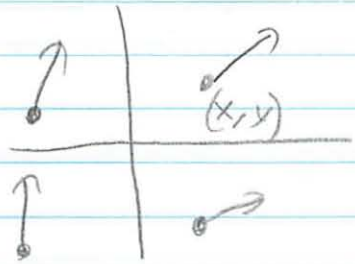
Lecture 20

Vector Fields + Line Integrals 2D

3/30

Geometrically (in 2D)



← at x,y there is a vector $\vec{F}(x,y)$

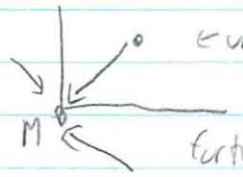
Analytic Representation

$$\vec{F}(x,y) = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$$
$$= \langle M, N \rangle$$

↑ functions of 2 variables

"Vector (valued) function of 2 variables"
- not just one vector, system of vectors
= a vector field
  - in a plane
  - 2D

Gravitational field (force field)
- 2D



← unit mass at x,y

further away it is, the smaller it is

Magnitude $(x,y) = \dfrac{GM}{r^2}$ ← r is distance to origin

Direction $\dfrac{-\langle x, y \rangle}{r}$

magnitude · dir

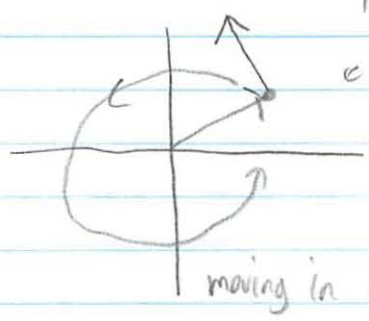
Field $\vec{F} = -\frac{GM}{r^2} \cdot \frac{\langle x, y \rangle}{r}$

$$= -\frac{GM(x\hat{i} + y\hat{j})}{(x^2+y^2)^{3/2}}$$

$$= -\frac{GM\langle x, y \rangle}{r^3}$$

Velocity Field
- of rotating disk
- counter clockwise
- ω = angular velocity

direction vector
= perpendicular direction
tangental circle



moving in circle

direction $\frac{\langle -y, x \rangle}{r}$

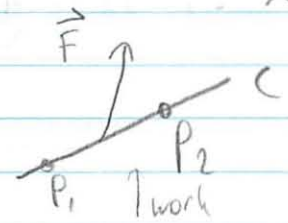
magnitude $\omega \cdot r$

$\vec{F} = \omega\langle -y, x \rangle$
$\omega(-y\hat{i} + x\hat{j})$

Line Integrals

Calculate work done $\boxed{\text{by}}$ a force field $\vec{F}$ along a curve

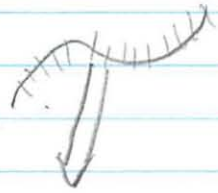
— simplest constant force over straight line
= Force · distance
- constant force → does not depend on x and y



Work by $\vec{F}$
P_1 to P_2 (unit mass)

$= \vec{F} \cdot \hat{u} \, |\overrightarrow{P_1 P_2}|$ ← $\hat{u}$ component of $\vec{F}$ · distance
$= \vec{F} \cdot \overrightarrow{P_1 P_2}$

- $\vec{F}$ non constant
C is real curve
now more complex
* view curve as tiny little line segment



parametrize curve
$r(t) = \langle x(t), y(t) \rangle$

$\vec{F} = \langle M(x,y), N(x,y) \rangle$

work by $\vec{F}$ on Δr_i

$\downarrow$

Choose a point on line

$\approx \langle M(x_i, y_i), N(x_i, y_i) \rangle \cdot \Delta \vec{r}_i$

the arrow vector

total work $\approx \sum_{1}^{n} \langle M(x_i, y_i), N(x_i, y_i) \rangle \cdot \Delta \vec{r}_i$

$\Delta r_i = \Delta x \, \vec{i} + \Delta y \, \vec{j}$

Total work along curve

$$\approx \sum_{i=1}^{n} M(x_i, y_i) \Delta x_i + N(x_i, y_i) \Delta y_i$$

Let n go to ∞

$$= \int_C \vec{F} \cdot dr$$

Express everything in terms of t

$x = x(t)$ $\quad$ $x_i = x(t_i)$

$y = y(t)$ $\quad$ $y_i = y(t_i)$

$$= \int_{t_1}^{t_2} \left[M(x(t), y(t)) \frac{dx}{dt} + N(x(t), y(t)) \frac{dy}{dt} \right] dt$$

$$\int_C \vec{F} \cdot d\vec{r} \qquad d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\parallel$$

$$\int_C M\,dx + N\,dy$$ ← what most people write for line integral

use t to do parametrisation ?

$$\int_{t_1}^{t_2} \left[\vec{F}(x(t), y(t)) \cdot \frac{d\vec{r}}{dt} \right] dt$$

use t w/ this equation

$$\int_{t_1}^{t_2} \left[M(x(t), y(t)) \frac{dx}{dt} + N(x(t), y(t)) \frac{dy}{dt} \right] dt$$

Get one massive function of t, integrate w/ respect to t

<u>3rd Way</u>

Intrinsic Calculation

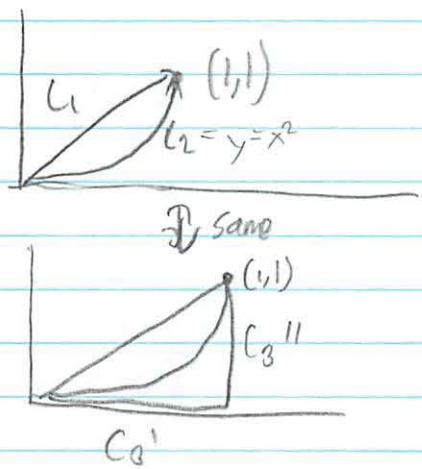
- insight + speed (if it works)

Use $t = s$ ← arc lenght on curve

$$\int_C \vec{F} \cdot dr = \int_C \left(\vec{F} \cdot \underbrace{\frac{dr}{ds}}_{\vec{T}} \right) ds$$

$\vec{T}$ the unit tangent vector

$$= \int_C \vec{F} \cdot \vec{T}\, ds$$

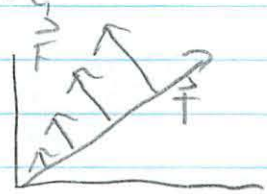


$\vec{F} = -y\hat{i} + x\hat{j}$

$\omega = 1$

rotating velocity field counter clockwise

$$\int_C \vec{F} \cdot d\vec{r} \rightarrow \underline{C_1} \int_{C_1} \vec{F} \cdot \vec{T}\, ds = 0$$



always perpendicular along line

$$\underline{C_2} \int_{C_2} \vec{F} \cdot \vec{T}\, ds = \int_{C_2} \left[M \frac{dx}{dt} + N \frac{dy}{dt} \right] dt$$

$$= \int_{C_2} -y\,dx + x\,dy$$

use x as parameter

$$= \int_0^1 -x^2\,dx + x\,2x\,dx$$

$$= \int_0^1 x^2 \, dx$$

$$= \frac{1}{3}$$

$\underline{C_3}$ Most complex of all

$$\int_{C3} -y\,dx + x\,dy$$

$$= \int_{C_3'} -y\,dx + x\,dy + \int_{C_3''} -y\,dx + x\,dx$$

$$= \int_{C_3'} 0 + x \cdot 0 + \int \underset{\substack{\uparrow \\ \text{since } x=1 \\ \text{along vertical} \\ \text{line}}}{0} + 1$$

$$= \int_0^1 dy$$

$$= 1$$

So $C_1 = 0$
$C_2 = \frac{1}{3}$
$C_3 = 1$

Lecture 21

[Gradient Fields + Conservative Fields
Potential Functions, FTC for line integrals]

4/2

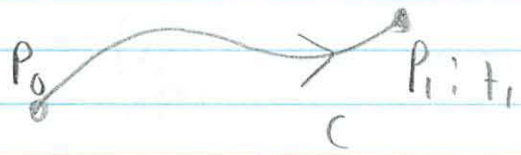
Test 2
Average
74

Mei 72

$\vec{F} = \langle M, N \rangle$

$C = \vec{r}(t)$
$= \langle x(t), y(t) \rangle$

t_0



	Fields	Differentials
Symbol	$\int_C \vec{F} \cdot d\vec{r}$	$= \int_C M\,dx + N\,dy$
Calc	$\int_{t_0}^{t_1} \left(\vec{F} \cdot \frac{d\vec{r}}{dt}\right) dt$	$= \int_{t_0}^{t_1} \left(M \frac{dx}{dt} + N \frac{dy}{dt}\right) dt$

↑ Physics + electrical engineering

↑ this quick answer
See what going on
* intuition *

↑ everyone else

I will use this for calculations

↑ * calculations *

Learn all 4 forms

5th form

$$\int_C (\vec{F} \cdot \vec{T})\, ds$$

↳ when $t = s$ arc lenght

cleaner parametrization
<u>Symbolic calculation</u>

since $\vec{T} = \frac{d\vec{r}}{ds}$

1. Remember

Math work done by the field
Physics <u>againts</u>

2. Direction you are transversing

If you are travelling the other way

c prime

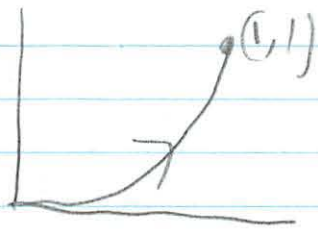
$$\int_C \vec{F} \cdot d\vec{r} = -\int_{C'} \vec{F} \cdot d\vec{r}$$

3.

$$\int_{t_1}^{t_0} \_\_\_\, dt = -\int_{t_0}^{t_1} \_\_\_\, dt$$

$\vec{F} = \langle y^2, 2xy \rangle$

C:



paramatization
$x = t^2$
$y = t^3$

(only one can calc arc lenght for)

$$\int_C y^2\,dx + 2xy\,dy \quad \text{(differential form)}$$

replace

$$\int_0^1 t^6 \cdot 2t\,dt + 2t^2 \cdot t^3 \cdot 3t^2\,dt$$

dt

$dx = 2t\,dt$
$dy = 3t^2\,dt$

Now normal (8.0) integral

$$\int_0^1 8t^7\,dt$$

$$t^8 \Big|_0^1$$

$$1$$

* ans to Scalar product is a scalar

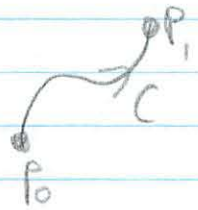
Fundamental theory of line integrals

Gradiant field - field $\vec{F}$ which has $\overrightarrow{\nabla f(x,y)}$

$$F = \left\langle \underset{M}{\frac{\partial f}{\partial x}}, \underset{N}{\frac{\partial f}{\partial y}} \right\rangle$$

Fundamental theory of calculus for line integrals

If f is a gradiant field $F = \vec{\nabla} f$



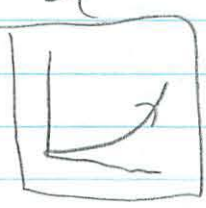
(x_0, x_1)

$$\int_C \vec{F} \cdot d\vec{r} = f(x,y)\Big|_{P_0}^{P_1}$$

$$= f(x_1, y_1) - f(x_0, y_0)$$

$$\langle y^2, 2xy \rangle = \vec{\nabla}(xy^2)$$

$$\int_C y^2\,dx + 2xy\,dy$$



$$= x\,y^2\Big|_{(0,0)}^{(1,1)}$$

$$= 1$$

That's the simple idea
But some more stuff to say about it
Basis for a lot of stuff

Matter i not what you expect

<u>Proof</u> (to memorize)

hypothesis $\vec{F} = \vec{\nabla} f$

$$\int_C \vec{F} \cdot d\vec{r} =$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \right) dt$$ calculating form

now 18.01 integral

$$= \int_{t_0}^{t_1} \left[\frac{df(x(t), y(t))}{dt} \right] dt$$

fundamental theorem of calculus (first)

$$= f(x(t), y(t)) \Big|_{t_0}^{t_1}$$

$$= f(x_1, y_1) - f(x_0, y_0)$$

$$\int_0^{\uparrow} \sec t \, dt$$

problem is finding anti deriv
should know

Problems for Tue

1. - given $\vec{F} = \langle M, N \rangle$
   how do you tell its a gradiant field?
   - extreamly rare, but important

2. If it is $\vec{F} = \vec{\nabla} f$ how do you find f?

3. Translate everything into language of differentials

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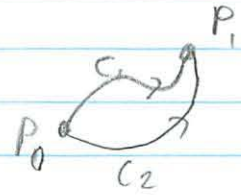
hypothesis
$\vec{F} = \vec{\nabla} f \Rightarrow$ [if then / implies] #1 and #2



#1.

start + end at same pt = same value of line integral

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

"path independent"

#2.

$$\oint_C \vec{F} \cdot d\vec{r}$$ that is closed

Γ sign means path is closed

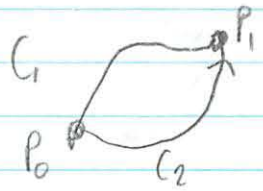
$= 0$

$P_0 = P_1$

"F is conservative"

$$\int_{P_1}^{P_1} = \int_{P_0}^{P_0} = 0$$

relation #1 $\Rightarrow$ #2 = we did it
#2 $\Rightarrow$ #1



hyp $\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$

$$\int_{C_1} + \int_{C_2'} = 0$$

by hypothesis since $C_1 + C_2$ is closed curve

$$\int_{C_1} - \int_{C_2} = 0$$

thus $\int_{C_1} = \int_{C_2}$

Recitation

Test Correction + Line Integrals 4/5

Test 2

#5 On test paper did not spend much time on

Lectures Line integrals

$\vec{F}$ is field $\vec{F}(x,y)$ = vector

$$\int_C \vec{F} \cdot d\vec{r}$$

$\vec{F}$: vector; $d\vec{r}$: vector; $\cdot$: dot product

$d\vec{r} = \langle x'(t), y'(t) \rangle$ is "position vector" moving along C — parameterization

P is function $P(x,y)$ = number

$$\int_C P\,dx + Q\,dy$$

P, Q: number

$dy = y'(t)\,dt$

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ex1 Let C be curve parametrized by $\vec{r}(t) = \langle t+1, t^2-1 \rangle$ $t \in [0,1]$

a) Compute $I = \int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle x+y, x \rangle$

b) Compute $J = \int_C P\,dx + Q\,dy$ where $P(x,y) = y$, $Q(x,y) = 1$

↑ 2 different equations want to compute, not 2 different witnesses

So curve has been given to us in terms of t already know

$$x = t+1$$
$$y = t^2-1$$

a Compute I

$$\vec{F} = \langle x+y, x \rangle$$
$$\langle t+1+t^2-1, t+1 \rangle$$
$$\langle t^2+t, t+1 \rangle$$

Now $\int_0^1 \langle t^2+t, t+1 \rangle \cdot \langle t+1, t^2-1 \rangle$

~~so just dot product now~~

~~$(t^2+t \cdot t+1) + (t+1 \cdot t^2-1)$~~

derivative of original x and y

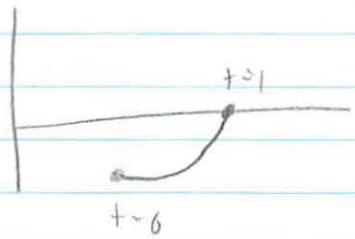
Need $x'(t) = 1$ $\quad d\vec{r} = \langle 1, 2t \rangle$

$y'(t) = 2t$

~~and then $t=1$ $t=0$~~

Can draw the curve

but don't have to



now find dot product

$$F \cdot dr = \langle t^2+t, t+1 \rangle \cdot \langle 1, 2t \rangle$$
$$(t^2+t \cdot 1) + [(t+1) \cdot 2t]$$
$$t^2+t + 2t^2+2t$$
$$3t^2+3t$$

$$I = \int_0^1 (3t + 3t^2)\,dt$$

$$\frac{3t^2}{2} + t^3 \Big|_0^1$$

$$\frac{3}{2} + 1 = \frac{5}{2}$$ ✓

b) Now a problem with differentials

$P = y = t^2 + 1$
$Q = 1$

$dx = x'(t)\,dt = 1\,dt$
$dy = y'(t)\,dt = 2t\,dt$

$$\int_0^1 (t^2 - 1)\,dt + 2t\,dt$$

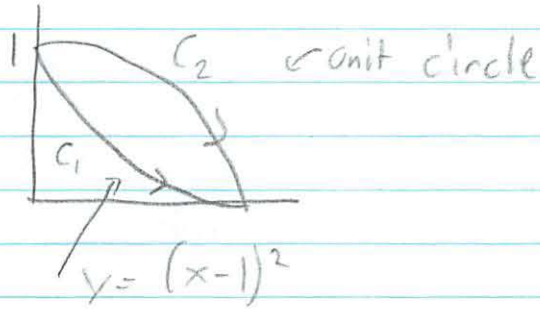
$$\frac{t^3}{3} + t^2 - t \Big|_0^1$$

$$\frac{1}{3}$$

Remark ✱ If $F = \langle M, N \rangle$ then — from definition

$$\int_C \vec{F} \cdot d\vec{r} = M\,dx + N\,dy$$

ex 2



$\vec{F} = \langle y, -1 \rangle$

a) $I = \int_{C_1} \vec{F} \cdot d\vec{r}$

b) $J = \int_{C_2} \vec{F} \cdot d\vec{r}$

Now need to parametrize the circle myself

a) $x(t)$ look for canidates
$y(t)$ What makes sense

$x(t) = t$
$y(t) = (t-1)^2$

Then can check that it satifies
$y = (x-1)^2$

Choose a parameter, that identifies a point
Let's say x is trial pt
So $t = x$
So what is $y \to (x-1)^2$
So plug in $t = x$ for $(t-1)^2$

x With this curve x is not a good choice

Could use y as parameter

$y(t) = t$

$x(t) = \ldots$

For our function

$y(t) = (1-t)$ ← to go right direction

$x(t) = \sqrt{(1-t)}$

But don't use - too complicated

$t \in (0,1)$

So now calculate

$\left(\frac{4}{3}\right)$

b) What parameter here?

$x^2 + y^2 = r^2$ unit circle

$x^2 + y^2 = 1^2$ →

Lets declare $x(t) = t$

$y(t) = \sqrt{1 - x^2}$

$\sqrt{1 - t^2}$

Could also do <u>angle</u> θ as parameter

$x(t) = \cos t$
$y(t) = \sin t$ $\qquad$ <u>t is not time</u>

t is CCW

But we are going clockwise, so

$x(t) = \cos\left(\frac{\pi}{2} - t\right)$
$y(t) = \sin\left(\frac{\pi}{2} - t\right)$ $\qquad t \in \left[0, \frac{\pi}{2}\right]$

get from graph or convert

$t = \boxed{1 + \pi/4}$

Free to take whatever parametrization
- it will work out in the end
- Relation between 2 parameterizations is relation of changing speed

new $x(t) = x(f(t))$
$y(t) = y(f(t))$

?

relation of parametrization

Result <u>does</u> depend on curve

Not path independent
since field is not conserved

Lecture 22

Criterion for conservative fields + potential functions 4/6

		Special kind	Fund. Th. Calculus $\int$
Fields	$M\hat{i} + N\hat{j}$ Function of 2 variables	gradient $\vec{\nabla} f = f_x \hat{i} + f_y \hat{j}$	$\int_{P_0}^{P_1} \vec{\nabla} f \cdot d\vec{r} = f(P_1) - f(P_0)$
Differentials	$M dx + N dy$	$df = f_x\, dx + f_y\, dy$ exact/total differential of f that make it path independent	$\int_{P_0}^{P_1} df = f(P_1) - f(P_0)$ path independent

Problem 1

Given $\begin{cases} \langle M, N \rangle \\ M dx + N dy \end{cases}$ how do we decide if it is special?

$\hookrightarrow \begin{cases} \vec{\nabla} f \\ df \end{cases}$

If it does pass the test, how to find the function?

