|$?

? no solutions ~~can redo now~~ ~~still don't get how all fit together~~ how about now

$1.2 = \vec{\nabla} w \cdot \hat{u}$

$1.2 = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle \langle \hat{i}, \hat{j} \rangle$

$|\Delta w| = \frac{\Delta w}{\Delta s} \rightarrow 1 = \frac{1.2}{\Delta s}$, $\Delta s = 1.2$

is that it?

d) At (1,2) how far to go in $\hat{i} - 2\hat{j} + 2k$ for increase .10

$\cancel{.1 = \frac{dw}{ds} = \frac{\Delta w}{\Delta s}} = \cancel{\frac{.1}{\Delta s}}$

$\sqrt{1^2 + 2^2 + 2^2} = 3$

~~$\Delta s = ?$~~ think I got this

ok should be it

$\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle \cdot \langle 1, -2, 2 \rangle =$

$3 = \frac{\Delta f}{\Delta s} \leftarrow .1$

$\Delta s = \frac{.1}{3} = \frac{1}{30}$

- still can't really visualize - oh is it how far to travel on graph for .1 temp change?

7. Suppose $\left.\frac{dw}{ds}\right|_{u} = 2 \qquad \left.\frac{dw}{ds}\right|_{v} = 1$ at P

where $u = \frac{\hat{i}+\hat{j}}{\sqrt{2}} \qquad v = \frac{\hat{i}-\hat{j}}{\sqrt{2}}$ Find $(\nabla w)_P$

Gradiant can be calculated knowing directional derivitives in any 2 non-parallel dir, not just $\hat{i}$ and $\hat{j}$

did we do in class

yeah just dir derivitive

At P $\vec{\nabla w} = a\hat{i} + b\hat{j}$

$a\hat{i} + b\hat{j} \cdot \frac{\hat{i}+\hat{j}}{\sqrt{2}} = 2$

$a\hat{i} + b\hat{j} \cdot \frac{\hat{i}-\hat{j}}{\sqrt{2}} = 1$

$a+b = 2\sqrt{2}$
$a-b = \sqrt{2}$

$a = \frac{3}{2}\sqrt{2} \qquad b = \frac{1}{2}\sqrt{2}$

Oh $\left(\frac{dw}{ds}\right)_{0,\hat{u}} = \left(\frac{\partial w}{\partial x}\right)\frac{dx}{ds} + \left(\frac{\partial w}{\partial y}\right)\frac{dy}{dx}$

know

but how find this

$\frac{dw}{ds} = \vec{\nabla w} \cdot \hat{u}$

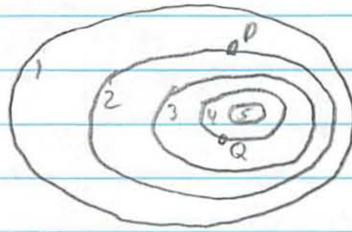
know

need — or do we have it as $\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \rangle$

don't have either

9b The picture shows level curve $w = f(x,y)$

w as marked



Find B where $w=3$ $\frac{\partial w}{\partial x} = 0$

Use $\frac{dw}{ds}\Big|_{u} \approx \frac{\Delta w}{\Delta s}$

travel in dir u from P to nearest curve
↳ perpendicular
Δs = distance traveled
Δw = distance w changes estimate

Ok, so I get it now.

Got to the curve 3
Find where in x dir it $= 0$
↳ but what is x dir so when  tangent $= 0$?

b) c) $\frac{\partial w}{\partial x} = \frac{dw}{ds}\Big|_{\hat{i}}$ $\frac{\partial w}{\partial y} = \frac{dw}{ds}\Big|_{\hat{j}}$

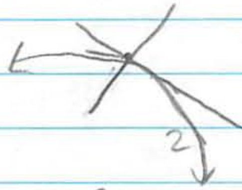
yeah
perp to $\hat{i}$
$\hat{j}$

So B is where $\hat{i}$ is tangent to level curve
C $\hat{j}$ " "

So yes ✓

C ✓

d At the point P estimate $\frac{\partial w}{\partial x}$ $\frac{\partial w}{\partial y}$