Criterion $M_y = N_x$ ← partial deriv

$$\begin{cases} \langle M, N \rangle = \vec{\nabla} f \\ M dx + N dy = df \end{cases}$$

But not true always
- it must be true in the whole x, y plane
- then it is always true

Ex $axy\,dx + (bx^2 + cy^2)\,dy$ exact

$\langle axy, bx^2 + cy^2 \rangle$

when is it a gradient field
aka when conservative
aka when path independent

$$ax \overset{?}{=} 2bx$$
$$a = 2b$$

<u>Hypothesis!</u>

$$\begin{cases} M = f_x \\ N = f_y \end{cases}$$ true in some region of a plane

Calculate

$$\left.\begin{aligned} M_y &= f_{xy} \\ N_x &= f_{yx} \end{aligned}\right\}$$ <u>thus</u> $f_{xy} = f_{yx}$

does not really look like it should be true - but it is
- if all 4 2nd order partial derivitive f_{xx} f_{xy} f_{yx} f_{yy} are contineous

Note totally cheating argument - circular

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

$$f(x,y) = f(0,0) + \left[(f_x)_0 + (f_y)_0\right] + \frac{1}{2!}\left[(f_{xx})_0 x^2 + 2(f_{xy})_0 xy + (f_{yy})_0 y^2\right] + \dots$$

Its all numbers....

$$= a_0 + (b_1 x + b_2 y) + (c_1 x^2 + c_2 xy + c_3 y^2) + \dots x^m y^n$$

For $x^m y^n = f$

$$f_{xy} = mn\, x^{m-1} y^{n-1} = f_{yx}$$

Problem #2 If $\langle M,N\rangle$ it passes the test, how do we find $f(x,y)$

2 ways 1/3 like first way (of people)
2/3 " second "

Method 1

$$\boxed{f(P) = f(P_0) + \int_{P_0}^{P} M\,dx + N\,dy}$$

P_1

Know $\langle M,N\rangle = \vec{\nabla} f$

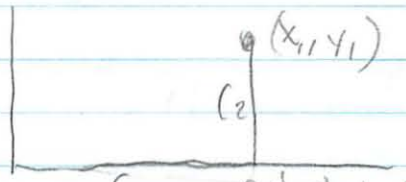
constant, not important when differentiating

Gradient field - so over any path

$\vec{F} = \langle 2xy, x^2+y^2 \rangle$

test $2x = 2x$ ✓ → so gradient field

$$f(x_1,y_1) = \int_{P_0}^{P_1} 2xy\,dx + (x^2+y^2)\,dy$$



simplest path to calculate

$$= \int_{C_1} 2xy\,dx + (x^2+y^2)\,dy + \int_{C_2} 2xy\,dx + (x^2+y^2)\,dy$$

$y=0$ $dy=0$
so it all disappears

$\underbrace{x=x_1}_{\text{drops out}}$ $\underbrace{dx=0}_{\text{what is left}}$

$$\int_0^{y_1} (x_1^2+y^2)\,dy$$

Parametrization

$x = x_1$

$y = y_1$

$= \quad x_1^2 y_1 + \frac{1}{3} y_1^3$

$f(x,y) = x^2 y + \frac{1}{3} y^3$

$\vec{\nabla} f = \langle 2xy, x^2 + y^2 \rangle$ checks ✓

<u>Method 2</u>

$\langle 2xy, x^2 + y^2 \rangle$

$\downarrow$

$f_x = 2xy$

$f_y = x^2 + y^2$

What was original function

$f = x^2 y + c$ ← any functions of y

$x^2 y + g(y)$ ↳ what is g?

do it systematically for when 3 variables

$f_y = x^2 + y^2$

$= x^2 + g'(y)$

↙ know

$g'(y) = y^2$

$g(y) = \frac{1}{3} y^3$

$f(x,y) = x^2 y + \frac{1}{3} y^3$

Pros of each method

Method 1 - theoritical → gives formula for answer
Calculation is easy, straight forward

Method 2 - algorithm for calculation
requires no knowledge of line integrals

Recitation

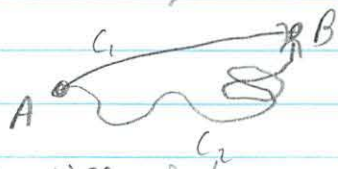
4/7

Lecture

Def Field $\vec{F}$ is a "gradient" field / "conservative" field if $\vec{F} = \vec{\nabla} f$

Prop $\vec{F} = \langle M, N \rangle$ is $\vec{\nabla} f$ if $M_y = N_x$ everywhere

Theorem If $\vec{F} = \vec{\nabla} f$ then
$$\int_C \vec{F} \cdot dr = f(B) - f(A)$$
for any curve from $A \to B$



Same with differentials

Def $M dx + N dy$ is "exact" if f such that
$$df = M dx + N dy$$

Prop $M dx + N dy$ exact and only if $M_y = N_x$

Theorem $\int_C M dx + N dy = f(B) - f(A)$
if $M dx + N dy = df$

---

ex 1 Let $\vec{F} = \langle e^y, x e^y + 1 \rangle$

a) Is $\vec{F}$ a gradient field?

~~$M_x = 0$~~ $0 = 0$ yes I did wrong $M_y = e^y$

~~$N_y = 0$~~ $N_x = e^y$ Yes

* can't switch *

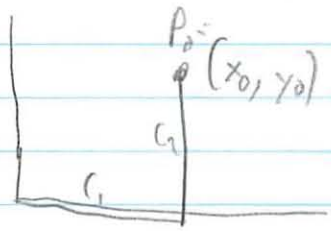
b) Find a function such that $\vec{F} = \nabla \vec{f}$ by FTC (method 1)

Let $P_0 = (x_0, y_0)$

$$f(P_0) = f(0) + \int_C \vec{F} \cdot d\vec{r}$$

for any curve C from 0 to P_0

- can pick any curve
- but we always use same curve



$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + f(0)$$

constant take as 0

Need to parametrize C_1 and C_2

doing this slightly differently - more work

C_1: $x(t) = t$, $y(t) = 0$ since constant, $\in [0, x_0]$, $dx = 1$, $dy = 0$

C_2: $x(t) = x_0$, $y(t) = t$, $\in [0, y_0]$, $dx = 0$, $dy = 1$

that is always the same for that curve

W/ surfaces need 2 parameters

$$\int_{C_1} \underbrace{\langle t, t+1 \rangle}_{\vec{F}} \cdot \underbrace{\langle 1, 0 \rangle}_{d\vec{r} = \langle dx, dy \rangle} dt + \int_{C_2} \langle e^t, x_0 e^t + 1 \rangle \cdot \langle 0, 1 \rangle$$

$$\int_0^{x_0} 1\, dt + \int_0^{y_0} x_0 e^t + 1\, dt$$

$x_0 \quad + \quad x_0 e^{y_0} + y_0 - x_0$

$\boxed{x_0 e^{y} + y}$ Drop the sub 0

Method 2

We want $\vec{\nabla} f = \vec{F}$

that is $\begin{cases} f_x = e^y \\ f_y = xe^y + 1 \end{cases}$

Step 1

a) $f = xe^y + c(y)$

↑ constant w/ respect to x
= function of y alone

Step 2

$f_x = xe^y + c'(y)$

Step 3

$c'(y) = 1$ ← since $\boxed{f_y = xe^y + c'(y) = xe^y + 1}$

$c(y) = y$

Step 4

$f = \boxed{xe^y + y} + c$

↑ can take this $= 0$
so forgot about it

Notice it matches

ex 2 Let $\vec{F} = \langle \log y + y, \frac{x}{y} + \alpha x + y \rangle$

a) For which value of α is it a gradient field?

$M_y = \frac{1}{y} + 1$ $\qquad$ $\frac{1}{y} + 1 = \frac{1}{y} + \alpha$

$M_x = \frac{1}{y} + \alpha$ $\qquad$ $\boxed{\alpha = 1}$ ✓

b) For that value find a function f such that $\vec{F} = \vec{\nabla} f$

$f_x = \ln(y) + y$
$f_y = \frac{x}{y} + x + y$

$f = \int \ln(y) + y \, dx$
$= \cancel{y \ln(y) + y - y + c(y)}$ $x \ln(y) + xy + c(y)$
$= \cancel{y \ln(y) + c(y)}$

$f_y = \frac{x}{y} + x + c'(y)$

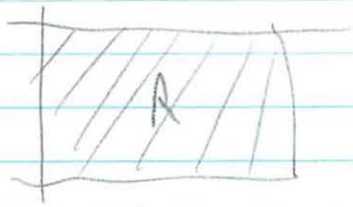
$c'(y) = \cancel{1}y$ $\quad$ typo wrong
$c(y) = y$ $\quad$ $\frac{y^2}{2}$

$f = \cancel{y \ln(y) + y + c}$

should check both differentiate
- think screwed this up somewhere

$x \ln(y) + xy + \frac{y^2}{2} + c$

ex3



$\vec{F} = \langle x, y \rangle$

max value of $\int_C \vec{F} \cdot dr$ over all possible curves contained in R

F gradient $\xrightarrow{so}$ $\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$

out of time

1

18.02A Problem Set 6B (DUE 4/8/10 10:45AM 2-106)

Hand this in along with Problem Set 6A; staple 6A and 6B together, in that order.

Part I (15 pts. + 10 pts. for Pset 6A)

Lecture 20. Tues. Mar. 30 Vector fields and line integrals in 2D
Read: Notes V1, Textbook: 21.1 Work: 4A-3bd, 4; 4B-1bef, 2, 3

Exam 2. Thurs. April 1 11:05-11:55 Walker 3rd floor

The exam will cover Lectures 9 - 19 on Partial Differentiation and Double Integration. No books, notes, calculators; no use of cell-phones.

Lecture 21. Fri. Apr.2 Gradient fields, conservative fields, path-independence; Fundamental theorem of calculus for line integrals.
Read: 21.2 to p. 762 Work: 4C-1, 2, 3

Lecture 22. Tues. Apr. 6 Criterion for conservative fields; potential functions.
Read: Notes V2 Work: 4C-5a, 6ab (do all three both by Method 1 and Method 2)

Lecture 43. Thurs. Apr. 8 Green's theorem (tangential form).
Read: 21.3 to middle p. 768

Part II (20 pts.)

Directions: Try each problem alone for 15 minutes. If you subsequently collaborate, solutions must be written up independently. It is illegal to consult problem sets from previous semesters. With each problem is the day it can be done.

Problem 1. (Tues. 3/30 3 pts.: 1,2)

a) Write down in xy-coordinates the vector field $\mathbf{F}$ whose vector at $P : (x, y)$ runs in the vertical direction from P to the line $L : x + y = 1$.

b) Show *without calculation* that for this field, $\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0$, if C is any positively oriented right triangle with legs parallel to the axes and hypotenuse on L.

Problem 2. (Tues. 3/30 3 pts.)

Imagine the x-axis is an infinitely long heavy wire, of uniform density. The gravitational force it exerts on a unit mass placed at a point $(y, z) \neq (0, 0)$ in the yz-plane is given by

$$\mathbf{F}(y, z) = -c \frac{y\,\mathbf{j} + z\,\mathbf{k}}{y^2 + z^2},$$

where c is a positive constant. Taking $c = 1$, find by direct calculation of the line integral (i.e., by using a parametrization of C) the work done *by* the gravitational field in moving a unit mass along each of the following paths C in the yz-plane:

a) half-line $z = 1$, $y \geq 0$; starting point $(0, 1)$;
b) circle of radius a with center at the origin, traced counterclockwise;
c) line from $(0, 1)$ to $(1, 0)$.

Problem 3. (Fri. 4/2 4 pts.: 1,1,2)

a) Let $\mathbf{F} = -\nabla(\ln r)$, where r is the distance from the origin in the yz-plane. Show that $\mathbf{F}$ is the field of Problem 2, when $c = 1$.

b) Show that for any path C joining two point P_1 and P_2 in the yz-plane, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the ratio r_2/r_1, where r_i is the distance of P_i from the origin. What points P_i should be excluded?

c) Let C be a directed line segment of length 1.
(i) If $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$, how is C placed?
(ii) What position of C gives $\int_C \mathbf{F} \cdot d\mathbf{r}$ the largest value?

Problem 4. (Fri. 4/2 3 pts: 1,2) Let $\mathbf{F} = \nabla(x^2y - 3xy^2)$. Let C be the quarter of the ellipse $9x^2 + 4y^2 = 1$ running from the positive x-axis to the positive y-axis.
a) Write the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ in the $\int_C M\,dx + N\,dy$ form
b) Evaluate it in two different but equally easy ways: using the Fundamental Theorem and using path-independence. (Both ways can be done mentally, but write down enough to indicate the reasoning.)

Problem 5. (Tues. 4/6 4 pts.: 1, 1.5, 1.5) Let $\mathbf{F} = xy\,\mathbf{i} + x^2\,\mathbf{j}$.
a) Verify that $\mathbf{F}$ is not a conservative field.
b) Try to find a potential function for $\mathbf{F}$ by using Method 1 in the Notes. What goes wrong?
c) Answer the same question for Method 2.

Problem 6. (Tues. 4/6 3 pts.: 1.5, 1.5)

The most general two-dimensional field $\mathbf{F}$ whose direction is radially outward, and whose magnitude depends only on r $(r = \sqrt{x^2 + y^2})$, is

$$\mathbf{F} = f(r)(x\,\mathbf{i} + y\,\mathbf{j})\ .$$

a) Assuming $f(r)$ is differentiable for all r, show that any such $\mathbf{F}$ is conservative.
(Use the chain rule and the standard formulas $\partial r/\partial x = x/r$, $\partial r/\partial y = y/r$.)

b) Find a potential function of the form $g(r)$, if $f(r) = r^n$, where n is a positive or negative integer.
(Solve $\nabla g(r) = \mathbf{F}$ for the function $g(r)$. One value of n has to be handled differently from all the others.)

P-Set 6B

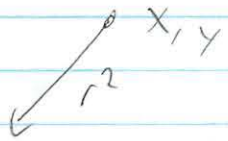
4/3

Part 1

<u>Lecture 20</u> Vector Fields + line integrals in 2D

4A-3b Write down an explicit equation for

The vector at (x, y) directed radially in towards origin w/ magnitude r^2



So, what?

Field = magnitude · direction

$r^2 \langle \_\_\_ \rangle$

What is radially inward in polar coords?

$r^2 \langle -r, 0 \rangle$

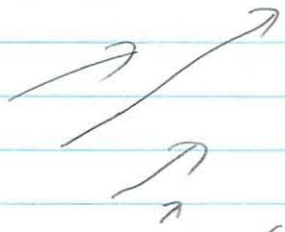
$-r\langle x\hat{i} + y\hat{j} \rangle$

* <u>ok</u> $-\frac{\langle x, y \rangle}{r}$ is radially inward *

so $r^2 \frac{\langle x\hat{i} + y\hat{j} \rangle}{r}$ ✓

d. Each vector is parallel to $\hat{i} + \hat{j}$
but magnitude varries.

$\hat{i} + \hat{j}$



$\text{Field}_{(x,y)}$ = magnitude · direction

↙ = magnitude

$m \langle \hat{i} + \hat{j} \rangle$

Is that all?

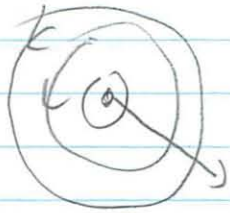
well $f(x,y) \langle \hat{i} + \hat{j} \rangle$ ✓

so m is a function of x and y

4. The electromagnetic Force field of a long straight wire along the z-axis carrying a uniform current is a 2D field, tangent to horizontal circles in dir of right hand rule

Just like physics qu now

Magnitude $\frac{k}{r}$



so $\frac{k}{r} \cdot \frac{\langle \hat{i} + \hat{j} \rangle}{r}$

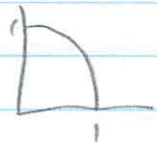
$\frac{k \langle \hat{i} + \hat{j} \rangle}{r^2}$ $\quad k \frac{\langle -y\hat{i} + x\hat{j} \rangle}{r^2}$

? why that $-y\hat{i} + x\hat{j}$ ✓

For each field $\vec{F}$ and curve C
evaluate $\int_C \vec{F} \cdot dr$

Line Integrals in the Plane use differentials $\int_C M\,dx + N\,dy$

4B-1b

$\vec{F} = xy\,\hat{\imath} - x^2\,\hat{\jmath}$ $C =$ 

$$\vec{F} = \langle M(x,y), N(x,y) \rangle$$

$$= \int_{t_1}^{t_2} \left[M(x(t), y(t)) \frac{dx}{dt} + N(x(t), y(t)) \frac{dy}{dt} \right] dt$$

But how do you actually solve, did not do example in class. will wait post recitation
Never really got that "parametrising" thing

Ok 4/6 - but what function did they give you?

have $\vec{F}$ but no actual function
well that is given by the line as it moves

$x^2 + y^2 = 1^2$

$x = \sqrt{1-y^2}$

$y = \sqrt{1-x^2}$

So just say

$x = t$

$y = \sqrt{1-t^2}$

Or is it (from notes)

$x = t$

$y = (t-1)^2$

well same thing right?

No

So its my original thought

Now rewrite $\vec{F}$ in terms of t

$$\langle t\sqrt{1-t^2},\ -t^2 \rangle$$

now $\displaystyle\int_0^1 \langle t\sqrt{1-t^2},\ -t^2 \rangle \cdot \overset{\langle x'(t),\, y'(t)\rangle}{\langle 1,\ \frac{-t}{\sqrt{1-t^2}} \rangle}$

$$\left(t\cdot\sqrt{1-t^2}\right) + \left(-t^2 \cdot \frac{-t}{\sqrt{1-t^2}}\right)\Bigg|_0^1$$

$$t\sqrt{1-t^2} + \frac{t^3}{\sqrt{1-t^2}}\Bigg|_0^1$$

$$1\sqrt{1-1^2} + \frac{1^3}{\sqrt{1-1^2}}$$

$$0 \quad + \text{ divide by 0 error}$$

They have totally different parameter

These are tese problems where you have to <u>guess</u> a parameter

$x = \cos t \quad \rightarrow dx = -\sin t\ dt$

$y = \sin t \qquad dy = \cos t\ dt$

And they use differentials $\displaystyle\int_C xy\,dx - x^2\,dy$

$$\int_{\pi/2}^{0} -\sin^2 t\cos t \cdot -\sin t - \cos^2 t \cdot \cos t$$

$$\int_{\pi/2}^{0} -\sin^2 t\cos t - \cos^3 t$$

Now just integrate ?
well simplify first

$$= \int_{\pi/2}^{0} \cos t \, dt$$

I don't get how they are simplifying - worked out on calculator

Now integrate

$$- \sin t \Big|_{\pi/2}^{0}$$

$$-\sin 0 - [-\sin \pi/2]$$

$$-0 + 1$$

$$1$$

But my question is how do you pick the parameterization. It does always = the same thing in the end, but the amount of work is different. I prob screwed mine up somewhere

e $\quad F = 6y\,\hat{\imath} + x\,\hat{\jmath} \qquad x = t^2,\ y = t^3 \qquad$ from $(1,1)$ to $(4,8)$

ok first write that

$x = t^2 \quad \rightarrow \quad dx = 2t\,dt$

$y = t^3 \qquad\quad dy = 3t^2\,dt$

was already parametrized for us

$$\int 6y\,dx + x\,dy \quad ✓$$

↑ these are x right?

$$\int_1^4 6 \cdot t^3 \cdot 2t + t^2 \cdot 3t^2 \quad ✓$$

$$12t^4 + 3t^4$$

$$15t^4 \quad ✓$$

$$\left.\frac{15t^5}{5}\right|_1^4$$

$$3t^5 \quad ✓$$

$$3(4)^5 - 3(1)^5$$

$$\cancel{3072} - 3$$

$$\cancel{3069}$$

$$93 \quad ✓$$

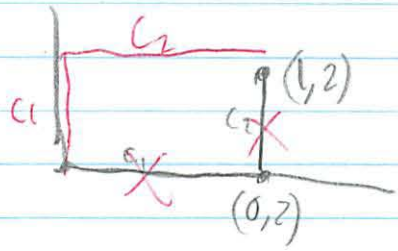
So I picked the wrong integral bounds - what I was not sure of - how did I screw up?

so it is $t = \sqrt{x}$ $\quad \sqrt{4} = 2$

$\sqrt{1} = 1$

f $\vec{F} = (x+y)\uparrow + xy\uparrow$

C broken line



$x = t$ ← how do we pick

$y =$

Well for this write it out first
↳ no clear eq of the line

$$\int_{C_1} = (x+y)\,dx + xy\,dy + \int_{C_2} (x+y)\,dx + xy\,dy$$

$x = 0 \quad dx = 0 \qquad x = 2 \quad dy = 0$

$$\int_{C_1} x\,\cancel{dx} + \int_{C_2} 2y\,dy \quad (x+2)$$

What did I do now? - went wrong way in picture

$$\int_0^2 x\,\cancel{dx} + \int_0^1 \cancel{2y\,dy} \quad (x+2)$$

$$\frac{x^2}{2}\Big|_0^2 + \frac{2y^2}{2}\Big|_0^1$$

$$\frac{2^2}{2} + \frac{2}{2} - 0$$

$$2 + 1$$

3

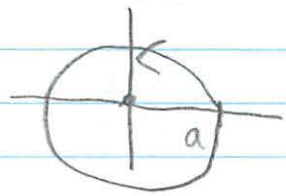
$$\frac{x^2}{2} + 2x\Big|_0^1$$

$\frac{5}{2}$

So far have not gotten a single problem, but gotten better

2. For the following fields and curves eval with geometry, not calculation

a) $\vec{F} = x\hat{i} + y\hat{j}$.



So closed path so 0

$\vec{F}$ points radially outwards
tangent always perpendicular to radius
$\vec{F} \cdot t = 0$
$\int_c \vec{F} \cdot dr = \int_c \vec{F} \cdot t \, ds = 0$

I thought it was just because it was a closed circle
- only true if $\vec{F} = \vec{\nabla} f$
which means that $f_{xy} = f_{yx}$?
and/or means field is conservative, or not dependent on x,y

check ? In this case the field is constant as you walk along the circle - which is the argument they gave.

b) $\vec{F} = y\hat{i} - x\hat{j}$ C is counterclockwise circle

$\vec{F}$ always tangent to circle in clockwise
$\vec{F} = -a\hat{t}$ ← so is that the parameterization

$\int_C \vec{F} \cdot dr$

$\int_C F \cdot \hat{t}\, ds$

$-\int_C a\, ds$

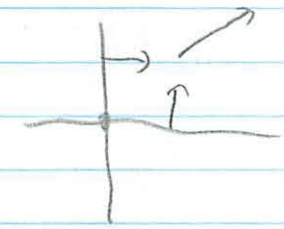
$-2\pi a^2$ ← so that is the line integral adding up force at each pt

what's that?

✓

3. Let $F = \hat{i} + \hat{j}$, How would you place a directed line segment C of lenght 1 so that $\int_C \vec{F} \cdot d\vec{r}$ would be a

a) maximum



Make it as large as possible, as far away from the origin as possible.

Maximum C if C in dir of field

$C = \frac{\hat{i} + \hat{j}}{\sqrt{2}}$

? Wait where did we cover this?
It sounds familiar from physics

– what is the

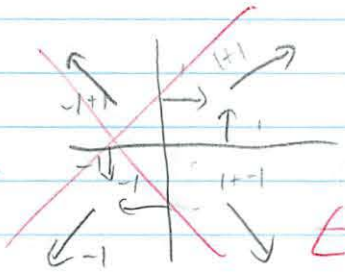
b) minimum

dir opposet field $C = \frac{-\hat{i} + \hat{j}}{\sqrt{2}}$

c zero?

perpendicular to the field $C = \pm \frac{\hat{i} - \hat{j}}{\sqrt{2}}$

But lets look at other regions



But then how do know where it starts
- oh it also says of lenght 1

Is that picture even what a field is?

Well direction → $\frac{\langle x, y \rangle}{r}$ - so that is what they did

So their max ans was ↗ $\hat{i}+\hat{j}$

min ↙ $-\hat{i}-\hat{j}$

zero ↘ or ↗

$\hat{i}-\hat{j}$ $-\hat{i}+\hat{j}$

So that was kind what I drew
- quadrants 2, 3, 4 may have been wrong

better

yeah that was direction
it was not $x\hat{i} + y\hat{j}$
which I drew
↳ think how we are doing in physics

d) What is max + min value of integral

→

max $= \sqrt{2}$
min $= -\sqrt{2}$

↑ right, same but opposite dir

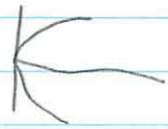
$$\int_C \vec{F} \cdot d\vec{r} = \pm |F| \cdot |C| = \pm\sqrt{2}$$

So what did they do
well did integral
but over what?

Lecture 21 Gradiant fields, conservative fields, path-independente
Fundamental theorem of calculus for line integrals

4C-1 Let $f(x,y) = x^3y + y^3$
$C = y^2 = x$ between $(1,-1), (1,1)$ upwards

a) Calculate $F = \nabla f$



$$\int_C x^3y\,dx + y^3\,dy$$

Now set some parameter
$x = t^2$ ← ?? how do we choose these?
$y = t^3$

Wait don't we just want gradiant

$\langle f_x, f_y \rangle$ ✓

$\langle 3x^2y, x^3 + 3y^2 \rangle$ or $3x^2y\,\hat{i} + (x^3+3y^2)\hat{j}$

b) Now calc $\int_C \vec{F} \cdot dr$ directly

Use y as parameter
$x = y^2$
$y = y$

thus $dx = 2y\,dy$

?? → So how do you decide what to pick again as parameter? - doesn't say - ask

$$\int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1} 3(y^2)^2 y \cdot 2y \, dy + \left[(y^2)^3 + 3y^2\right] dy$$

replace $x + ys$

combie simplify

$$\int_{-1}^{1} 6y^6 \, dy + y^6 + 3y^2 \, dy$$

note $(x^5)^2 = x^{10}$

$$\int_{-1}^{1} 7y^6 + 3y^2 \, dy$$

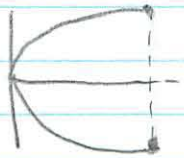
$$\frac{7y^7}{7} + \frac{3y^3}{3} \Big|_{-1}^{1}$$

$$y^7 + y^3$$

$$1^7 + 1^3 - \left[[-1]^7 + [-1]^3\right]$$

$\cancel{0}$ 4 addition error

ii) by using path independence to replace C w/ a simpler path



so new path is 2

$$\int \vec{F}\cdot d\vec{r}$$

$$\underbrace{x^3 y + y^3}_{\text{what do I do w/ this?}} \cdot 2$$

They still have some formal process

Use y as parameter $C_2: \; x = 1 \quad dx = 0$

$y = y$

$$\int_{-1}^{1} (1 + 3y^2)\, dy$$

← where all in world do they get this ?!?

$$y + y^3 \Big|_{-1}^{1}$$

④

From here need to describe path in terms of parameter

iii) By FTC for line integrals

$$\boxed{\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= f(x, y) \Big|_{P_0}^{P_1} \\ &= f(x_1, x_2) - f(x_0, x_0)\end{aligned}}$$

?? So how in all world does that work

$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)}$$

$A = (1, -1)$

$B = (1, 1)$

$$\int_C = (1 + 1) - (-1 - 1)$$

④ that was easy - but what did they do?

✓

will rest after Office hrs

2. Let $f(x, y) = x e^{xy}$
$C = 1/x$ from $(1,1)$ to $(0, \infty)$

So gradient field means constant/conservative field = means path independence

$f_x = 1 e^{xy}$ $\langle e^{xy}, 0 \rangle$
$f_y = 0$

? are these type of problems solved any different ?

Need a parameter
$y = 1/x$
$x = t$
$y = 1/t$

Now dx, dy
$dx = 1\ dt$
$dy = \frac{1}{-t^2}\ dt$ ✓ (they do x, but its comparable)

Ask: how do gradient field problems differ from other problems

Now write with this

$\int_C x e^{xy}\ dx$ ←??

<u>Wrong</u> $\nabla f = xye^{xy} + e^{xy}\uparrow + x^2 e^{xy}\nearrow$

? where in all world did they get this ??

$$\int_C \vec{F} \cdot d\vec{r}$$

$$\int_{-1}^{0} (e + e)\,dx + (x^2 e) \frac{-dx}{x^2}$$

So they must have done
written in gradient field

$$\int_{1}^{0} xye^{xy} + e^{xy} \uparrow dx + x^2 e^{xy} \uparrow dy$$

replaced w/ parameters

$$\int_{1}^{0} t \cdot \frac{1}{t} e^{t \cdot 1/t} + e^{t \cdot 1/t} dt + t^2 e^{t \cdot 1/t} \uparrow -\frac{1}{t^2} dt$$

$$\int_{1}^{0} e + e\,dt - t^2 e \cdot \frac{1}{t^2} dt$$

$$\int_{1}^{0} 2e\,dt - e\,dt$$

$$\cancel{\int_{1}^{0} e\,dt}$$ can't do

$$2et - et \Big|_{1}^{0}$$

$$-[2e - e]$$

$$-e$$ ✓

Once I got started, I got it
One more like this

3. $f(x,y) = \sin x \cdot \cos y$

$f_x = \cos x \cos y$
$f_y = -\sin x \sin y$

$\nabla f = \langle \cos x \cos y, -\sin x \sin y \rangle$ ✓

$x = \sin x$
$y = \cos y$

e ? Is that right
- do they have to work out to something

$dx = \cos x$
$dy = -\sin y$

$\int_C \cos x \cos y \, dx + -\sin x \sin y \, dy$

$\int_C \cos(\sin x) \cos(\cos y) \cdot \cos x - \sin(\sin x) \sin(\cos y) - \sin y$

$\int_C \cos(\sin x) \cdot \cos^2 y \cdot \cos x + \sin^2 x \sin(\cos y) \sin y$

$\int_C \cos(\sin x) \cdot \cos x + \sin(\cos y) \sin y$

No limits given
∴ Is that correct - can't go further

Path independent of course

$\int \sin x_1 \cos_1 - \sin x_0 \cos y_0$

FTC of line integrals
↳ oh the really easy way to do it

The difference on right hand side is maximized if $\sin x_1 \cos y_1$ is maximized and $\sin x_0 \cos y_0$ is minimized - yeah <u>duh</u>

Since $|\sin x \cos y| = |\sin x||\cos y| \leq 1$
max difference is $1 - -1 = 2$

↳ Running from $\left(-\frac{\pi}{2}, 0\right)$ to $\left(\frac{\pi}{2}, 0\right)$ ✓

Lecture 22 <u>Criterion for conservative fields, potential functions</u>

4C-5a via both methods 1 + 2

For each of the following tell for what value of the constants the field will be a gradient field and for this value find a corresponding mathmatical potential function.

$$\vec{F} = (y^2 + 2x)\hat{i} + axy\hat{j} = \vec{\nabla} f$$

$\vec{F}$ is gradient field if $M_y = N_x$

$M_y = 2y$ $\quad$ $2y = ay$

$N_x = ay$ $\quad$ $a = 2$

Ok so now we know what a is

$$\vec{F} = (y^2 + 2x)\hat{i} + 2xy\hat{j}$$

Potential function $= f(x,y) = xy^2 + x^2 + C$

what is?

that long formula?

or

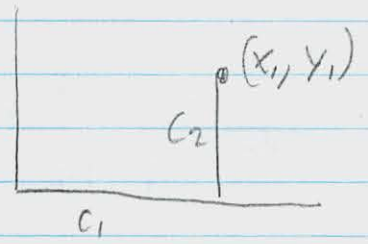
this is all method 1

$$f(P_1) = f(P_0) + \int_{P_0}^{P_1} M\,dx + N\,dy$$

$$= f(P_0) + \underbrace{\int_{P_0}^{P_1} y^2 + 2x\,dx + 2xy\,dy}_{\text{need to find this}}$$

- can graph

Gradient field so path independent



$$= \int_{C_1} (x^2+y^2)\,dx + 2xy\,dy + \int_{C_2} (x^2+y^2)\,dx + 2xy\,dy$$

$y=0 \quad dy=0 \qquad\qquad x=x_1 \quad dx=0$

$$\int_0^{x_1} x^2\,dx + \int_0^{y_1} 2x_1 y\,dy$$

$$\frac{x^3}{3}\Big|_0^{x_1} + \frac{2x_1 y^2}{2}\Big|_0^{y_1}$$

$$\frac{x_1^3}{3} + x_1 y_1^2$$

? should be

$$x^2 + xy^2 + C$$

—close, but why not exact

New Method 2

$$f = \int y^2 + 2x \; dx$$

$$f = xy^2 + x^2 + \boxed{g(y)}$$

$$f = \int 2xy \; dy$$

$$f = xy^2$$

$$f_y = 2x + g'(y)$$

$$g'(y) = y$$
$$g(y) = \frac{y^2}{2}$$

$$f(x,y) = xy^2 + x^2 + \frac{y^2}{2}$$

✓ checks out

cool

like method 2 better - faster ✓

6a. Decide which of the following differentials is exact. For the one that is exact, express it in terms of df.

$$y\,dx - x\,dy$$

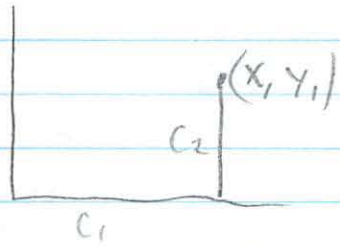
so is it $\nabla f = \langle y, -x \rangle$

Method 1

$$f(P) = f(P_0) + \int_{P_0}^{P} M\,dx + N\,dy$$

$$f(P_0) + \int_{P_0}^{P} y\,dx - x\,dy$$

Now need to draw a path independent way



Can always use this method.

$$\int_{C_1} y\,dx - x\,dy + \int_{C_2} y\,dx - x\,dy$$

$x=0 \quad dy=0$ $\qquad$ $x=x_1 \quad dx=0$

$$0 + \int_0^{y_1} -x_1\,dy$$

Now integrate

$$-x_1 y \Big|_0^{y_1}$$

$$\boxed{-x_1 y_1}$$

actually that is easier to figure out, but more writing

<u>Method 2</u>

$f_x = y$
$f_y = -x$

$f = xy + g(y)$
$f = -x + g(y)$

$g(y) = y$
$g'(y) = 1$

$$\boxed{f = xy + 1}$$

$y\,dx - x\,dy$ is not exact
since $M_y = 1$ $N_x = -1$

(so that is why I had sign problems)

So is it just unanswerable?

6b. $y(2x+y)\,dx + x(2y+x)\,dy$

<u>Method 1</u>

$$f(P) = f(P_0) + \int_{P_0}^{P} M\,dx + N\,dy$$

$$\langle y(2x+y),\ x(2y+x) \rangle$$

$$\int_{C_1} y(2x+y)\,dx + x(2y+x)\,dy + \int_{C_2} y(2x+y)\,dx + x(2y+x)\,dy$$

$y=0$ $dy=0$ $\qquad$ $x=x_1$ $dx=0$

$$0 + \int_0^{y_1} 0 + x_1(2y+x_1)\,dy$$

$$\int_0^{y} 2x_1 y + x_1^2\,dy$$

$$\frac{2x_1 y^2}{2} + x_1^2 y \Big|_0^{y_1}$$

$$\boxed{x_1 y_1^2 + x_1^2 y_1 + c}$$ ✓ Perfect

Method 2

$$\langle y(2x+y),\ x(2y+x) \rangle$$

This will be complicated

$$f = \int y(2x+y)\,dx$$

$$f = x(x+y)y + g(y)$$

$$f_y = x(2y+x)$$
$$f = y(y+x)x$$
$$= x(x+y)y$$

How is this going to go?

$$x^2 + xy + xy + y^2 + g(y)$$
$$x^2 + 2xy + y^2 + g(y)$$

- Not close to what I got above

Should do more pratice to memorize test cases

Did not check if exact
$M_y = 2x + 2y = N_x$

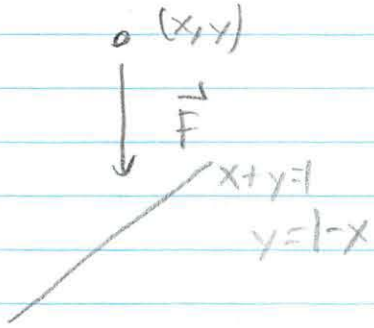
↑ But what went wrong here?

✓

Part 2

(working outside!)

1a. Write down in x-y coordinates the vector field $\vec{F}$ whose vectotor at $P(x,y)$ runs from P to $L \ni x+y=1$



So we are going from any point → down
but the vector is not F
instead it is a curve C which runs vertically

$\int_C \vec{F} \cdot d\vec{r}$

r is the curve

$- \langle 0, y \rangle$

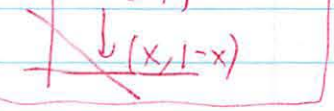
✓

oliver ok!

$\langle 0, y_0 - (1-x) \rangle$

from y_0 to $1-x$

$\boxed{F = \langle 0, 1-x-y \rangle}$



Or is it just a field - which has magnitude + direction

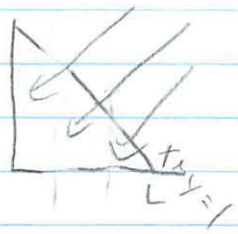
What would magnitude be

$\sqrt{0^2 + (y_0 - (1-x))^2}$

$y_0 - (1-x) \langle 0, y_0 - (1-x) \rangle$

b. Show that w/ a calculation - for this field

$$\int_C F \cdot dr \neq 0 \text{ if } C \text{ is a right triangle}$$



-1.5

Well the field is non constant. It is dependent on x in the calculation of it. This means that the line integral along a closed path is not nessarily 0.

? should we write any thing else
- there are parts which are perpendicular to the field - these don't count at all
only the degree each line is parallel counts (dot product) There are segments of the line which meet this criteria

? alright

2. Imagine the x-axis is an ∞ long heavy wire of uniform density. The gravitational force exerted on it is

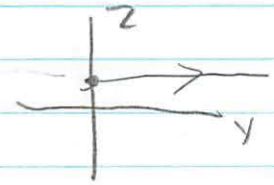
Oliver
office
Fix

$$F(y,z) = \frac{-c(y\hat{j} + z\hat{k})}{y^2+z^2} \qquad (y,z) \neq (0,0)$$

c is a positive constant $= 1$

Find line integral - by using a parametrization of C - the work done by moving field along some path in yz plane

a) half-line $z=1$ $y \geq 0$ start $(0,1)$



$$\vec{r} = \langle x, 0 \rangle \times [0, \infty)$$

$y = t \qquad dy = 1$
$z = 1 \qquad dz = 0$

$$\int_0^\infty \vec{F} \cdot \vec{T}$$

$$\frac{\langle -y, -z \rangle}{y^2+z^2} \cdot \langle 1, 0 \rangle$$

$$\frac{\langle -1, -1 \rangle}{t^2} \cdot \langle 1, 0 \rangle$$

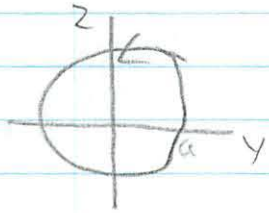
$$\frac{-t}{t^2} \cdot 1$$

$$\int_0^{\infty} \frac{-1}{t}\, dt \rightarrow -\ln(t) \Big|_0^{\infty}$$

$$-\ln(\infty) + \ln(0)$$

$-\infty$ ✓ not closed loop ✓

b) circle w/ radius a, centered at origin, counterclockwise



so this one should be cos, sin

$$a^2 = y^2 + z^2$$
$$a = \sqrt{y^2 + z^2}$$

$y = \cos t \qquad dy = -\sin t$
$z = \sin t \qquad dz = \cos t$

$$\int_0^{2\pi} \frac{\langle -y, -z \rangle}{y^2 + z^2} \cdot \langle dx, dy \rangle$$

$$\int_0^{2\pi} \frac{\langle -\cos t, -\sin t \rangle}{\cos^2 t + \sin^2 t} \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$(-\cos t \cdot -\sin t) + (-\sin t \cos t)\, dt$$

$$\int_0^{2\pi} -2 \sin t \cos t\, dt$$

$$+\cos 2t \Big|_0^{2\pi}$$

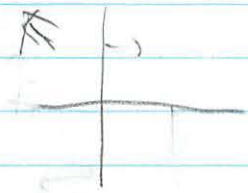
$$\cos^2(2\pi) - [\cos^2(0)]$$

$$1 - 1$$

$$0$$

✓

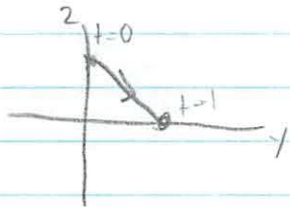
Closed loop - field prob constant
Something about field being tangent



- in that model of betor

$$\text{dir}(-\hat{y}, \hat{z})$$

c) line from $(0,1)$ to $(1,0)$



$$r = \frac{\langle 0-1 \rangle}{\langle 1-0 \rangle} = \langle \hat{y}, -\hat{z} \rangle$$

$[0,1]$

~~$y = t$~~
~~$z = t+1$~~ ; what is parameter

~~C_1~~
~~parameter~~
oliver OH

$y = t$
$z = -t+1$

✓ bingo (took like 10 min to find)
hate these problems

$dy = 1$
$dz = -1$

$$\int \frac{\langle -y, -z \rangle}{y^2+z^2} \cdot \langle dy, dz \rangle$$

$$\int_0^1 \frac{\langle t, -t+1 \rangle}{t^2+(-t+1)^2} \cdot \langle 1, -1 \rangle \, dt$$

$$t^2 + (-t)^2 - 2 \cdot 1 \cdot t + 1^2$$
$$t^2 + t^2 - 2t + 1$$
$$2t^2 - 2t + 1$$

$$\int_0^1 \left(\frac{t}{2t^2-2t+1} \cdot 1 \right) + \left(\frac{-t+1}{2t^2-2t+1} \cdot -1 \right) dt$$

$$\int_0^1 \frac{t + t - 1}{2t^2-2t+1} \, dt$$

$$\int_0^1 \frac{2t-1}{2t^2-2t+1} \, dt$$

$$\frac{-\ln(2t^2-2t+1) + \cancel{2\tan^{-1}(2t-1)}}{4} \Big|_0^1$$

-0.5

$\cancel{\frac{\pi}{4}}$ 0

Should be same for a, b, c

3a Let $\vec{F} = -\nabla(\ln r)$ where r is the distance from the origin in the y,z plane. Show that F is field of problem 2 when $c=1$

So we need to show $M_y = N_x$
or $M_z = N_y$ in this case

$$\vec{F} \quad \frac{y\hat{j} + z\hat{k}}{y^2+z^2}$$

$$\vec{F} \quad \left\langle \frac{y}{y^2+z^2}, \frac{z}{y^2+z^2} \right\rangle$$

$$y(y^2+z^2)^{-1}$$

$$\boxed{M_z = \frac{-2zy}{(y^2+z^2)^2} \qquad N_y = \frac{-2y \cdot z}{(y^2+z^2)^2}}$$

know already →

they are the same
So is a gradiant field

Or did I screw this up?
M and N are from $\vec{\nabla} f = \langle M, N \rangle$
but I calculated them
-so its ok - should be good

Oliver OH

; Wait am I good?

~~take gradient~~ $\cancel{\left\langle \frac{-(y^2-z^2)}{(y^2+z^2)^2}, \frac{-(z^2-y^2)}{(y^2+z^2)^2} \right\rangle}$

should $= \frac{\ln r}{r = \sqrt{x^2+z^2}}$

$$\cancel{\left\langle \frac{-y^2+z^2}{r}, \frac{-z^2+y^2}{r} \right\rangle}$$

Want $\nabla(\ln r)$

$\nabla(\ln \sqrt{y^2+z^2})$

g_y $\qquad$ g_z

$$\left\langle \frac{y}{y^2+z^2}, \frac{z}{z^2+y^2} \right\rangle \checkmark$$

solved now

matches $\vec{F}$ ✓

$\vec{F}$ = field

f = function

$\vec{F} = \nabla(f)$

$$\left\langle \frac{y}{r^2}, \frac{z}{r^2} \right\rangle$$

3b Show that for any path C joining 2 points P_1 and P_2 in the x, y plane the value of $\int_C F \cdot dr$ depends on $\frac{r_2}{r_1}$ where r_i is distance from origin.

What pts P_i should be excluded?