? so what am I looking for

redo

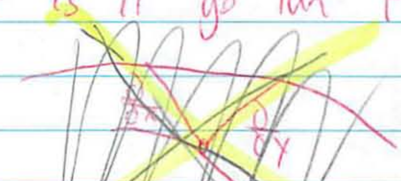
$\frac{\partial w}{\partial x} = \frac{dw}{dx}\Big|_p \approx \frac{\Delta w}{\Delta s}$; so in ↔ dir how far to a gradient change

$\frac{-1}{5/6} = -6$

$\frac{\partial w}{\partial y} = \frac{dw}{dy}\Big|_p \approx \frac{\Delta w}{\Delta s}$ so in ↕ dir how often it changes

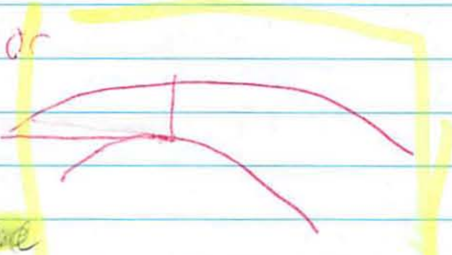
$\frac{-1}{1} = -1$

? ; or is it go tan from that pt



← ~~I think that is it yeah it has to be~~

or



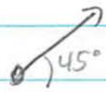
← ~~can't be~~

→

it's this one

-Oliver

e. At Q estimate $\frac{dw}{ds}$ in dir $\hat{i}+\hat{j}$



How much w change in that dir

$$\frac{1}{1/2} = 2$$

Lecture 11 3D gradiants, level surfaces, tan planes

2D-1b $w = \frac{xy}{z}$ $(2,-1,1)$ $\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

~~$\vec{\nabla} w = \langle 1, 1, z^{-2} \rangle$~~

~~at pt $= \langle 1, 1, -1 \rangle$~~

$\vec{\nabla} w = \langle \frac{y}{z}\hat{\imath} + \frac{x}{z} - \frac{xy}{z^2}\hat{k} \rangle$

$\vec{\nabla} w_{pt} = \langle -1, 2, 2 \rangle$

directional derivitive

$\langle -1, 2, 2 \rangle \cdot \dfrac{\langle 1\hat{\imath} + 2\hat{\jmath} - 2\hat{k} \rangle}{\sqrt{1^2 + 2^2 + 2^2}}$

$\langle -1, 2, 2 \rangle \cdot \frac{1}{3} \langle 1\hat{\imath} + 2\hat{\jmath} - 2\hat{k} \rangle$

$(-1 \cdot \frac{1}{3}) + (2 \cdot \frac{2}{3}) + (2 \cdot \frac{-2}{3})$

$-\frac{1}{3}\hat{\imath} + \frac{4}{3}\hat{\jmath} - \frac{4}{3}\hat{k}$

$-\frac{1}{3}$

↑ actually can put to 1 value

2b $\quad W = xy + yz + xz \qquad (1,-1,2)$

max $\hat{u}$ is $\hat{u} = \text{dir}(\vec{\nabla W})$

$\vec{W} = \langle y + 0 + z,\ x + z + 0,\ 0 + y + x \rangle$

or is it $x'y + y'x$ $\qquad$ $x'y + y'x$

$1 \cdot y + 0 \cdot x$ $\qquad$ $0y + 0x = 0$

so works out

$\langle -1 + 2,\ 1 + 2,\ -1 + 1 \rangle$

$\langle 1,\ 3,\ 0 \rangle$ ✓

$\text{dir} = \left\langle \dfrac{1, 3, 0}{\sqrt{1^2 + 3^2 + 0^2}} \right\rangle$

$\left\langle \frac{1}{10}, \frac{3}{10}, 0 \right\rangle$ ✓ $\dfrac{\hat{i} + 3\hat{j}}{10}$

now I understand this better

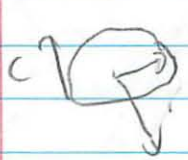
min is the $\ominus$ of that

0 is when perpendicular $\dfrac{\cancel{-3\hat{i}} + \hat{j}}{10}$

$\dfrac{-3\hat{i} + \hat{j} + c\hat{k}}{\sqrt{10^2 + c^2}}$ for all c

?? how got ?