↳ this first

P_0 which is $(0,0)$ should be ✓

Well FTC (line integrals) says that

$$\int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = f(y_2, z_2) - f(y_1, z_1)$$

where gradient field

Well if you plug points in

$$y_2\ z_2 \rightarrow \ln\sqrt{y_2^2 + z_2^2} = \ln r_2$$

$$y_1\ z_1 = \ln\left(\sqrt{y_1^2 + z_1^2}\right) = \ln r_1$$

$$= \ln(r_2) - \ln(r_1)$$ ✓

$$= \frac{r_2}{r_1}$$

rule of ln

$\ln a + \ln b = \ln ab$

$\ln a - \ln b = \ln \frac{a}{b}$

r_i

am I getting confused?

oliver OH fixed self

c) Let C be a directed line segment of lenght 1

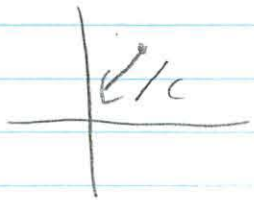
i) If $\int_C \vec{F} \cdot d\vec{r} = 0$, how is C placed?

C must be placed perpendicular to the field ✓

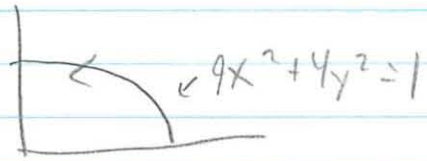
ii) What position of C gives $\int_C \vec{F} \cdot d\vec{r}$ the largest value?

In the direction of the field, parallel with it

-0,5



4. Let $\vec{F} = \nabla(x^2 y - 3xy^2)$ Let C be 1/4 ellipse $9x^2 + 4y^2 = 1$ from x → y axis



a) Write integral $\int_C \vec{F} \cdot d\vec{r}$ in the $\int_C M\,dx + N\,dy$ form

$f_x = 2xy - 3y^2$ copy error

$f_y = x^2 - 3x \cdot 2y$

$\vec{\nabla} f = \langle 2x - 3y^2, x^2 - 6xy \rangle$

$\int_C (2x - 3y^2)\,dx + (x^2 - 6xy)\,dy$

no parametrizing done, right?

correct - oliver oh

b) Evaluate using FTC + Path Independent

FTC → $f(x_2, y_2) - f(x_1, y_1)$

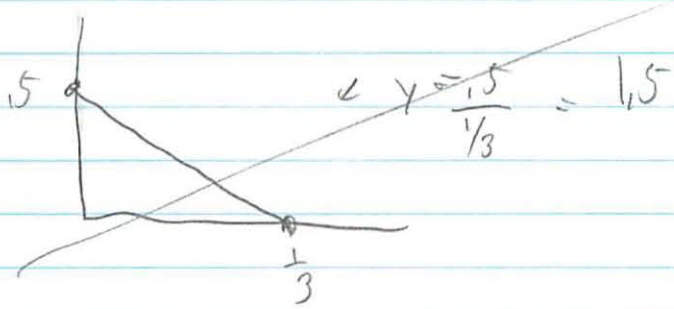
$x_2^2 y_2 - 3x_2 y_2^2 - [x_1^2 y_1 - 3x_1 y_1^2]$

$x_2 = 0$ $x_1 = 1/3$ $0 - 0 - 0 - 0$

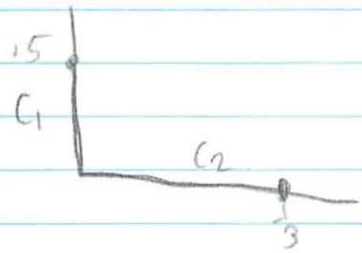
$y_2 = 15$ $y_1 = 0$ 0 ?

Capital + lower f confusing me - see chart a few pg back

b) Path independence



Lets just split it



$$\int_{C_1} (2xy - 3y^2)\,dx + (x^2 - 6xy)\,dy + \int_{C_2} (2xy - 3y^2)\,dx + (x^2 - 6xy)\,dy$$

Can keep dx?
- pretty sure

$x = 0 \quad dx = 0$ $\qquad$ $y = 0 \quad dy = 0$

$$\int_{C_1} 0 + 0\,dy + \int_{C_2} \cancel{2xy}\,dx + 0$$

0

$$\cancel{\int_0^{1/3} 2x\,dx \quad \frac{2x^2}{2}\Big|_0^{1/3} \quad \frac{1}{6}}$$

0 ✓ copy error in part a

$\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$

5. $F = \underbrace{xy}_{M}\,\hat{i} + \underbrace{x^2}_{N}\,\hat{j}$

a) Verify $\vec{F}$ is not conservative

FYI

$\boxed{\# \; M_y = f_{xy} \; \#}$

You can see that x and y are terms of the field this means it is not conservative

$\cancel{f_x = y\,\hat{i} + 2x\,\hat{j}}$

$\cancel{f_{xy} = 1\,\hat{i}}$

But they are =

help Kimberly

$\cancel{f_y = x\,\hat{i}}$

$\cancel{f_{yx} = 1\,\hat{i}}$

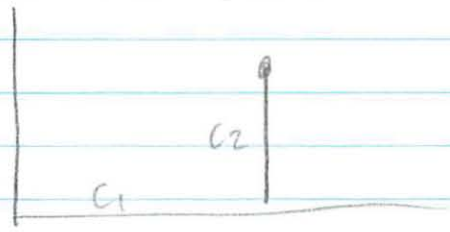
$M = f_x$

You can take the gradient of a vector field w/o it being a gradient field

So $M_y = x$
$N_x = 2x$ $\hat{j} \neq$ ✓

b) Try to find a potential function for $\vec{F}$ by using method 1. What goes wrong?

Potential Function $= \int_{P_0}^{P} M\,dx + N\,dy$



$$\int_{C_1} xy\,dx + x^2\,dy + \int_{C_2} xy\,dx + x^2\,dy$$

$y=0 \quad dy=0 \qquad\qquad x=x_1 \quad dx=0$

$$0 + \int_0^{x_1} 0 + x^2\,dy$$

$$x^2 y \Big|_0^{y_1}$$

$$x^2 y$$

✓

¿ - But I found it ¿ What was supposed to go wrong? (Now, if you take $\nabla(x^2y)$, you don't get F)

probably blew past some assumption I was supposed to make

why is it working

but as I am doing the integral it is not an even field

Or is it in the way I write the integral w/ $M\,dx + N\,dy$

Yeah I assumed

$$\vec{F} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$ which was invalid

- only true for gradient fields

But then why did we bother trying it?

Or is it that result = variable ← but that is ok ¿

c. Now Method 2
- this will prob fail more

$f_x = xy$
$f_y = x^2$

$f_x = \int xy \, dx$
$f = x^2 y + g(y)$

$f_y = x^2 +$
$= x^2 + g'(y)$

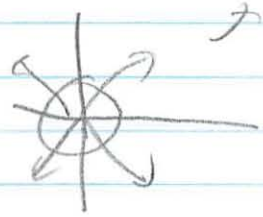
$g'(y) = 0$
$g(y) = c$

$f(x,y) = x^2 y + c$

same answer

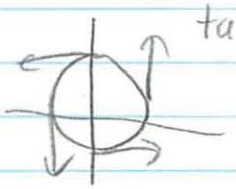
Why is it working? It's not, again $\nabla(x^2y)$ is not F

6. The most 2D fiels whose direction is radially outward and whose magnitude depends only on r $(r=\sqrt{x^2+y^2})$ is



$$\vec{F} = f(r)(x\hat{\imath} + y\hat{\jmath})$$

↑ varries magnitude



tangent

a) Assuming $f(r)$ is differentiable for all r, show that any such $\vec{F}$ is conservative

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r}$$

Well $f(r)$ just disappears. coefficent - No, indirectly depends on x, y

-1.5

$$\frac{\partial F}{\partial r} = \frac{\partial x}{\partial r} + \frac{\partial y}{\partial r}$$

- but how does that use our hint?

well $\vec{F}$ is conservative if $\vec{F} = \nabla f$

or $f_x = 1\hat{\imath}$
$f_y = 1\hat{\jmath}$

$\langle 1, 1 \rangle$

$M_y = 0$ ✓
$N_x = 0$

or $f_x = \frac{\partial f}{\partial x}$

want $\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} = \frac{\partial f}{\partial r}$

or is it

so it would be

$h = \langle 1 \cdot \frac{r}{x}, 1 \cdot \frac{r}{y} \rangle$

And check

$h_y = 0$ still same?
$h_x = 0$

?? ¿does that answer the question?

b) Find a potential function of the form $g(r)$ if $f(r) = r^n$ where n is ⊕ ⊖ integer

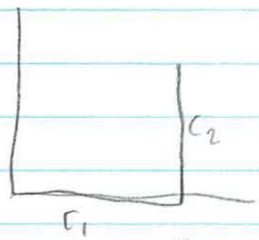
(Hint: Solve $\nabla g(r) = \vec{F}$ for the function $g(r)$. One value of n has to be handled differently from all of the others) so ∫ back

-1.5

see solu

$$g(r) = \int_{C_1} r^n x\,dx + r^n y\,dy + \int_{C_2} r^n x\,dx + r^n y\,dy$$

$y=0 \quad dy=0 \qquad\qquad x=x_1 \quad dx=0$



$$\int_{C_1} \sqrt{x^2+y^2}^{\,n} x\,dx + \sqrt{x^2+y^2}^{\,n} y\,dy$$

$$+ \int_{C_2} \sqrt{x^2+y^2}^{\,n} x\,dx + \sqrt{x^2+y^2}^{\,n} y\,dy$$

$$\int_0^{x_1} x^{n+1}\,dx + \int_{C_2} \sqrt{x_1^2+y^2}^{\,n} y\,dy$$

$$\frac{x^{n+2}}{n+2}\Big|_0^{x_1} + \int_0^{y_1} (x^2+y^2)^{n-1} y\,dy$$

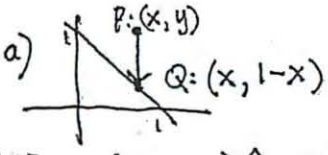
$$\frac{x^{n+2}}{n+2} + \frac{(y^2+x^2)^n}{2n}\Big|_0^{y_1}$$

Well when $n-2=0$ it will be different since $n=2$ the answer will always be 1. Is this what is different?

what is diffent

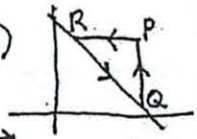
Solutions – Prob. Set 6B

1 a)



$\vec{F} = \vec{PQ} = (1-x-y)\hat{j}$ OR
$= \langle 0, 1-x-y \rangle$

b)

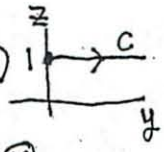


A typical C is shown

on $\vec{PR}$, $\vec{F}\cdot\vec{T} = 0$ since $\vec{F} \perp \vec{PR}$
on $\vec{RQ}$, $\vec{F}\cdot T = 0$ since $\vec{F} = \vec{0}$ on $\vec{RQ}$
On $\vec{QP}$, $\vec{F}\cdot\vec{T} < 0$ since $\vec{F}$ and $\vec{QP}$ have opposite directions.

$$\therefore \int_C \vec{F}\cdot\vec{T}\,ds = \int_{\vec{QP}} \vec{F}\cdot\vec{T}\,ds < 0.$$

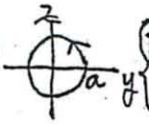
2 a)



Parametrizn: $\begin{cases} z = 1 & (\text{so } dz = 0) \\ y = y & (\text{or } y = t) \end{cases}$

$$\text{Work} = -\int_C \frac{y\,dy + z\,dz}{y^2+z^2} \overset{(*)}{=} -\int_0^\infty \frac{y\,dy}{y^2+1}$$
$$= -\tfrac{1}{2}\ln(y^2+1)\Big]_0^\infty = -\infty$$

b)

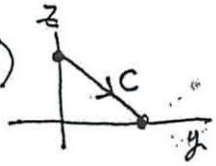


$\begin{cases} y = a\cos t; & dy = -a\sin t\,dt \\ z = a\sin t & dz = a\cos t\,dt \end{cases}$

Using same integral as in (a), (*)

$$\text{Work} = -\int_0^{2\pi} \frac{-a^2\sin t\cos t + a^2\sin t\cos t}{a^2}\,dt$$
$$= -\int_0^{2\pi} 0\,dt = 0.$$

c)



Use: $y = y$, $dy = dy$
$z = 1-y$, $dz = -dy$

$$\text{Work} = -\int_0^1 \frac{y\,dy - (1-y)\,dy}{2y^2-2y+1} \quad [\text{using } (*)]$$
$$= -\int_0^1 \frac{(2y-1)}{2y^2-2y+1}\,dy$$
$$= -\tfrac{1}{2}\ln(2y^2-2y+1)\Big]_0^1 = 0-0 = 0.$$

3 a) $r = \sqrt{y^2+z^2}$

$$-\nabla \ln(y^2+z^2)^{1/2} = -\tfrac{1}{2}\nabla \ln(y^2+z^2)$$
$$= -\tfrac{1}{2}\left\langle \frac{2y}{y^2+z^2}, \frac{2z}{y^2+z^2} \right\rangle$$
$$= -\left(\frac{y\hat{i} + z\hat{j}}{y^2+z^2}\right)$$

b) Using the Fundamental Thm of Calculus for line integrals,

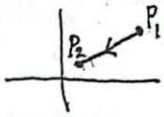
$$\int_{P_1}^{P_2} \vec{F}\cdot d\vec{r} = -\ln r\Big]_{P_1}^{P_2} = -[\ln r_2 - \ln r_1]$$
$$= -\ln(r_2/r_1), \quad \text{where } r_i = r(P_i).$$

Have to exclude pts where $r(P_i) = 0$, i.e., $P_i \neq (0,0)$.

c) (i) C is placed so that its endpoints are equally distant from (0,0).

(ii) There is no maximum value for $\int_C \vec{F}\cdot d\vec{r}$:

if we place C as shown (radially)



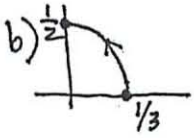
then $\frac{r_2}{r_1} \to 0$ as $P_2 \to (0,0)$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = -\ln\left(\frac{r_2}{r_1}\right) \to \infty \text{ as } P_2 \to (0,0).$$

4 a) $\nabla(x^2y - 3xy^2) = \langle 2xy - 3y^2, x^2 - 6xy \rangle$

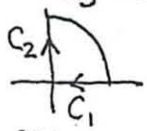
$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C (2xy-3y^2)\,dx + (x^2-6xy)\,dy$$

b)



Using the F.T.: $\int_C \vec{F}\cdot d\vec{r} = x^2y - 3xy^2\Big]_{(1/3,0)}^{(0,1/2)}$
$= 0 - 0 = 0$

Using the path:



C_1: $x = x$, $y = 0$ $\qquad$ C_2: $x = 0$, $y = y$

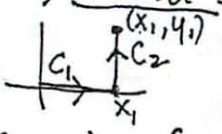
$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} 0\cdot dx = 0, \quad \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_2} 0\cdot dy = 0.$$

$\boxed{5}$ a) $\vec{F} = xy\hat{\imath} + x^2\hat{\jmath}$

$$\frac{\partial(xy)}{\partial y} = x, \quad \frac{\partial(x^2)}{\partial x} = 2x$$

Since these are not equal, the field is not conservative.

b) Method 1:



C_1: $y=0$ $x=x$

C_2: $x=x_1$, $y=y$

$$f(x_1, y_1) = \int_{C_1+C_2} xy\,dx + x^2\,dy$$

$$\int_{C_1} = \int_0^{x_1} 0\,dx = 0$$

$$\int_{C_2} = \int_0^{y_1} x_1^2\,dy = x_1^2 y_1$$

(since $dx=0$ on C_2)

$\therefore f(x_1, y_1) = x_1^2 y_1$, or $\boxed{f = x^2 y}$

but $\nabla(x^2 y) = 2xy\hat{\imath} + x^2\hat{\jmath} \neq \vec{F}$

Method 2:

$$\frac{\partial f}{\partial x} = xy \quad (\text{from } \vec{F})$$

$$\therefore f = \tfrac{1}{2}x^2 y + g(y)$$

$$\therefore \frac{\partial f}{\partial y} = \tfrac{1}{2}x^2 + g'(y)$$

$$= x^2 \quad (\text{from } \vec{F})$$

But $x^2 \neq \frac{1}{2}x^2 + g'(y)$ for any $g(y)$, since

$$\underbrace{\tfrac{1}{2}x^2}_{\text{function of } x \text{ alone}} \neq \underbrace{g'(y)}_{\text{function of } y \text{ alone}}$$

$\boxed{6}$ $\vec{F} = f(r)(x\hat{\imath} + y\hat{\jmath})$

a) Using $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$:

$$\frac{\partial}{\partial y} f(r)x = f'(r)\frac{\partial r}{\partial y} \cdot x = f'(r) \cdot \frac{yx}{r}$$

$$\frac{\partial}{\partial x} f(r)y = f'(r)\frac{\partial r}{\partial x} \cdot y = f'(r) \cdot \frac{xy}{r}$$

Since these are equal, $\vec{F}$ is conservative

b) $\nabla g(r) = r^n(x\hat{\imath} + y\hat{\jmath})$

⊛ $\therefore g'(r)\frac{\partial r}{\partial x} = g'(r) \cdot \frac{x}{r} = r^n x$

⊛ $g'(r)\frac{\partial r}{\partial y} = g'(r) \cdot \frac{y}{r} = r^n y$

These will be both satisfied if

$\frac{g'(r)}{r} = r^n$, so $g'(r) = r^{n+1}$

$g(r) = \boxed{\frac{r^{n+2}}{n+2}}$.

[if $n = -2$,

$g'(r) = \frac{1}{r}$, so $g(r) = \boxed{\ln r}$

Lecture 23

Green's Therm

4/8

- next 3 lectures

Hypothesis: - simple curve, does not overlap (self intersection)
- no end points "closed"
- piecewise, smooth $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$
- differentiable except at certain points + shapes

the curve divides plane into 2 parts
- ∞ large area exterior
- enclosed region, finite area R
  - hard to prove, but we don't care

curve must be oriented in a positive oriented
- aka go counterclockwise
- but sometimes that is not clear
- so the interior is always on your left

$\vec{F} = M\hat{i} + N\hat{j}$
↳ must be differentiable in R

Green's therm relates the line integral around the curve

$\oint$ ← closed path

$\oint M\,dx + N\,dy$ - could be work integral

Green's Theorm

$$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\cdot dA$$

$$\text{curl}\,\vec{F} = N_x - M_y$$

ex $\oint 3x^2y^2\,dx + 2x^2(1+xy)\,dy$

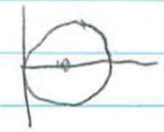
doing $\iint$ may be hard

So try Green's theorm

$$\iint_R \underbrace{4x + \cancel{6x^2y}}_{N_x} - \left[\cancel{6yx^2}\right]\,dA$$

$$4\iint_R x\,dA$$

- could use com or avg value over

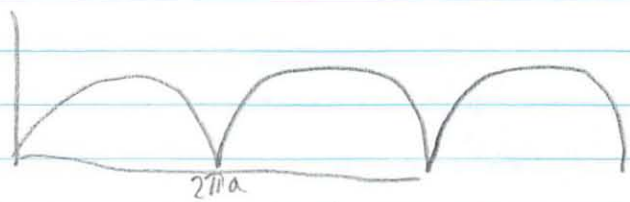


$$\text{Average of } x = \frac{\iint_R x\,dA}{\text{avg } R = \pi a^2}$$

$$\iint_R x\,dA = \underbrace{(\text{avg of } x)}_{a} \cdot \pi a^2$$

$$= 4\pi a^3$$

Some are more easily calculated as a line integral



cycloid

did by vector methods before →

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases}$$ deriv of

Area under 1 arch

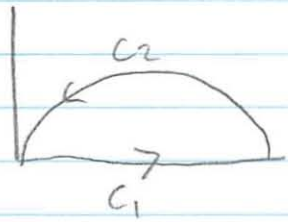
Area by line integral

$$-\int y\,dx = \iint_R 1\,dA$$

Want $N_x - M_y = 1$

$\int x\,dy$ or $\frac{1}{2}$ (both) ← what is this for??

$$\oint_{C_1+C_2} -y\,dx = \int -y\,dx + \int_{C_2} -y\,dx$$



$$0 \qquad -(a(1-\cos\theta))$$

$$dx = y\,d\theta$$

$$= -\int_0^{2\pi} a^2(1-\cos\theta)^2\,d\theta$$

calculate

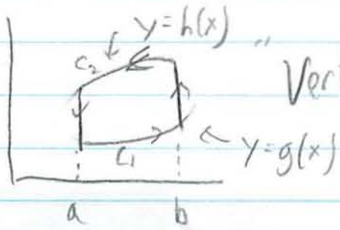
$$= -a^2\int_{2\pi}^{0} (1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$\downarrow \quad + \quad \downarrow$$

$$2\pi + 0 + \pi$$

$$\boxed{3\pi a^2}$$

3 times the area of the circle
- very mysterious when it came out
- important
- not obvious



"Vertically simple" R
entering + leaving once each

Proof

- calculate both sides of the theorm
- double integrals + line integrals

$$\oint_C M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dA$$

On C_2, C_4 - vertical lines will contribute nothing to line integral since $x = a$ or $x = b$ and $dx = 0$

$$\int_{C_1} M\,dx + \int_{C_2} M\,dx$$

$C_1: x = x$, $y = g(x)$ can change to t - but harder

$$\int_a^b M(x, g(x))\,dx + \int M(x, h(x))\,dx$$

$$\oint M\,dx = \int_a^b \left[M(x, g(x)) - M(x, h(x))\right] dx$$

$$\iint_R \frac{\partial M}{\partial y}\,dy\,dx = \int_a^b \int_{g(x)}^{h(x)} \frac{\partial M}{\partial y}\,dy\,dx$$

<u>Inner</u> $M(x,y)\Big|_{g(x)}^{h(x)}$

$= M(x, h(x)) - M(x, g(x))$

<u>Outer</u>
= same thing

So same answer

Lecture 24

Flux, Green's Theorem in normal form

4/4

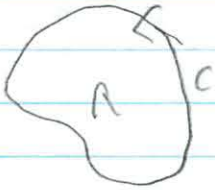
Review

$\vec{F} = M\hat{i} + N\hat{j}$

force field

$\oint \vec{F} \cdot \hat{t}\, ds$

unit tangent



tangential form

$$\boxed{= \oint \vec{F} \cdot d\vec{r} = \iint_R (\text{curl } \vec{F})\, dA}$$

$$\oint M\,dx + N\,dy \qquad \iint_R (N_x - M_y)\, dA$$

Green's Theorem

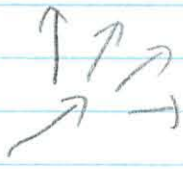
work energy

↑ why??

but how does path relate to inside of region

$\vec{F}$ velocity field

- less common in 1st year
- common in fluid mechanics



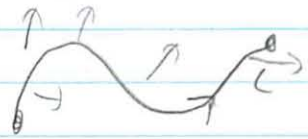
velocity of incompressible fluid

constant over time
- not a function of t

$\vec{F}(x, y)$

Flux

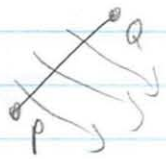
- analog to work



- net flow rate across C
  - cm^2/sec
  - face into C
  - ⊕ flux is left to right
    ⊖ " " right to left

- net = ⊕ − ⊖ flows

a)

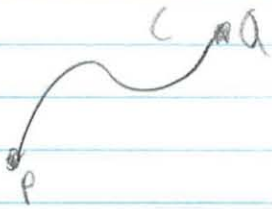


$\vec{F}$ is constant
- does not vary w/ position

resolve F into 2 components
- perpendicular + parallel to lines
- only the perpendicular $\vec{F}$ counts as flux
- in the $\hat{n}$ (unit normal) direction

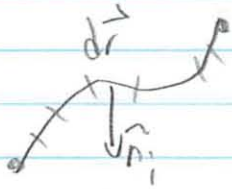
$$\text{Flux across } C = (\vec{F} \cdot \hat{n})\,|PQ|$$

b)



but now $\hat{n}$ keeps changing direction
magnitude still the same though

Divide curves into little pieces



$$\sum_{i=1}^{n} \vec{F}(x_i, y_i) \cdot \Delta r_i = \text{total flux}$$

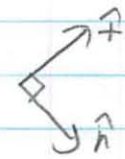
now as n (# of pieces) $\to \infty$

$$= \int_C \left(\vec{F} \cdot \hat{n}\right) ds$$

usual notation - not all may agree

flux across C

$\hat{T} = \left\langle \frac{dx}{ds}, \frac{dy}{ds} \right\rangle$ unit tangent vector



$\hat{n} = \left\langle -\frac{dy}{ds}, \frac{dx}{dt} \right\rangle$

Note its on the other side as compared to start of year

- 2 different conventions for seperate problems

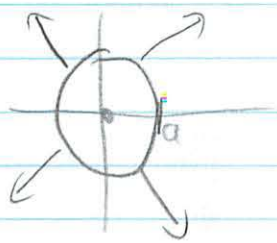
$$\int_C \vec{F} \cdot \hat{n} \, ds$$

flux

$$= \int_C \left(\frac{M\,dy}{ds} - \frac{N\,dx}{ds} \right) ds$$

But can never use s as a parameter
- only in formulas

Symbolic flux integral $= \int_C M\,dy - N\,dx$



$\vec{F}$ is always out

$\vec{F} = x\hat{i} + y\hat{j}$

Flux over C

$x = a\cos\theta$ $\quad$ $dx = -a\sin\theta$

$y = a\sin\theta$ $\quad$ $dy = a\cos\theta$

$\vec{F} = x\hat{i} + y\hat{j}$

$$\text{flux over } c = \int_C x\,dy - y\,dx$$

$$= \int_0^{2\pi} \left(a^2\cos^2\theta + a^2\sin^2\theta\right) d\theta$$

$$= \underset{\substack{\uparrow \\ \text{lenght } O \\ \text{circle}}}{2\pi a} \cdot \underset{\substack{\uparrow \\ \text{velocity} \\ \text{flow}}}{a} = 2\pi a^2$$

Green's theorm → flux
"normal form of Green's theorm"

* notice the order * ☺

$$\oint_C -N\,dx + M\,dy = \iint_R M_x - (-N)_y \, dA$$

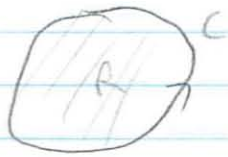
applied to this case
- only 1 Green's theorm, but multiple meanings

$$\oint M\,dy - N\,dx = \iint (M_x + N_y)\, dA$$

$$\oint F \cdot \hat{n}\, ds = \iint_R (\underbrace{\operatorname{div} \vec{F}}_{\text{divergence}})\, dA$$

Flux over C = ?? have to know divergence
- but is physical interpertation

So what is div?
- $M_x + N_y$
- but what does it mean?

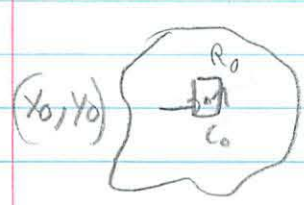


$f(x,y) = \operatorname{div} \vec{F}$

$f(x_0, y_0) = ??$

$$\oint_C \vec{F} \cdot \hat{n}\, ds = \operatorname{div}(x\hat{i} + y\hat{j})$$

$$= 2 \quad \text{why?}$$



By Green

$$f(x,y) = \text{div } \vec{F}$$
$$f(x_0,y_0) = ??$$
$$f(x,y) \approx f(x_0,y_0)$$

By Greens

$$\oint_{C_0} \vec{F} \cdot \hat{n} \, ds = \iint_{R_0} f(x_0,y_0) \, dA$$
$$= f_{\text{div}}(x_0,y_0) \text{ area. } R_0$$

$$(\text{div } \vec{F})_{(x_0,y_0)} = \frac{\text{net flux across } C_0}{\text{Area of } R_0}$$
$$\parallel$$
$$f(x_0,y_0)$$

= rate (cm^2/sec) to witch fluid is added to tank at (x_0, y_0) (or $\ominus$ if leaking)

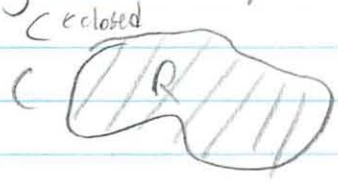
Recitation

4/12

Lectures

Greens Theorm

$$\oint_{C \text{ closed}} M\,dx + N\,dy = \iint_R N_x - M_y \, dA$$



Consequences for fields

"work" $\oint_C \vec{F} \cdot dr = \iint_R \text{curl } \vec{F} \, dA$

"flux" $\oint_C \vec{F} \cdot \hat{n} \, ds = \iint_R \text{div } \vec{F} \, dA$

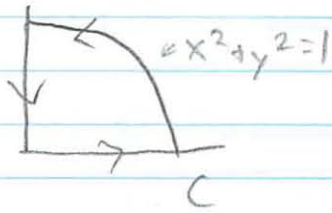
$dr \equiv \langle dx, dy \rangle = \langle x', y' \rangle \, dt$

$\hat{n} \, ds = \langle dy, -dx \rangle = \langle y', x' \rangle \, dt$

$\hat{n}$ = normal unit vector

ex 1 Compute a line integral in 2 different ways

$\vec{F} = x\hat{\imath} + xy\hat{\jmath}$



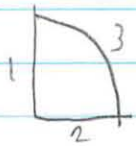
did not start Pset due to cplx so very bad at this

Compute $I = \oint_C \vec{F} \cdot d\vec{r}$

a) directly
b) using Green's Theorm

So need to cut the curve into 3 pieces

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} + \int_{C_2} + \int_{C_3}$$



C_1 $\int_{C_1} \vec{F} \cdot d\vec{r} = 0$ because $x = 0$ on C_1
So $f = 0$ so $\int_{C_1} F \cdot dr = 0$

C_2 $\int_{C_2} \vec{F} \cdot d\vec{r}$ = need to parametrize

$x(t) = t$ or $x(x) = x$
$y(t) = 0$ $\quad$ $y(x) = 0$

↑ Oliver TA $\quad$ ↑ Prof. Mattick

$d\vec{r} = \langle x', y' \rangle dt \quad t \in [0,1] \quad x \in [0,1]$
$\langle 1, 0 \rangle dt$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=1} \langle t, 0 \rangle \cdot \langle 1, 0 \rangle dt$$

$$= \int_0^1 t\, dt$$

$$\frac{t^2}{2} \Big|_0^1$$

$$\frac{1}{2}$$

C_3 $\int_{C_3} F \cdot dr$ = needs parametrization

$x(t) = \cos t$
$y(t) = \sin t$

- others are possible - this is best

$$\int_{C_3} F \cdot dr = \int_0^{\pi/2} \langle \cos t, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{\pi/2} -\cos t \sin t + \cos^2 t \sin t \, dt$$

$$= \left. \frac{\cos t^2}{2} + \frac{\cos t^3}{3} \right|_0^{\pi/2}$$ Same sort of identies

$$= -\frac{1}{2} + \frac{1}{3}$$

$$= -\frac{1}{6}$$

$$I = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$$0 + \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

b) $$I = \oint_C F \cdot d\vec{r} \underset{\text{greens theorm}}{=} \iint \text{curl } \vec{F} \, dA$$

what is?

$$= \iint \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x) =$$

$$= \iint_R y \, dx \, dy$$

look at picture $\square$

$$= \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-x^2}} y \, dx \, dy$$

$$= \int_{x=0}^{1} \frac{y^2}{2} \Big|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{1-x^2}{2} dx$$

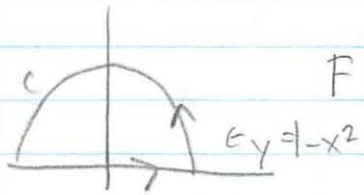
$$= \frac{x^2}{} - \frac{x^3}{6} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{6}$$

$$= 1/3$$

✓ Same answer both methods

ex2



$F = \langle y^2 + y, 2xy \rangle$

Set up integrals $\oint_C \vec{F} \cdot \hat{n}\, ds$

a) directly
b) using green's theorm

a) $\int_{C_1} = 0$ because $y = 0$ so $F = 0$

$\int_{C_2} \vec{F} \cdot \hat{n} =$ needs parameters
what do we have?

$x(t) = t$ $\quad x = x$
$y(t) = 1 - t^2$ $\quad y = 1 - x^2$

that parametrized $-C_2$

$$\int_{C_2} \vec{F}\cdot\hat{n}\, ds = -\int_{-C_2} \vec{F}\cdot\hat{n}\, ds$$

$\hat{n}\, ds = \langle y', -x' \rangle\, dt = \langle -2t, -1 \rangle$ in terms of t

$$I = \int_{t=-1}^{1} \langle (1-t^2)^2 + (1-t^2), 2t(1-t^2) \rangle \cdot \langle -2t, -1 \rangle\, dt$$

b) $I = \iint \operatorname{div} \vec{F}\, dA$

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x^2+y) + \frac{\partial}{\partial y} 2xy$$
$$= 2x$$

Proving a theorm
- doing in recitation

<u>Property</u> if $\vec{F} = \langle M, N \rangle$ satisfies $M_y = N_x$ on $\mathbb{R}^2$ then $\vec{F}$ is conservative (aka its a gradient field $\to f$ such that $\vec{F} = \vec{\nabla} f$

<u>Proof</u> We suppose $\vec{F} = \langle M, N \rangle$ satisfies $M_y = N_x$

By Green's Theorm we get $\oint_C \vec{F}\cdot d\vec{r} = 0$

along closed path of gradient field

(because $\oint_C \vec{F} \cdot d\vec{r} = \iint_R N_x - M_y \, dA$

$= \iint_R 0 = 0$)

- Thus if C_1, C_2 one continuous curve with same endpoints then $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$

(because $\int_{C_1} \vec{F} \cdot dr - \int_{C_2} \vec{F} \cdot d\vec{r}$

$= \oint_{C_1 - C_2} \vec{F} \cdot d\vec{r} = 0$

- We define a function f by
  $f(P) = \int_C \vec{F} \cdot dr$ where C is going from $O \to P$

(defined by preceeding point)

- [illegible] $\vec{\nabla} f = \vec{F}$

Review
in Lecture

4/13

$$\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_R \text{curl}\,\vec{F}\,dA$$

$$dr = \hat{T}\,ds = \langle x', y' \rangle$$

tangent

$$\oint M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$$

Work form / tangential form

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \text{div}\,\vec{F}\,dA$$

$$\oint M\,dy - N\,dx = \iint_R (M_x + N_y)\,dA$$

normal form

Flux form /

Green's Theorm

Lecture 25

Extension + applications of Green's Theorm 4/13

An example of Green's Theorm
- very important

Last recitation
- proff was wrong direction

Thur

$$\begin{cases} N_x = M_y \\ \text{curl } \vec{F} = 0 \end{cases} \rightarrow \vec{F} \text{ is conservative (any simple closed } C) \rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$$

So must be true in whole x, y plane

Proof = Green's Theorm

Example

$$\vec{F} = \frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$$

- perpendicular to normal of circle
- so tangental to circle

$$\begin{cases} \text{direction: tangental to circle} \\ \text{magnitude: } \frac{1}{r} = \frac{1}{a} \end{cases}$$

Calculate curl
(using chain rule)

Remember

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$N_x = \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right)$$

$\leftarrow N$

* practice

$$= \frac{r^2 \cdot 1 - x \cdot 2r \cdot \left(\frac{x}{r}\right)}{r^4}$$

$$= \frac{r^2 - 2x^2}{r^4}$$

$$-M_y = -\frac{\partial}{\partial y}\left(\frac{y}{r^2}\right) = \frac{r^2 - 2y^2}{r^4}$$

Now add them

$$\frac{r^2 - 2x^2 - r^2 - 2y^2}{r^4}$$

$$= 2r^2 - 2r^2$$

$$= 0$$

Supposed to be a conservative field
- is it?

$$\oint_C F \cdot t \, ds = |\vec{F} \cdot \vec{t}| \cdot \text{lenght } c$$

scalar products

$$= |\vec{F}|$$

$$= \frac{1}{a} 2\pi a$$

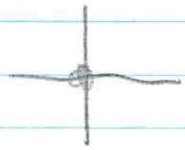
$$= 2\pi$$

Paradox : Green's Therom said 0
But we got 2π
and get 2π with sin + cos

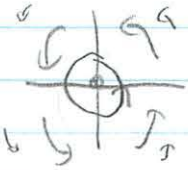
Why ???

So Green's Theorm not applicible
- $\vec{F}$ is not defined at the origin
- green's theorm does not apply

$\vec{F}$ is magnetic field of a wire coming out of page $\hat{z}$



unit DC current in + $\hat{z}$ / ⊙ dir

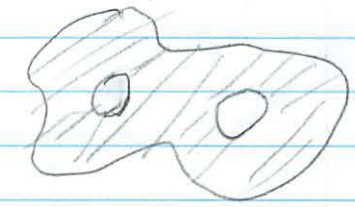


2π to go around

Problem - holes

Suppose curl $\vec{F} = 0$ in R, but R has "holes"

places where curl $\neq 0$

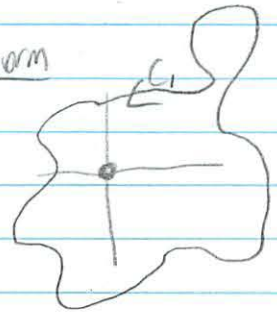


When no "holes" R is "simply connected"
- can be shrunk to a pt w/o leaving the region

1 hole = "one connected"

$\vec{F} = \frac{\langle -y, x \rangle}{r^2}$ defined in whole x-y plane without (0,0)
curl = 0
"one connected" at (0,0)

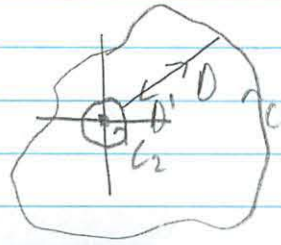
Theorem



For any such C

$$\oint_C \vec{F} \cdot d\vec{r} = 2\pi$$

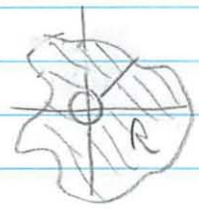
- can't go through pt b/c can't calculate line integral there
- but can be any path



← draw small second curve
- draw a line between them D

$$\underbrace{C_1 + D' + C_2' + D}_{C}$$

kinda overlaps when return on D but think of it as a 2nd line infinitessimally far away



So region is interior of C
- must walk in ⊕ dir with interior of left
R is always on the left

$\vec{F}$ defined in R
curl $\vec{F} = 0$ in R

Now Green's Theorem works!

this is cool!

$$\iint_R \text{curl } \vec{F} \cdot d\vec{r} = 0$$

$$\int_C F \cdot dr = \oint_{C_1} + \int_{D'} + \oint_{C_2'} + \int_D$$

Cancles

So cut disappears

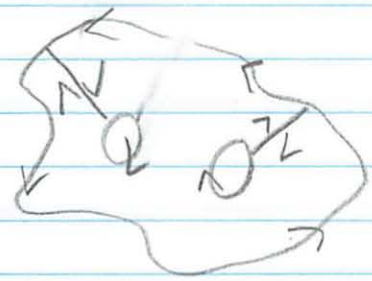
$$= \oint_{C_1} - \oint_{C_2} = 0$$

So $\oint_{C_1} \vec{F} \cdot d\vec{r} = \oint_{C_2} \vec{F} \cdot d\vec{r} = \boxed{2\pi}$ ← That is what we want

This is called "<u>extended Green's Theorem</u>"

(both work + flux forms)

If R has holes, bounded by closed curve



$$\int_{C_1+C_2+C_3} \vec{F} \cdot d\vec{r} = \iint_R \text{curl } \vec{F} \, dA$$

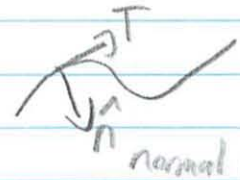
Proved i make cuts

Recitation

4/14

New stuff from last lecture: the holes

Remember $d\vec{r} = \vec{T}\, ds$
$= \langle x', y' \rangle\, dt$
$\hat{n}\, ds = \langle y', -x' \rangle\, dt$



normal to tangent vector on the right

<u>ex1</u> $\vec{F} = \langle -y, x \rangle$

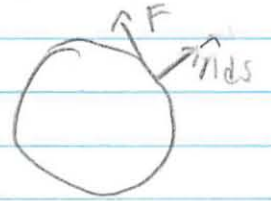
Compute $\int_C \vec{F} \cdot \hat{n}\, ds$ (flux) and $\int_C \vec{F} \cdot d\vec{r}$ (work)

a) C is circle of radius a around O
b) C is line going from O to (a,b)

<u>Hint</u> Look at geometric properties of F

a) $\int_C \vec{F} \cdot \hat{n}\, ds$

will be 0
because flux is normal at every point



$\int \vec{F} \cdot d\vec{r}$
- do parametrization + calculation

= maximum
= product of the lenght

$\vec{F}$ is $\parallel$ to $d\vec{r} = \vec{T}\,ds$

$\vec{F} \cdot d\vec{r} = |\vec{F}| \cdot |d\vec{r}| = |\vec{F}|\,ds$

$= \int_C \sqrt{y^2 + x^2}\,ds$

$= \int_C a\,ds$

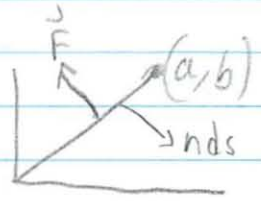
$= a \int_C ds$

= circumference of a circle

$= a \times 2\pi a$

$= 2\pi a^2$

b) $\int_C \vec{F} \cdot \hat{n}\,ds$



$= \int_C -|\vec{F}|\,ds$

Lets do by parametrization

$x(t) = at \qquad t \in [0,1]$
$y(t) = bt$

$n\,ds = \langle y', -x' \rangle\,dt$
$= \langle b, -a \rangle\,dt$

$\int_C \vec{F} \cdot \vec{n}\,ds = \int_0^1 \langle -bt, at \rangle \cdot \langle b, -a \rangle\,dt$

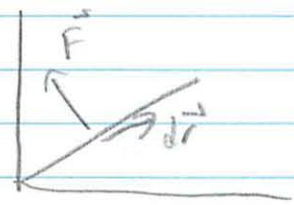
$$= \int_0^1 -(a^2+b^2)\,t\,dt$$

$$-(a^2+b^2)\frac{t^2}{2}\Big|_0^1$$

$$-\frac{a^2+b^2}{2}$$

$$\int_C \vec{F}\cdot d\vec{r} = 0$$

because they are $\perp$



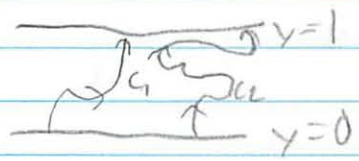
Work = along curve $\oint_C \vec{F}\cdot d\vec{r}$ think F → force field

flux = across curve $\oint_C \vec{F}\cdot\hat{n}\,ds$ velocity

---

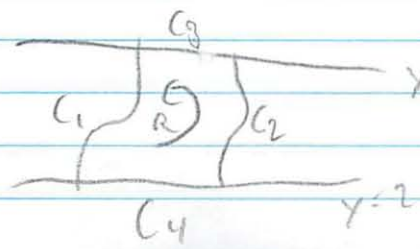
ex2 $\vec{F} = \langle 2xy^2 - 2xy,\ 2x^2y - x^2 + y^2 \rangle$

Graduate Student Pratiding Teaching :(



Work done by F same along both C_1 and C_2

We integrate over Region between C_1 and C_2



$C = -C_1 - C_3 + C_2 + C_4$

$$\oint \vec{F}\, d\vec{r} = -\int_{C_1} - \int_{C_3} + \int_{C_2} + \int_{C_4} \vec{F}\, d\vec{r}$$

$$= \iint_R \text{curl } \vec{F}\, dA$$

$$\vec{F} = \langle M, N \rangle$$

$$\text{curl } \vec{F} = N_x - M_y$$

$$= (4xy - 2x) - (4xy - 2x)$$

$$= 0$$

↑ oh so one of those tricky
it will go to 0

To show $\int_{C_3}, \int_{C_4} \vec{F}\, d\vec{r} = 0$

$$\int_{C_3} \vec{F}\, d\vec{r} =$$

$$\begin{pmatrix} x(t) = t \\ y(t) = 1 \end{pmatrix} \qquad t \in [x_0, x_1]$$

$$dr = \langle x(t)', y(t)' \rangle\, dt$$

$$\langle 1, 0 \rangle\, dt$$

$$\int_{C_3} \langle 2t - 2t, 2t^2 - t^2 + 1 \rangle \cdot \langle 1, 0 \rangle$$

$$\int_{C_3} 0\, dt$$

so 0

Similarly for C_4

$$\int_{C_1} \vec{F}\, dr = \int_{C_2} \vec{F}\, d\vec{r}$$

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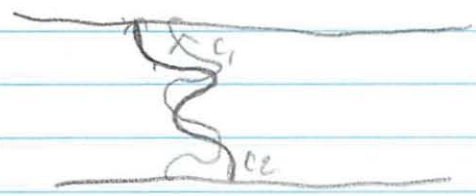
ex3 More General Case



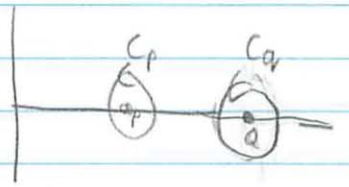
To do this
make new curve C_3 that is competely independent $\int_{C_1} \int_{C_3}$

---

ex4 Regions w/ holes

Suppose curl $\vec{F} = 0$ everywhere except at 2 points
$(a,0)$
$(a+2,0)$

$$\oint_{C_p} \vec{F} \cdot d\vec{r} = 3 \qquad \oint \vec{F} \cdot d\vec{r} = 1$$



Value of $\int_C \vec{F} \cdot dr$ where C is circle of radius r around 0

$r \neq a, a+2$

if $r < a$ $\quad \oint_C = 0$ b.

because $\int_C = \iint_R \underset{\neq 0}{\text{curl } \vec{F}} = 0$

if $a < r < a+2$

$\oint = 3$

if $r > a+2$ $\quad \oint = 4$

18.02 Problem Set 7 (DUE THURS. APR. 15 10:45 2-106)

Part I (15 pts.)