Oliver OH



everything perp

- full cycle of possibilities

$\nabla f \cdot v = 0$

$\langle 1, 3, 0 \rangle$ ← notice horiz

↳ do the switch, paramatrise c, + −

$\dfrac{\langle -3, 1, c \rangle}{\sqrt{10^2 + c^2}}$

- Oliver had trouble finding - not a core theme of the class

3a. Find tangent pt

$xy^2z^3 = 12 \qquad (3,2,1)$

↳ need the normal vector to the plane

$\vec{\nabla w} = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$

$(\nabla w)_P = \langle 2^2 1^2, 2(3)(2)(1)^3, 3(3)(2)^2(1)^2 \rangle$
$\langle 4, 12, 36 \rangle$ ✓

So need to find normal vector

is the gradiant reduced $\langle 1, 3, 9 \rangle$

Now plug in
$3(x-1) + 2(y-3) + 1(z-9) = 0$
$3x - 3 + 2y - 6 + z - 9 = 0$
$3x + 2y + z = 18$ ✓

5b $T = x^2 + 2y^2 + 2z^2$ temp in space

At $(1,1,1)$ where for most rapid decrease?

just more practice of last section w/ extra work 3D does not do much

$\vec{\nabla w} = \langle 2x, 4y, 4z \rangle \qquad -\text{dir}(\vec{\nabla w})$

$-\text{dir}' = \dfrac{\langle 2x, 4y, 4z \rangle}{\sqrt{2^2+4^2+4^2}}$

$= -\langle \tfrac{1}{3}x, \tfrac{2}{3}y, \tfrac{2}{3}z \rangle$

5c How far to get a ↓ of 1.2 in T?

$|\vec{\nabla w}| = -1.2$

$-1.2 = \left|\frac{dw}{ds}\right| = \frac{\Delta w}{\Delta s}$

$\Delta s = -1.20$

$\Delta s \approx .2$

where is this from?

8. The atmospheric pressure $P = 30 + (x+1)(y+2)e^{z}$
Where is pt closest to origin where $p = 31.1$

$P(0,0,0) = 32$

plug the pts in

$\int e^{2z} = 2e^{2z}$

want to travel to 31.1

Oh right want to go $-\mathrm{dir}(\vec{\nabla w})$ for the fastest ↓

$\vec{\nabla w} = \langle (y+2)e^{x}, (x+1)e^{z}, (x+1)(y+2)e^{z} \rangle$

$(\nabla w)_{P_0} = \langle 2, 1, 2 \rangle$

$|\nabla w|_{P_0} = 3$

$-3 \cdot \Delta s = -.9$

$\hookrightarrow \Delta s = .3$

so travel $.3$ in dir $-\nabla w$

$|-\langle 2,1,2\rangle| = 3$

distance $.3$ is $\frac{1}{10}$ from $(0,0,0)$ to $(-2,-1,-2)$

so $(-.2, -.1, -.2)$

$3BP1 = \frac{15}{15}$

MatLab Directions for 18.02

Access MatLab by clicking on MatLab on the Athena screen, or by typing:
% **add matlab** [return]   % **matlab** [return]

Entering matrices and vectors; Basic operations

In MatLab the variables represent matrices and vectors. The symbol = is used to assign values to the variables. To see how this works, type each of these lines in order; remember: **always hit [return] or [enter] to end a line.**

$A = [1\ 2\ 3;\ 4\ 5\ 6;\ 7\ 8\ 9]$ *(you can use commas instead of spaces: 1,2,3;)*
$b = [1\ 0\ 1]$
b'
eye(3)   *(eye = I, the identity matrix)*

Try a mistake: $C = [1, 2, 3;\ 4, 5]$; to correct it, press any arrow key to get the line back.