Lecture 23. Thurs. Apr. 8 Green's theorem (tangential form).
Read: 21.3 to middle p. 768 Work: 4D-1bc, 2, 3, 5

Lecture 24. Fri. Apr. 9 Flux; Green's Theorem in normal form.
Read: Notes V3, V4 Work: 4E-1ac, 2, 5 (without calculation); 4F-4, 5

Lecture 25. Tues. Apr. 13 Extensions and applications of Green's Theorem
Read: Notes V5, V6 (pp. 1,2) Work: 4G-1, 4, 5

Lecture 26. Thurs. Apr. 15 Triple integrals: rectangular and cylindrical coordinates
Read: Notes I.3, 20.5, 20.6

Part II (20 pts.)

Directions: Try each problem alone for 20 minutes. If you subsequently collaborate, solutions must be written up independently. It is illegal to consult problem sets from previous semesters. With each problem is the day it can be done.

Problem 1. (Thurs. 4 pts.: 1, 2, 1)

a) Let C be a simple, closed, smooth, positively oriented curve and let R be its interior. Show that

$$\frac{1}{3}\oint_C -y^3\,dx + x^3\,dy$$

is equal to the polar moment of inertia about the origin (cf. p. 725) of the plane region R, assuming its density $\delta = 1$.

b) Use part (a) to calculate the polar moment of inertia about the origin of the region inside the ellipse given parametrically by $x = a\cos t,\ y = b\sin t,\ (a,\ b$ positive constants).

(Hint: Begin by sketching the ellipse. By symmetry it is enough to calculate the polar moment of inertia for the portion of the ellipse inside the first quadrant. This simplifies the boundary, and also permits the use of the formulas for $\int_0^{\pi/2}\sin^n t\,dt$ and $\int_0^{\pi/2}\cos^n t\,dt$ at the beginning of exercises 3B.)

c) Check your answer in part (b) by seeing what it gives when $b = a$, i.e., the ellipse becomes a circle of radius a. Then confirm this by calculating the polar moment of inertia of a circular disc of radius a, using a double integral in polar coordinates.

Problem 2. (Thurs. 3 pts.) Prove the other half of Green's theorem

$$\oint_C N(x,y)\,dy = \iint_R \frac{\partial N}{\partial x}\,dA$$

with the usual hypotheses, except assume the interior R of the curve is *horizontally simple*, i.e. is the region lying between the graphs of the smooth functions $x = g(y)$ on the left and $x = h(y)$ on the right, and the horizontal lines $y = a$ below and $y = b$ above.

if split through day does not seem so long!

Problem 3. (Thurs. 3 pts.)

Find the simple, closed, positively oriented curve C for which

$$\oint_C x^2(y+1)\,dx + 4x(1-y^2)\,dy$$

A - need to find
- oh like my ex find area

$1-x^2$

has the maximum value. Indicate reasoning.

(Use Green's theorem; note that the integrand of $\iint_R$ takes on both positive and negative values.)

want region for biggest integrand

Problem 4. (Fri. 4 pts.: 3,1) Let $\mathbf{F} = 2xy(1-x^2)\,\mathbf{i} + y^2(3x^2-1)\,\mathbf{j}$

a) Show that the flux of $\mathbf{F}$ across any smooth curve $y = f(x)$ running from the line $x = -1$ to the line $x = 1$ has always the same value, i.e., the flux does not depend on the choice of curve.

(Take two such curves C_1 and C_2; assume first that one lies entirely above the other and apply Green's Theorem in the normal form.)

b) Determine this constant value of the flux.

$\int_a^b$ not $-\infty$ b/c then $\int \ominus$

So want from -1 to 1

$4x^2 + y^2 = 0$

$4 - x^2 - y^2$

$4 - x^2 - 2y^2$

Problem 5. (Fri. 2 pts.)

A function $f(x,y)$ defined in a region S of the xy-plane is said to be **harmonic** in S if at every point of S it satisfies the two-dimensional form of **Laplace's equation**:

$$f_{xx} + f_{yy} = 0.$$

Prove that if the field $\mathbf{F}$ is the gradient of a harmonic function in S, the flux of $\mathbf{F}$ across any simple closed curve lying entirely inside S is zero.

Problem 6. (Tues. 4 pts) Let

$$\mathbf{F} = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}.$$

Show that the flux of $\mathbf{F}$ across any simple closed curve C surrounding the origin is zero.

(Note that the field is not defined at the origin. Study and imitate the analogous argument in the notes involving the calculation of work done around a simple closed curve by a plane vector field whose curl vanishes, but which is not defined at the origin.)

18.02 P-Set 7

34/35

Michael Plasmeier

4/13

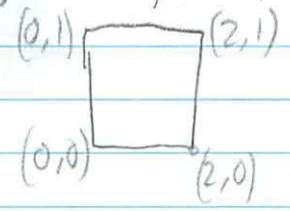
Lecture 23 Green's Theorm Tangental Form

4D-1b For the following fields F, curves C $\int_C \vec{F} \cdot d\vec{r}$

- directly as line integral
- green's theorm

$F = x^2(\hat{\imath} + \hat{\jmath}) = \langle x^2, x^2 \rangle$

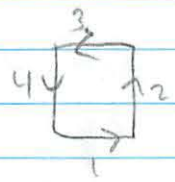
C = rectangle



So we want to do

$\oint_C M\,dx + N\,dy$ → but have to do it for each section

$$\int_{C_1}^{2} x^2\,dx + \int_{C_2}^{1} 4\,dy + \int_{C_3}^{0} x^2\,dx + \int_{C_4}^{0} 0\,dy$$



$dy = 0$
$y = 0$ & does not matter
x changing
~~as~~ x^2

$dx = 0$
$x = 2$
y changing
~~as~~ x^2

$dy = 0$
x changing
x^2

$x = 0$
$x^2 = 0$

and then it adds

4

Ok - this was last P-set right?

Now Green's theorem - which is really simple to calculate

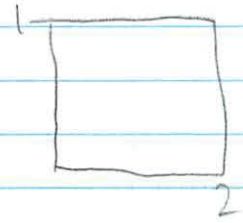
$$\iint_R N_x - M_y \, dA$$

$N = x^2 \qquad N_x = 2x$

$M = x^2 \qquad M_y = 0$

$$\iint_R 2x - 0 \, dA$$

now need to integrate over R
- again last P-set or one before that?



$$\int_0^1 \int_0^2 2x \, dx \, dy$$

y x

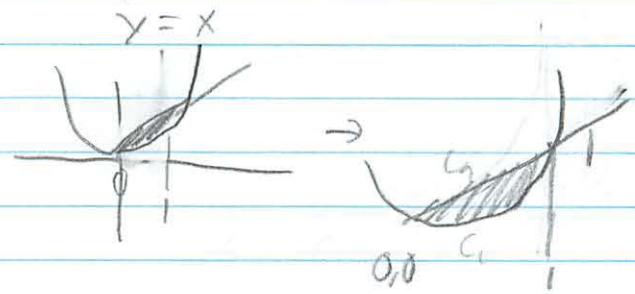
$$\frac{2x^2}{2}\Big|_0^2$$

$$\int_0^1 4 \, dy$$

4

Ok - so learned both before methods
- know how methods are different

c) $F = \langle xy, y^2 \rangle$ $\quad C = y = x^2 \quad 0 \le x \le 1$



a) Directly

$$\int_C M\,dx + N\,dy$$

$$\int_{C_1} xy\,dx + y^2\,dy + \int_{C_2} xy\,dx + y^2\,dy$$

	$y = x^2$	$x = x$
parametrize	$x = x$	$y = x$
	$dx = 1$	$dx = 1$
	$dy = 2x$	$dy = 1$

thats kinda easy

$$\int x \cdot x^2 \cdot 1 + x^4 \cdot 2x + \int x x \cdot 1 + x^2 \cdot 1$$

now what?

or did I convert wrong? - or it would work when $\int_0^1$ & forgot definite integral

$$\int \vec{F} \cdot d\vec{r} \qquad \int \vec{F} \cdot d\vec{r}$$

$$\int_0^1 \langle x^2 x, x^4 \rangle \langle 1, 2x \rangle + \int_0^1 \langle xx, x^2 \rangle \langle 1, 1 \rangle$$

$$\int_0^1 x^3 + 8x^5 + \int_0^1 x^2 + x^2$$

did not actually integrate

$\cancel{9} + \cancel{2}$

$\cancel{11}$

next

$\frac{x^4}{4} + \frac{x^6}{3}\Big|_0^1$ $\qquad$ $\frac{2}{3}x^3\Big|_0^1$ $\qquad$ don't forget to actualy integrate!

$\frac{7}{12}$ $\qquad$ $-\frac{2}{3}$

$-\frac{1}{12}$

b) Green's Theorm

$M = xy$ $\qquad$ $\cancel{M_x = 1_y}$ $\qquad$ note backwards

$N = y^2$ $\qquad$ $\cancel{N_y =}$ $\qquad$ $N_x = 0$

$M_y = 1x$

$$\iint_R 0 - 1x \, dA$$

$$\int_{x^2}^{x}\int_0^1 -1x \, dx \, dy$$

$$-\frac{x^2}{2}\Big|_0^1$$

$$\int_{x^2}^{x} -\frac{1}{2} \, dy$$

$$-\frac{1}{2}y\Big|_{x^2}^{x}$$

$$-\frac{1}{2}x - \left[-\frac{1}{2}x^2\right]$$

$$\frac{-x + x^2}{2}$$

$$\frac{x^2 - x}{2}$$

but need #

-switch integral x and y

$$\int_0^1 \int_{x^2}^{x} -x \, dy \, dx$$

$$-xy \Big|_{x^2}^{x}$$

$$-xx - [-xx^2]$$

$$-x^2 + x^3$$

$$\int_0^1 x^3 - x^2 \, dx$$

$$\frac{x^4}{4} - \frac{x^3}{3} \Big|_0^1$$

$$\frac{1}{4} - \frac{1}{3}$$

$$\frac{3}{12} - \frac{4}{12} = -\frac{1}{12}$$

2. Show that $\oint_C 4x^3 y \, dx + x^4 \, dy = 0$ for all closed C

Well isn't it a closed path - yeah by icon
But isn't the field not conservative, so it is not

Oh no - look what it is

Green's Theorm

$M = 4x^3 y \qquad N_x = 4x^3$

$N = x^4 \qquad M_y = 4x^3$

$$\iint_R 4x^3 - 4x^3 = 0$$

I guess you just have to see that

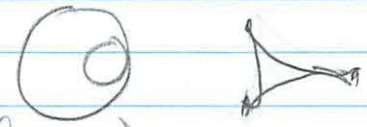
3. Find the area inside the hypocycloid $x^{2/3} + y^{2/3} = 1$
by using green's theorm
Paramatrize $x = \cos^3\theta$
$y = \sin^3\theta$

Hint: $\vec{F} = \frac{1}{2}\langle y, x\rangle$

What's a hypocycloid?

WP

covered these before



$$x(\theta) = (R-r)\cos\theta + r\cos\left(\frac{R-r}{r}\theta\right) = M$$

$$y(\theta) = (R-r)\sin\theta - r\sin\left(\frac{R-r}{r}\theta\right) = N$$

own

So how would I do this?

shape →

and why for area isn't that normal double area or do it for flux

$$\iint_R \left[M_x + N_y\right] dA$$

↑ still need to $\int$ over the area!

Use symetric form for integrand since parametrization of curve does not favor x or y

?? symetric form

?? What is this so called symmetric form?

$$\text{Area} = \frac{1}{2}\oint_C -y\,dx + x\,dy$$

?? Don't get their solution

$$\frac{1}{2}\int_0^{2\pi} 3\sin^4 t\cos^2 t\,dt + 3\sin^2 t\cos^4 t\,dt$$

Where in all world did they get that? What were the original functions they were using? The parameterizations?

$$\frac{3}{2}\int_0^{2\pi} \sin^2\cos^2 t\,dt$$

Now using $\sin^2 t\cos^2 t = \frac{1}{4}(\sin 2t)^2$

$$= \frac{1}{4}\cdot\frac{1}{2}(1-\cos 4t)$$

Now above $= \frac{3}{8}\left(\frac{t}{2} - \frac{\sin 4t}{8}\right)\Big|_0^{2\pi}$

$$\frac{3\pi}{8}$$

Do not get whatsoever!

3

again
office
Hrs
Oliver

$$\text{Area} = \iint_R 1\, dA = \oint \vec{F} \cdot d\vec{r}$$

if $\text{Curl } \vec{F} = 1$

$\quad \| \; N_x - M_y$

On lecture day 2 but marked as day 1

need to find function

some easy fields prof gave us

$$\vec{F} = \langle 0, x \rangle \text{ or } \langle -y, 0 \rangle \text{ or } \langle -\tfrac{y}{2}, \tfrac{x}{2} \rangle$$

Use this trick to compute area

They gave parametrization

$x = \cos^3\theta \qquad \theta \in [0, 2\pi]$

$y = \sin^3\theta$

$$\int_0^{2\pi} \langle 0, \cos^3\theta \rangle \cdot \overset{dr}{\langle -3\sin\theta\cos^2\theta, 3\cos\theta\sin^2\theta \rangle}$$

$$\int_0^{2\pi} 3\cos^4\theta \sin^2\theta \, d\theta$$

but how to $\int$ - no clear anti deriv

know $\int_0^{\pi/2} \cos^n\theta \, d\theta$ cut integral into parts so get this

$$\int_0^{2\pi} 1 - \cos^2\theta$$

5. Show that the value $\oint_C xy^2\,dx + (x^2y+2x)\,dy$

around any square depends only on size of square - not position

So is this another trick

$M = xy^2$ $\quad$ $\cancel{N_x = x^2}$ $\quad$ $N_x = 2xy+2$

$N = x^2y+2x$ $\quad$ $\cancel{M_x = y^2}$ $\quad$ $M_y = 2xy$

↑ get this right

$$\cancel{\iint_R x^2 - y^2}$$

$$\cancel{\int_0^y \int_0^x x^2 - y^2\,dx\,dy}$$ ← think did this wrong

$$\cancel{\frac{x^3}{3} - xy^2 \Big|_0^1}$$

$$\cancel{\int_0^y \frac{1}{3} - y^2\,dy}$$

$$\cancel{\frac{1}{3}y - \frac{y^3}{3} \Big|_0^y}$$

$$\cancel{\frac{y}{3} - \frac{y^3}{3}}$$

$$\boxed{\frac{-y^3+y}{3}}$$ ← so what is y

Try again

$$\iint_R (2xy+2) - \cancel{x^2}\ 2xy$$

$$\int_0^y \int_0^x 2xy + 2 - y^2 \, dy \, dx$$

$x \qquad y$

$$\frac{2x^2y}{2} + 2y - \frac{y^3}{3} \Big|_0^x$$

$$x^3 + 2x - \frac{x^3}{3}$$

$$\int_0^y x^3 + 2x - \frac{x^3}{3} \, dx$$

$$\frac{x^4}{4} + \frac{2x^2}{2} - \frac{x^4}{3 \cdot 4} \Big|_0^y$$

$$\frac{y^4}{4} + y^2 - \frac{y^4}{12}$$

so how does that help???

Note have

$$\iint_R 2 \, dA = 2(\text{area of } R)$$

Lecture 24 Flux, Green's Theorem in normal form

4E-1a Let $F = -y\hat{i} + x\hat{j} = \langle -y, x \rangle$

Recall what this is

tangent to circle all all times

thanks repitation

And remember $\hat{n}$ is straight out (normal to tangent vector on right)

So it will be $\perp$ at all points and thus $= 0$ flux

c) C is line from $(0,0)$ to $(1,0)$

$ds = \vec{T}\,dr = \langle x', y' \rangle$

↑ tangent

$= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$

flux form

$\oint_C \langle -y, x \rangle \cdot \langle 0, 0 \rangle$

$= \langle 0, 0 \rangle$

$\hat{n} = -\hat{j}$ ← just the normal vector to curve

$F \cdot \hat{n} = x\hat{j} \cdot -\hat{j}$

$= -x$

$\int_0^1 -x\,dx = -\frac{1}{2}$

2. Let $\vec{F}$ be the constant vector field $\hat{i} + \hat{j}$
Where would you place a directed line segment C so <u>flux</u> is

a) Maximal - in the same direction of the vector
-(since dot product) aka northwest ✓

b) Minimal - if opposet vector - southeast

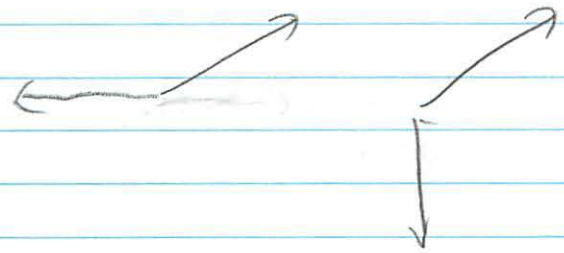
c) Zero - perpendicular to vector northeast
South west

d) -1 - if C has direction + magnitude of $-\hat{i}$ or $-\hat{j}$

$$(\hat{i} + \hat{j}) \cdot -\hat{i} = -1$$
$$(\hat{i} + \hat{j}) \cdot -\hat{j} = -1$$

But why

~~Why?~~



↑ but how is that ⊖?

Don't think of flux in terms of Guassian surface - just vector addition
Have to integrate, right?

d) What are values of min and max?

$|F| \cdot |n| \cdot \cos\theta$ $\qquad$ $|F| \cdot |n| \cdot \cos\theta$

$\sqrt{2} \cdot 1 \cdot \cos 0$ $\qquad$ $\sqrt{2} \cdot 1 \cdot \cos \pi$

$\sqrt{2}$ $\qquad$ $\sqrt{2}$

5. without calculation: Let $\text{dir}(F)$ = radially out

$|F| = r^m$ ← integer

a) Evaluate the flux of F across circle at origin radius a CCW

Always tangent to circle so 0?

← where is this from

No $\quad a^m \cdot 2\pi a$ $\quad$ - I thought it was ⊥ to

$2\pi a^{m+1}$

OH ∥ to $\hat{n}$

which means it does add

$\longrightarrow n \longrightarrow F$ so does add since parallel

But you get the 2π from integrating?

Not shown in solution

did in lecture w/ paramatrization

- good to see 2 forms of it

b) For what values of m will the flux be independent of a?

a^m

$a^1 = a$ $\quad a^0 = 1$ always

$a^2 = a \cdot a$

So when $m = 0$

they say -1

but that is $\frac{1}{a}$ and thus still depends on a

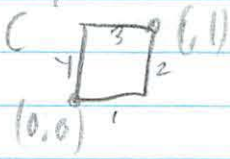
1 ans they gave

no in whole function

$2\pi a^{m+1}$

so m should be -1 for exponent $= 0$

4F-4 Verify Green's Theorm in normal form by calculating both sides and showing that they are =
if $F = x^2 \hat{i} + xy \hat{j}$ and C



Ok so normal form means for flux,

$$\cancel{\int_C F \cdot \hat{n} = \langle x^2, xy \rangle \cdot \hat{n}}$$

$$\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$\cancel{\int_0^1 \langle x^2, xy \rangle \langle \tfrac{1}{2}, -\text{something} \rangle}$$

$$\cancel{+ \int}$$

Note the other form
$\oint -N dx + M dy$

$M = x^2$
$N = xy$

C_1 $\int_0^1 -xy\,dx + x^2\,dy = 0$ ✓
$x=0$ $dy=0$

C_2 $\int_0^1 -xy\,dx + x^2\,dy$
$dx=0$ $x=1$ actually put it in

$$\int_0^1 x^2\,dy \rightarrow x^2 y \Big|_0^1 \rightarrow x^2 \rightarrow 1$$

C_3 $\int_1^0 -xy\,dx + x^2\,dy$

$\cancel{y=1}$ $\quad dy=0$

↑ thought error

$\int_0^1 -x\,dx$ ✓

$-\frac{x^2}{2}\Big|_0^1$

$+\frac{1}{2}$ should always have ∫

C_4 $\int_1^0 -xy\,dx + x^2\,dy$

$dx=0 \quad \cancel{x=0}$

↑ thought error again

$\cancel{\int_0^1 x^2\,dy}$

$\cancel{x^2 y\Big|_0^1}$

$\cancel{-x^2}$ 0

So $\cancel{0 + x^2 + y - x^2}$ $\quad 1+\frac{1}{2}$

✓ $\quad \frac{3}{2}$

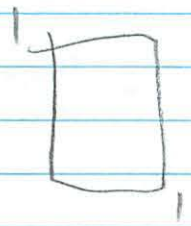
Now other way after screwing that up fairly bad

$\iint_R M_x - (-N)_y$

$M = x^2 \rightarrow M_x = 2x$

$N = xy \rightarrow -N = -xy \rightarrow (-N)_y = -1x$

$$\iint_R 2x - -x \, dA$$



$$\int_0^1 \int_0^1 3x \, dx \, dy$$

$$\frac{3x^2}{2}\Big|_0^1$$

$$\int_0^1 \frac{3}{2} \, dy$$

$$\frac{3y}{2}\Big|_0^1 = \frac{3}{2}$$ much better

5. Calc the div $\vec{F}$ for $\vec{F} = r^n(x\vec{i} + y\vec{j})$
curl $\vec{F}$ $= \langle r^n x, r^n y \rangle$

Simplify differentiation $r_x = \frac{x}{r}$ $r_y = \frac{y}{r}$

$M = \frac{x}{r} x = \frac{x^2}{r}$ $\frac{x^2}{\sqrt{x^2+y^2}}$ $N_x = \frac{-xy^2}{(y^2+x^2)^{3/2}}$

$N = \frac{y}{r} y = \frac{y^2}{r}$ $\frac{y^2}{\sqrt{x^2+y^2}}$

Well $r = \sqrt{x^2+y^2}$ $M_y = \frac{-x^2 y}{(x^2+y^2)^{3/2}}$

$$\text{curl } \vec{F} = \frac{-xy^2}{(x^2+y^2)^{3/2}} - -\frac{x^2y}{(x^2+y^2)^{3/2}}$$

$$\frac{-xy^2 + x^2y}{(x^2+y^2)^{3/2}}$$

div $\vec{F}$

$$M_x = \frac{x(x^2+2y^2)}{(x^2+y^2)^{3/2}}$$

$$(-N)_y = \frac{-y(y^2+2x^2)}{(x^2+y^2)^{3/2}}$$

$$\text{div } \vec{F} = \frac{x(x^2+2y^2)}{(x^2+y^2)^{3/2}} - \left(\frac{-y(y^2+2x^2)}{(x^2+y^2)^{3/2}}\right)$$

$$\frac{x^3+2xy^2+y^3+2x^2y}{(x^2+y^2)^{3/2}}$$

Here they explain how they got the hint

$$r = (x^2+y^2)^{1/2}$$

$$r_x = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \frac{x}{r}$$ ← Oh that was pre derived
- not to make deriv easier

So $M = r^n x$
$N = r^n y$

So by chain rule

$$M_x = r^n + nr^{n-1} x \cdot \frac{x}{r}$$

$$= r^n n^{r-2} x^2$$

$$N_y = r^n + nr^{n-1} y \cdot \frac{y}{r}$$
$$= r^n + nr^{n-2} \cdot y^2$$

$$\text{div } F = M_x + N_y$$ ← I thought it was $-(-N)_y$

$$= 2r^n + nr^{n-2}(x^2+y^2)$$
$$= r^n(2+n)$$ ← flux, area

$$N_x = nr^{n-1} \cdot \frac{x}{r} \cdot y$$

$$M_y = nr^{n-1} \cdot \frac{y}{r} \cdot x$$

$$\text{curl } \vec{F} = N_x - M_y$$
$$nr^{n-1} \frac{xy}{r} - nr^{-1} \frac{x \cdot y}{r}$$
$$= 0$$

← work, line

Lecture 25 Extension + applications of Green's Theorm

46-1 Using the criterion of this section - tell which are conservative or not

each problem is so slow but whole thing by fast
- since staying on fast

- so this is with holes
- but what is conservative

Conservative = can use any line and get same answer

But how do I tell from looking at it?

a) $(y^2+2)\,i + 2xy\,j$

~~[illegible]~~ Off right → curl $\vec{F} = 0$ if conservative

$M = y^2+2 \qquad N_x = 2y$
$N = 2xy \qquad M_y = 2y$

$\int 2y - 2y = 0$ Conservative ✓

$\vec{F} = \nabla f$

b) $x(\cos y)\,dx + y(\cos x)\,dy$

$M = x(\cos y) \qquad N_x = -xy \sin x$
$N = y\cos x \qquad M_y = +xy \sin y$

$\int -xy \sin x - [-xy \sin y]$

$-y$

$\int -xy(\sin x + \sin y)$ not conservative

$-y\sin x - x\sin y$

c) $\dfrac{x\hat{i} + y\hat{j}}{\sqrt{r^2-1}}$ this seems like it will work looks familiar

$r^2 = x^2+y^2$

$M = \dfrac{x}{\sqrt{x^2+y^2-1}}$ $\quad N = \dfrac{y}{\sqrt{x^2+y^2-1}}$

$x(x^2+y^2-1)^{-1/2}$ $\quad y(x^2+y^2-1)^{-1/2}$

$N_x = \dfrac{-xy}{(x^2+y^2-1)^{3/2}}$ $\quad M_y = \dfrac{-xy}{(x^2+y^2-1)^{-1/2}}$

So 0 so field conservative
Does not matter what path you take
but R is exterior of unit circle
- not simply connected ← ??

these are kinda fun

d) $\dfrac{x\,dx + y\,dy}{\sqrt{1-r^2}}$

$M = \dfrac{x}{\sqrt{1-x^2-y^2}}$ $\quad N = \dfrac{y}{\sqrt{1-x^2-y^2}}$

what is the difference

$N_x = \dfrac{xy}{(1-x^2-y^2)^{3/2}}$ $\quad M_y = \dfrac{yx}{(1-x^2-y^2)}$

same → conservative
interior of unit circle
is simply connected

e) $\sqrt{x}\,i + \sqrt{y}\,j$

$M = \sqrt{x}$
$N = \sqrt{y}$

$N_x = 0 \qquad M_y = 0$

Conservative
and R is in the first quadrant
is simply connected
F is a gradiant field

I don't get all of this how R must be
Simply connected. I know no holes
but what does inside of circle mean

4. Even though the field of example 1

$\vec{F} = xy\,\hat{i} + x^2\,\hat{j}$ not a gradient field

Show that $\oint F \cdot dr = 0$
around every simple closed curve

So this is work/curl

$M = xy \qquad N_x = 2x \qquad d_x = y$
$N = x^2 \qquad M_y = 1x \qquad d_y = 0$

or is it that $dr = 2\pi r$

~~so $\oint \langle xy, x^2 \rangle \cdot 2\pi r$~~

~~or $\oint xy \cdot y + x^2 \cdot 0$~~ Use the right form

$\int xy^2$

$$\boxed{\iint_R 2x - x \; dA}$$

$= \underset{\uparrow}{x}\,(\text{area of } R)$

x component of center of mass

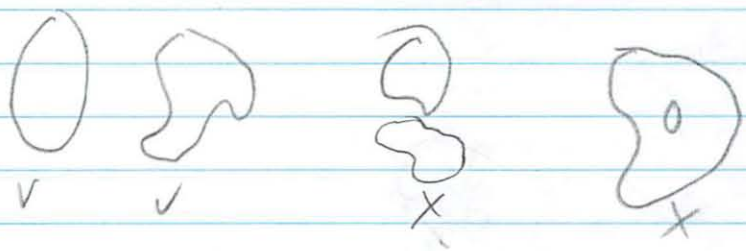
Region symmetric w/ respect to y axis
COM on y-axis

So $\bar{x} = 0$ and $\iint x\,dA = 0$

5. Which of the following regions are simply connected?

- the ones w/o holes
- again, is there anything else?
- oh it can be shrunk to one point

wolfram alpha



a) Half plane above x axis

- does going off to ∞ matter
- if not simply connected ✓ yes simply connected

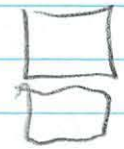
b) Plane minus line segment $(0,0) \to (0,1)$

No → hole ✓
a circle surrounding R is included
but not the interior
so like

c) Plane − (+) x axis

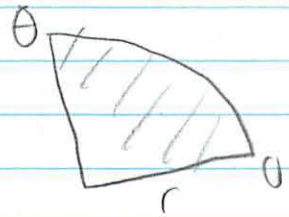
yes ✓

d) Plane – entire x axis



No ✓

e) region $r > 0$, $0 \le \theta < \theta_0$

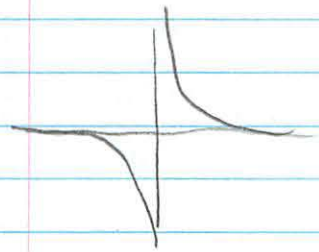


yes ✓

f) Region b/w 2 concentric circles

~~yes~~ No – if ~~in~~ hole its a no
even if they ask for region around

g) Region b/w 2 branches of $xy = 1$
$y = \frac{1}{x}$



No – not connected

yes for same reason – they
must have different picture
than I do

Part 2

#1. Let $\oint C$ be a simple, closed, smooth, ⊕ oriented curve

$$\frac{1}{3}\oint_C -y^3\,dx + x^3\,dy$$

is = to the polar moment of inertia

$$M_z = \iint_R r^2 \sigma(x,y)\,dA$$

$$\iint_R r^2\,dA$$

? so what

Show give = polar moment of inertia

Now $\iint_R N_x - M_y\,dA = \iint_R r^2\,dA = \frac{1}{3}\oint -y^3dx + x^3dy$

So $N_x - M_y = r^2$

$N_x - M_y = x^2 + y^2$

or am I messing up what = what

$M = -y^3$ $\qquad N_x = 3x^2$

$N = x^3$ $\qquad M_y = -3y^2$

$$\frac{1}{3}\iint_R \underset{\uparrow\, x^2}{3x^2} - \underset{\uparrow\, y^2}{-3y^2}\,dA$$

$$\iint_R x^2 + y^2\,dA = \iint_R r^2\,dA$$ ✓

b) Use part a to calculate the polar moment of inertia

$x = a\cos t$ ← given parametrization
$y = b\sin t$
↑ constants

Hint: start by sketching the ellipse. By symmetry only need portion of ellipse of 1st quad

$$\int_0^{\pi/2} \sin^n t \, dt$$

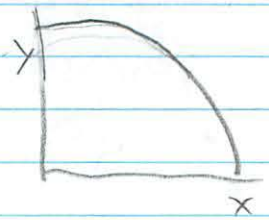
$$\int_0^{\pi/2} \cos^n t \, dt$$

$$\frac{1}{3}\oint -y^3 dx + x^3 dy$$

↑ don't know original x and y but what to do because M, N leads to $N_x M_y$ which is $\int x^2 + y^2$

Also do we know what the region is?
Oh duh ellipse

$$\int_0^x \int_0^x x^2 + y^2$$



$$\int_0^{b\sin t} \int_0^{a\sin t} a^2 \sin^2 t + b^2 \cos^2 t \, dx$$

$$\int_0^{a\sin t} a^2\sin^2 t + b^2\cos^2 t \, dt$$

$$t\left(\frac{a^2}{2}+\frac{b^2}{2}\right) - \frac{\sin t \cos t\,(a^2-b^2)}{2}\Bigg|_0^{a\sin t}$$

Dependent limit - unsolvable

Unless they want to $\int_0^{\pi/2}$ but I was not doing polar

$$\frac{\pi(a^2+b^2)}{4}$$

$$\int_0^{b\sin t} \frac{\pi(a^2+b^2)}{4}$$

No I think we wanted a

$$t\left(\frac{a^2}{2}+\frac{b^2}{2}\right) - \frac{\sin t \cos t\,(a^2-b^2)}{2}\Bigg|_0^{a}$$

$$\frac{a(a^2+b^2)}{2} - \frac{(a^2-b^2)\sin a \cdot \cos a}{2}$$

Does not look like it either

Back to above

$$\frac{\pi(a^2+b^2)}{4} \cdot b\sin t$$

??

Does not seem right eiter

Try

try again

$$= \frac{1}{3} \oint -y^3\,dx + x^3\,dy$$

$$= \frac{1}{3}\int_0^{\pi/2} -b^3\sin^3 t(-a\sin t)\,dt + a^3\cos^3 t(b\cos t)\,dt$$

$$= \frac{ab}{3}\int_0^{\pi/2} b^2\sin^4 t\,dt + a^2\cos^4 dt$$

$$= \frac{ab^3}{3}\int_0^{\pi/2}\sin^4 t\,dt + \frac{a^3b}{3}\int_0^{\pi/2}\cos^4 t\,dt$$

$$= \frac{ab^3}{3}\left(\frac{3!!}{4!!}\right) + \frac{a^3b}{3}\left(\frac{3!!}{4!!}\cdot\frac{\pi}{2}\right)$$

the rule

$$= \left(\frac{\pi}{16}(ab^3+a^3b)\right)\cdot 4 \text{ for full ellipse}$$

$$= \frac{\pi}{4}\left(ab^3+a^3b\right) \checkmark$$

c) Check your answer to part b, by seeing $b=a$
aka a circle of radius a
Then confirm polar moment of inertia

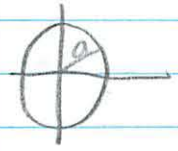
If $b=a$, then

$$\frac{\pi}{4}\left(a^4+a^4\right)$$

$$\frac{2a^4\pi}{4} = \frac{a^4\pi}{2}$$ ✓

$$I = \int_0^{2\pi}\int_0^a r^2 \cdot r \cdot dr\, d\theta$$

$\frac{a^4\pi}{2}$



$$= \int_0^{2\pi}\int_0^a r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{4} r^4 \Big|_0^a d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} a^4\, d\theta$$

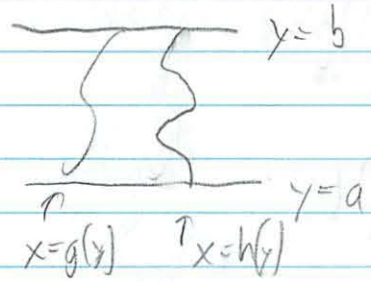
$$= \frac{1}{4}\cdot a^4 \cdot 2\pi$$

$$= \frac{a^4\pi}{2}$$ ✓ confirms ✓

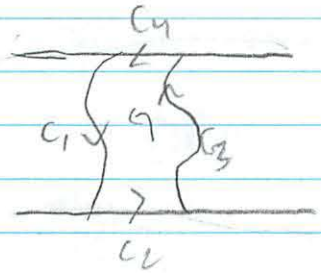
2. Prove the other half of Green's Theorem

$$\oint_C N(x,y)\,dy = \iint_R \frac{\partial N}{\partial x}\,dA$$

w/ usual hypothesis
except assume interior R is horizontally simple



Was totally confused till saw this picture b/c we did in recitation



Want to integrate over Region between C_1 and C_3

Never learned how to write a proof — or even what one is

$$C = -C_1 + C_3 + C_2 - C_4$$

$$\oint \vec{F}\,d\vec{r} = -\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} - \int_{C_4} \vec{F}\cdot d\vec{r}$$

$$= \iint_R \text{curl}\ \vec{F}\,dA$$

$$= \iint_R N_y - M_x\,dA$$

know that $\int_{C_1}, \int_{C_3} \vec{F}\cdot dr = 0$

$x = x \qquad dx = 1$

$y =$ fixed # constant $\quad dy = 0$

$$\int M\,dx + N\,dy$$

$$\int M\,dx$$

↑ some function of $y =$ constant either a or b

So $= 0$

So it does not matter

C_1 and C_3 are in different directions
So are $-$

And are over a conservative field
so it does not matter which path you take

$$\int_{C_1} \vec{F}\,dr = -\int_{C_3} \vec{F} \cdot d\vec{r}$$

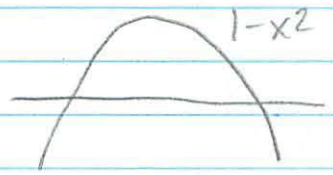
? is there more I have to do
? show area of inside
? count up the edges

3. Find the simple, closed, ⊕ oriented curve C for which ∮ has max value

$$\oint x^2(y+1)\,dx + 4x(1-y^2)\,dy$$

Use Green's Theorm
Note it has both ⊕/⊖ values

office hrs
olive



Want the biggest integrand

Don't want because that will be ⊖ so want from −1 to 1

? the above is the flux
max flux = max area

but then what are the constraints ?

??
don't get how

try again

$$\oint F \cdot dr = \oint_C x(y+1)\,dx + 4x(1-y^2)\,dy$$

$$= \oint M\,dx + N\,dy$$

Know the rules

$$= \iint_R (N_x - M_y)\,dA$$

$N = 4x - 4xy^2$
$M = x^2y + x^2$

then can write this

$$= \iint_R (4 - 4y^2 - x^2)$$

← and find it

So we know

$$\text{curl } \vec{F} = 4 - 4y^2 - x^2$$

$$4 - 4y^2 - x^2 = 0$$

$$\frac{1}{4}x^2 + y^2 = 1$$

$$\left[\frac{x^2}{4} + \frac{y^2}{1} = 1\right]$$

R is biggest inside ellipse ✓

Max curl will = max work = $\oint F dr$

4. $F = 2xy(1-x^2)\,\hat{i} + y^2(3x^2-1)\,\hat{j}$

$\langle 2xy - 2x^3y,\ 3x^2y^2 - y^2 \rangle$

a) Show that the flux of $\vec{F}$ across any smooth curve $y = f(x)$ running from $x=-1$ has same value $x=1$

So basically field conservative

$M = 2xy - 2x^3y$
$N = 3x^2y^2 - y^2$

$M_x = 2y - 6x^2y$
$N_y = 6x^2y - 2y$

$$\iint_R 2y - 6x^2y + 6x^2y - 2y$$

See next page

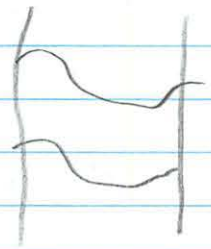
0

So field is conservative

Take 2 such curves C_1 and C_2 assume one lies entirely above the other

Green's Theorm in normal form

Taka flux form

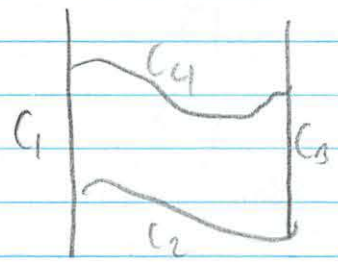


So this is basically the other problem $\circlearrowleft 90°$ and similar analysis works, (by symmetry) not exactly

1

b) Determine flux?

Isn't flux 0
by what I showed



$C_1 + C_3$ are 0
by symmetry

But we only want flux of one curve

But when I calculated it came out as 0 ✓

<u>Conservative</u> if $M_y = N_x$
which for us
$M_y = -N_x$

Ok so I did do it right. But what next?

Well you know

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot d\vec{r}$$

may be −

5. A function $f(x,y)$ defined in a region S of the xy plane is said to be harmonic in S if every point of S it satisfies the 2D form of Laplace's equation

$$f_{xx} + f_{yy} = 0$$

Prove that if $\vec{F}$ is the gradient of a harmonic function in S the flux of $\vec{F}$ across any simple closed curve lying entirely in S is 0

do not really get about what is what for gradient + non gradient fields

? Did we learn about "harmonic functions" - or they are teaching here

Is it that flux is 0 if inside conservative field?

Lets test $f = x^2 + xy$

$f_x = 2x + y$ $\quad f_{xx} = 2$ ⊗ not harmonic
$f_y = x$ $\quad f_y = 0$

$f = x^2 + y^2$

$f_x = 2x$ $\quad f_{xx} = 2$ ⊗
$f_y = 2y$ $\quad f_{yy} = 2$

$f = x^2 - y^2$
$f_x = 2x$ $\quad f_{xx} = 2$ ✓ harmonic
$f_y = -2y$ $\quad f_{yy} = -2$

Now $\vec{F} = \nabla f$

F is take gradient of f

$\vec{F} = \langle 2x, -2y \rangle$

$M = 2x \qquad M_x = 2$
$N = -2y \qquad M_y = -2$

$\iint 2 + -2 = 0$

Oh - its the same thing

$F = \langle f_x, f_y \rangle$

$M_x = f_{xx}$
$N_y = f_{yy}$

so its $\iint_R f_{xx} + f_{yy}$

And if it $= 0$ then no flux ✓
field conservative

? → ? Am I making the right conclusions?