Sum, difference $A + B$, $A - B$ (matrices must be same size)
Product $A{*}B$ (matrices must be compatibly sized)
Powers $A\,\hat{}\,n$ (A times itself n times; A must be square)
Quotient   left: $A\backslash b$ (the solution to $A\mathbf{x} = \mathbf{b}$)
right: b/A (the solution to $\mathbf{x}A = \mathbf{b}$)
Transpose A'
Inverse   inv(A)

Try typing (use the values of A and b above): $A +$ eye(3) $A{*}b$ $A{*}(b')$ $A{*}b'$ $3{*}b$

Special MatLab Operators

Array Operators: Use dots to make component-wise operations. Let $x = [x_1\ x_2\ \dots\ x_n]$.
$x.\hat{}\ m = [x_1^m\ \dots\ x_n^m]$ (m can be 0)
$x. * y = [x_1 y_1\ \dots\ x_n y_n]$
$f(x) = [f(x_1)\ \dots\ f(x_n)]$, $f = \sin$, cos, log, polynomials, etc.

Colon operator This generates a vector with equally spaced entries; for example,
$[0 : 2 : 12] = [0\ 2\ 4\ 6\ 8\ 10\ 12]$; $[2 : -.1 : 1.6] = [2.0\ 1.9\ 1.8\ 1.7\ 1.6]$

Two-dimensional plots in MatLab

Let $x = [x_1\ x_2\ \dots\ x_n]$, and $y = [y_1\ \dots\ y_n]$; then

plot (x, y) (plots the n points (x_i, y_i), joined by solid line segments)

plot $(x, y, '--')$ (plots the n points, joined by dashed line segments)

plot $(x, y, '*')$ (plots the n points as individual stars (or dots or circles, etc))

hold   (toggles between on and off (at the start it's off); when off, the new plot replaces the old; when on, the new
plot is superimposed on the old)

print   (gives a print-out of the current screen plot)

Try in order (read L to R; commands are separated by spaces; press [return] after each):
$x = [0 : .1 : 2]$ plot $(x,\ \sin(x))$ plot$(x,\ \cos(x),\ '*')$ hold
plot $(x,\ \sin(x),\ '--')$ hold
plot $(x,\ 4 * x.\hat{}\ 3)$ (plots $y = 4x^3$; note the need for the array operator)

Directions for 3D Graphs in MatLab

To plot the 3D graph of $z = f(x, y)$, you specify:

the grid (x_i, y_j) of lattice points: give the vectors $x = [x_1 \ldots x_n]$ and $y = [y_1 \ldots y_n]$.

Example: To make a grid with spacing .1, over the interval $[-2, 2]$ on both axes, type (in what follows, $\gg$ is the matlab prompt; don't type it — type the semicolon at the end so Matlab won't print out all the numbers — remember [return] at the end)

$\gg x = [-2 : .1 : 2];$ min interval max
$\gg y = [-2 : .1 : 2];$
$\gg [x, y] = \text{meshgrid}(x, y);$ ← make the grid

the function $z = f(x, y)$ For example, to graph the function $f(x, y) = x^2 - y^2$, type

So 5 commands

$\gg z = x.\hat{}\ 2 - y.\hat{}\ 2;$ oh the pt just means 2.0

plot the graph either as a mesh of lines, or as a filled-in surface (the color indicates the value of z, i.e., the height of the graph above the xy-plane); type first

$\gg$ mesh(x, y, z) then $\gg$ surf(x, y, z)

change the viewpoint To change the viewpoint (i.e., rotate the graph left or right, up or down), type

$\gg$ rotate3d

then place the mouse cursor in the graph region, hold down left button, move mouse, release button. The two numbers on the screen are the *azimuth*: angle in degrees from the negative y-axis to the line of sight, and the *elevation*, the angle in degrees from the xy-plane to the line of sight. To turn off rotation, type again: $\gg$ rotate3d

hidden lines Try typing: $\gg$ hidden (type it again to change back)

changing scale To change the x-axis scale to $[-4, 4]$, the y-axis to $[-5, 5]$, and the z-axis to $[-20, 20]$, type

$\gg$ axis([−4 4 −5 5 −20 20])

level curves To get a 2D plot with 20 level curves, type: $\gg$ contour$(x, y, z, 20)$

contour curves To get a 3D plot with 20 contour curves, type: $\gg$ contour3$(x, y, z, 20)$

in 3D