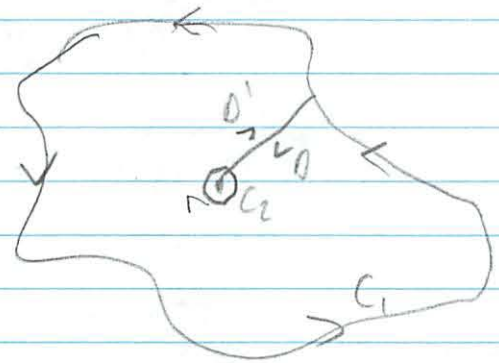
6. $F = \frac{-y}{x^2+y^2}\,\hat{i} + \frac{x}{x^2+y^2}\,\hat{j}$

Show that flux of F across any simply closed C surrounding origin $= 0$

Need to look closly at all of the assumptions

Note → field is not defined at origin
- Study notes

How does this differ from notes?



$$\oint_C = \oint_{C_1} + \int_D + \int_{D'} + \oint_{C^2}$$

circle

$$\oint_{C_1} - \oint_{C_2} = 0$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} = 2\pi$$

So is $dr = 2\pi r$?

Oh field is the same as example

$$F = \frac{\langle -y, x\rangle}{r^2}$$

$$= \iint_R \text{curl } \vec{F}\, d\vec{A}$$

$$\iint_R (N_y + M_x)\, dA$$

$$N = x\left(x^2+y^2\right)^{-1}$$
$$M = -y\left(x^2+y^2\right)^{-1}$$

$$N_y = -x\left(x^2+y^2\right)^{-2}\cdot 2y$$
$$M_x = y\left(x^2+y^2\right)^{-2}\cdot 2x$$

$$\text{flux} = \iint_R N_y + M_x$$
$$= \iint_R \left(\frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2}\right) dA$$
$$= \iint_R 0 \, dA$$
$$= 0 \quad \checkmark$$

18.02 Problem Set 7 Solns Spring 2010

1 a) $\oint_C \frac{1}{3}(-y^3dx + x^3dy)$ (R) C

$= \iint_R (x^2+y^2)\,dA$, by Green's theorem:

polar mom. of inertia of R about origin

$\frac{\partial}{\partial x}(x^3/3) = x^2$

$-\frac{\partial}{\partial y}(-\frac{y^3}{3}) = y^2$

b) $x = a\cos t \quad dx = -a\sin t\,dt$

$y = b\sin t \quad dy = b\cos t\,dt$

$\therefore \frac{1}{3}\oint_{C_1+C_2+C_3} -y^3dx + x^3dy = \frac{1}{3}\int_{C_1} -y^3dx + x^3dy$

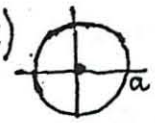
since integrand is 0 on C_2 ($x=0$, $dx=0$) and C_3 ($y=0$, $dy=0$)

$\int_{C_1} \frac{1}{3}(-y^3dx + x^3dy) = \frac{1}{3}\int_0^{\pi/2}(ab^3\sin^4 t + a^3b\cos^4 t)\,dt$

$= (ab^3 + a^3b)\cdot\frac{1}{3}\cdot\frac{3!!}{4!!}\cdot\frac{\pi}{2}$ $\frac{3!!}{4!!} = \frac{3\cdot 1}{4\cdot 2}$

$= ab(a^2+b^2)\frac{\pi}{16}$ Ans: $\boxed{ab(a^2+b^2)\cdot\frac{\pi}{4}}$

(multiply by 4 to include all 4 quadrants)

c) 

disc

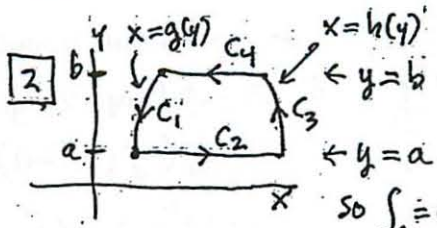
Putting $b=a$ in part (b) gives polar m. of i. $= \frac{2a^4\pi}{4}$ $\left(=\frac{\pi a^4}{2}\right)$

Calculating directly:

$= \int_0^{2\pi}\int_0^a r^2\cdot r\,dr\,d\theta$

$= 2\pi\cdot\frac{a^4}{4}$, which checks.

2



$\leftarrow y=b \quad dy=0$

$\leftarrow y=a \quad dy=0$

so $\int_{C_2}=0$, $\int_{C_4}=0$

$\therefore \oint_{C_1+C_2+C_3+C_4} N\,dy = \int_{C_1} N\,dy + \int_{C_3} N\,dy$

$= \int_b^a N(g(y),y)\,dy + \int_a^b N(h(y),y)\,dy$ (*)

Double integral: integrate along horiz. lines:

$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_a^b\int_{g(y)}^{h(y)} \frac{\partial N}{\partial x}\,dx\,dy$

Inner: $N(x,y)\Big]_{x=g(y)}^{x=h(y)} = N(h,y) - N(g,y)$

Outer: $\int_a^b [N(h,y) - N(g,y)]\,dy$,

which agrees with (*).

$\left(\int_b^a N(g(y),y)\,dy = -\int_a^b N(g,y)\,dy\right)$

3 By Green's theorem, (R) C

(*) $\oint_C \overset{M}{x^2(y+1)}\,dx + \overset{N}{4x(1-y^2)}\,dy$

$= \iint_R 4(1-y^2) - x^2\,dA$

$= \iint_R 4 - x^2 - 4y^2\,dA$

To maximize the integral, R should be the region where $4 - x^2 - 4y^2 \geq 0$

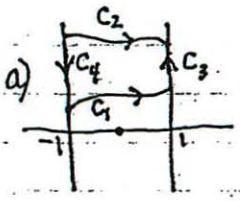
i.e., where $x^2 + 4y^2 \leq 4$

This is the region <u>inside</u> the ellipse C:

$x^2 + 4y^2 = 4$

So this C is the closed curve that maximizes the line integral.

$\boxed{4}$ a)



Use green's thm (normal form):

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(2xy - 2x^3y) + \frac{\partial}{\partial y}\left(y^2(3x^2-1)\right)$$

$$= 2y - 6x^2y + 6yx^2 - 2y$$

$$= 0$$

By Green's thm:

$$\therefore \oint_C \underset{\substack{\text{flux} \\ (\text{or } \vec{F}\cdot\hat{n}\,ds)}}{M\,dy - N\,dx} = \int_{C_1} + \int_{C_3} + \int_{C_2'} + \int_{C_4} = 0$$

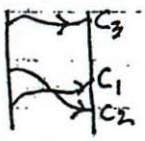
($C_2' = C_2$ oppositely oriented)

C_3: $x=1$, $dx=0$ $\quad \int_{C_3} M\,dy = \int \pm 2\cdot 1\cdot y\left(1-(\pm 1)^2\right)dy$

C_4: $x=-1$, $dx=0$ or C_4 $\quad = 0$

$$\therefore \int_{C_1} M\,dy - N\,dx = -\int_{C_2'} M\,dy - N\,dx = \int_{C_2} M\,dy - N\,dx$$

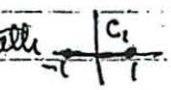
If C_1 and C_2 intersect, choose a C_3 above both of them:



Then by the previous:

$$\int_{C_1} M\,dy - N\,dx = \int_{C_3} M\,dy - N\,dx = \int_{C_2} M\,dy - N\,dx$$

b) Use as the path

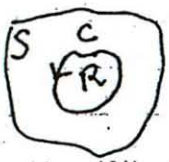


$y=0$, $dy=0$
$x=x$

$$\int_{C_1} M\,dy - N\,dx = \int_{C_1} 0\,dx = 0, \text{ since } N(x,0)=0.$$

$\boxed{5}$ Let the function $f(x,y)$ be harmonic in S, and suppose

$$\vec{F} = \vec{\nabla} f = \langle f_x, f_y \rangle$$



Flux of $\vec{F}$ across C

$$= \oint_C f_x\,dy - f_y\,dx$$

(by green's thm in normal form)

$$= \iint_R (f_x)_x + (f_y)_y \, dA$$

$$= \iint_R (f_{xx} + f_{yy})\, dA$$

$= 0$, since f is harmonic.

$\boxed{6}$ $\operatorname{div}\vec{F} = 0$ by calculation: (except at the origin, where $\vec{F}$ is undefined)

$$\frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) = -y\cdot\frac{\partial}{\partial x}(x^2+y^2)^{-1}$$

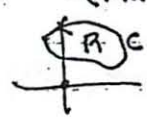
$$= -y\cdot\frac{-2x}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = x\cdot\frac{\partial}{\partial y}(x^2+y^2)^{-1}$$

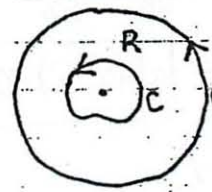
$$= x\cdot\frac{-2y}{(x^2+y^2)^2}$$

These have opposite signs, so their sum is 0.

(Thus $\oint_{C\ \text{flux}} \vec{F}\cdot\hat{n}\,ds = \iint_R \operatorname{div}\vec{F}\,dA = 0$



for any C not surrounding the origin, but we want a C which does surround the origin.)



Surround C with a large circle C_1 centered at the origin.

By the extended Green's thm (normal form)

$$\oint_{C_1 + C'} \vec{F}\cdot\hat{n}\,ds = \int_{C_1} \vec{F}\cdot\hat{n}\,ds - \int_C \vec{F}\cdot\hat{n}\,ds$$

$$= \iint_R \operatorname{div}\vec{F}\,dA = 0$$

$$\therefore \oint_C \vec{F}\cdot\hat{n}\,ds = \oint_{C_1} \vec{F}\cdot\hat{n}\,ds$$

However, the integral on the right is 0: for at each point of the circle, (say it has radius a)

$\vec{F} = \dfrac{-y\hat{i} + x\hat{j}}{a^2}$, which is tangent to the circle, $\therefore \perp$ to the normal, $\therefore \vec{F}\cdot\hat{n} = 0$

$\hat{n} = \dfrac{x\hat{i} + y\hat{j}}{a}$ (or note that $\vec{F}\cdot\hat{n} = 0$ from these two formulas)

[or you can calculate

$\oint_{C_1} M\,dy - N\,dx$ using $\begin{cases} x = a\cos\theta \\ y = a\sin\theta \end{cases}$

and get 0]

Lecture 26

Triple Integrals + Rect + Cylindrical coords

4/15

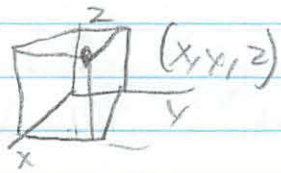
Exam a week from today
- vector integral calculus
- Green's Theorem
- triple integrals

Really need to study →

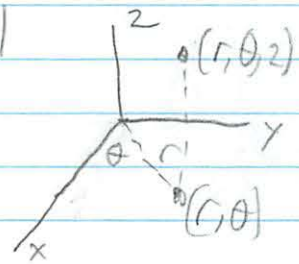
Now moving into space (3D)

Hard → setting up
figuring out equations of shape
limits

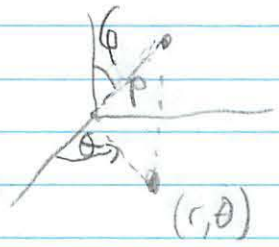
Coordinate systems
- rectangular



- cylindrical



- spherical



ρ is like r = distance from origin

$\theta = \measuredangle$ from x-axis (in xy plane)

φ = phi
angle from z axis
co-latitude
$\varphi = 0$ north pole
$\varphi = \frac{\pi}{2}$ equator

Look at the grid lines for a coord system
- but grid lines are surfaces
- level surfaces of the coordinates

Cect
$x = C_1$) vert planes
$y = C_2$
$z = C_3$ horiz plane

cyl
$r = C_1$ cylinder of radius C_1 axis: z-axis
$\theta = C_2$ like a page in a book (both sides)  top view
$z = C_3$ horiz plane

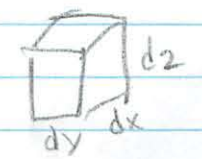
sph
$\rho = C_1$ sphere centered at origin
$\theta = C_3$ same; like a page in a book ← both sides
$\varphi = C_2$ half cone / nappe

Need to know how to convert between
- must memorize or be able to derive

can remember the triangle

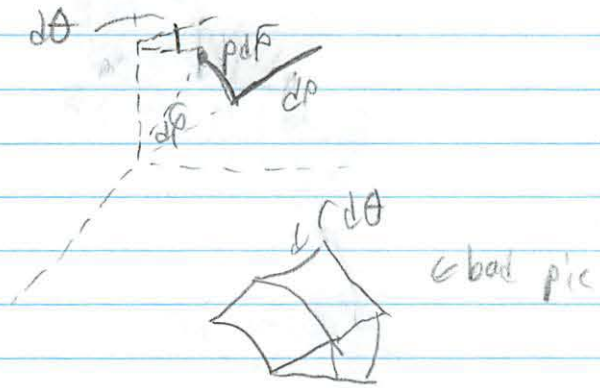
$x = r\cos\theta$ $\quad r = \rho\sin\varphi$
$y = r\sin\theta$ $\quad z = \rho\cos\varphi$

$dV = dx\,dy\,dz$
a little rectangular box
infantessamly small



θ is the longditude

Volume - $dV = dz\, r\, dr\, d\theta$

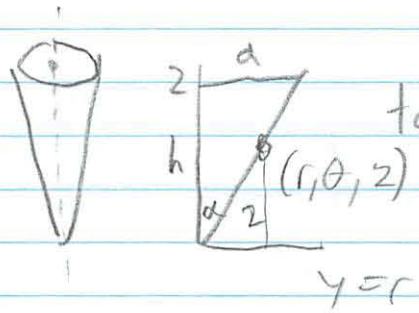


think of large melon
first cut slice b/w 2 book pages
then divide φ into values of ρ
- "mised rect


← better pie

$dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

Cone



$\tan\alpha = \frac{a}{h}$

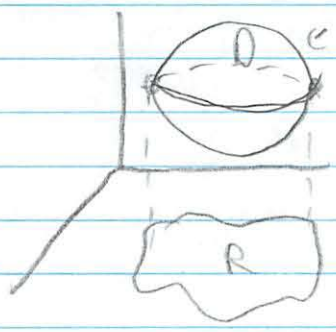
spherical $\varphi = \alpha$
cylindrical $\frac{z}{r} = \frac{h}{a}$ $\quad z = \frac{h}{a} r$
rect coord $z = \frac{h}{a}\sqrt{x^2+y^2}$

Lecture 27

Spherical Coords, Gravitational Attraction

4/16

Doing Triple Integrals
- harder than we will do so encounter all aspects
- read the book w/ simple examples
- no time for simple examples



← 2 surfaces converge ← $z = h(x,y)$, $z = g(x,y)$

← D = domain

← aux thing to use when calculating

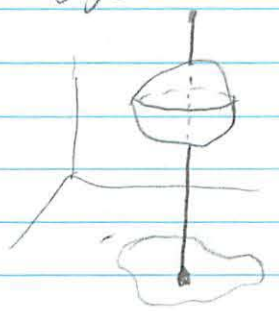
why did prof draw R like this wouldn't it be the shadow

D = volume
R = area

$$\iiint_D f(x,y,z)\, dV$$

↑ dom'n

$$= \iiint f\, dz$$



integrate from where enters ($z = g(x,y)$) to where it leaves ($z = h(x,y)$)

Use all the vertical lines in the shadow of D
- which is where x, y are in R

$$= \iiint_{g(x,y)}^{h(x,y)} f \, dz \cdot dA$$

0. Volume element
   - spherical $\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

1. Equations for $g(x,y)$ $h(x,y)$

2. Finding R
   - most students this is mistifying the most
   - intersection of two surfaces

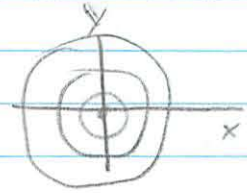
3. Put in limits on 3 integral signs
   - Use symmetry of the figure to simplify the limits
     - some are trivial
     - $\int_0^{\pi/2} \sin^n x \, dx$ $\int_0^{\pi/2} \cos^n x \, dx$ | hard to $\int$ from scratch
     - a lot of problems will have some symmetry

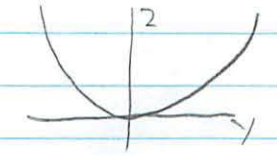
In each problem one of them is harder than the others

---

ex 1 Paraboloid $z = x^2 + y^2$
Plane $z = 2y$

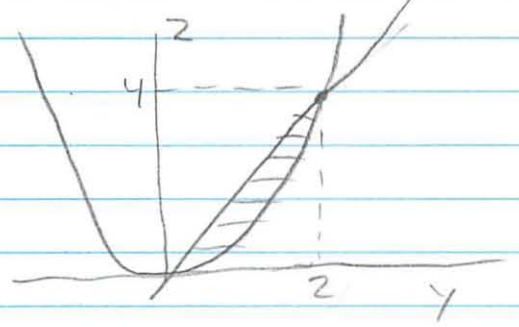
Find volume enclosed between them

level curves 

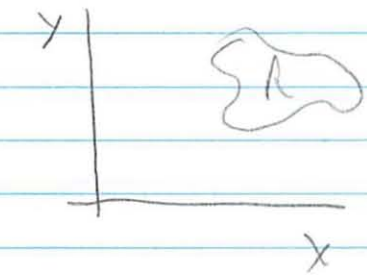


$z = y^2$

Cross Section of both



Find R



(x_1, y_1) in R

$\leftrightarrow$ there is a z_1 such that (x, y, z) lies in intersection of 2 surfaces

Find boundry of R

$\downarrow$ if and only if

$$\begin{cases} z = x^2 + y^2 \\ z = 2y \end{cases}$$ have solutions

Points that satisfy both equations
- non linear NLS problem

eliminate z

only where (x_1, y_1) satisfies
$x^2 + y^2 = 2y$

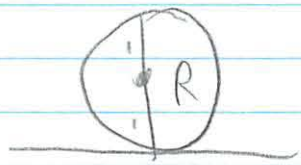
Single most essential step $\downarrow$

Boundry of R

$x^2+y^2-2y \rightarrow x^2+(y-1)^2$

complete square

- circle



to integrate over this region → natural to use polar coords

$r = 2\sin\theta$

$\text{vol} = \iiint 1 \, dz \cdot r \, dr \, d\theta$ ← dV in cylindrical coordinates

$$\int_0^{\pi} \int_0^{2\sin\theta} \int_{x^2+y^2}^{2y} 1 \, dz \, r \, dr \, d\theta$$

$\theta \quad r \quad z$

Don't do fancy set-ups
just answer the qu

In general set up + eval problems different to minimize mistakes

$$\int_0^{\pi} \int_0^{2\sin\theta} \int_{r^2}^{2r\sin\theta} dz \cdot r \, dr \, d\theta$$

Now inner, middle, outer integrals

<u>Inner</u>

$$zr \Big|_{r^2}^{2r\sin\theta}$$

$$r\left[2r\sin\theta - r^2\right]$$

<u>Middle</u>

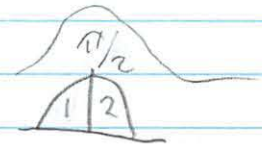
$$\frac{2}{3} r^3 \sin\theta - \frac{r^4}{4} \Big|_0^{2\sin\theta}$$

$$\frac{16}{3} \sin^4\theta - \frac{16}{4} \sin^4\theta$$

$$\frac{16}{12} \sin^4\theta$$

<u>Outer</u>

use symmetry



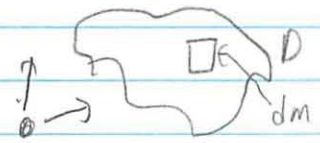
$$2\int_0^{\pi/2} \frac{4}{3} \sin^4\theta \, d\theta$$

$$\frac{8}{3} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \boxed{\frac{\pi}{2}}$$

↑ if even

This HW might be longer than others

Gravitational attraction in spherical coords



unit mass 1

$$\vec{F} = \langle F_x, F_y, F_z \rangle$$

Calculate F_z - total z-comp force on the mass at O

$$|\vec{F}| = \frac{G \cdot 1 \cdot dm}{\rho^2}$$ ← 2 masses

$$F_z = \frac{G\,dm}{\rho^2} \cos\varphi$$

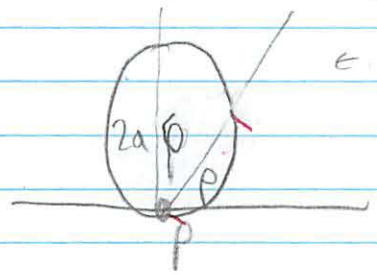
$$\text{Total } F_z = \iiint_D \frac{G \cos\varphi}{\rho^2} \delta \, dV$$ ← $\delta(x,y,z)$ usually constant though

for spherical coords - small miracle

$$\iiint_D \frac{\cos\varphi}{\cancel{\rho^2}} \cdot \cancel{\rho^2} \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Problem: find gravitational attraction of solid sphere
radius = a
$\delta = 1$
on a point circumference

Place point on origin



← eq of sphere in spherical coordinates?

ρ = lenght of rod

ρ φ coordinates

Dont memorize - be able to derive

$\rho = 2a\cos\varphi$

$6 \iiint \cos\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta$

$$6 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\varphi} \cos\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Practice for exam 3

Ex1. Let $f(x,y,z) = x + \sqrt{x^2+y^2} + x^2 + y^2 + z^2$. Express this function in terms of
a) cylindrical coordinates.
b) spherical coordinates.

Ex2. Let S be the solid in first quadrant between planes $z = 1$ and plane $x + y + z = 2$.
1) Set up limits for computing the integral of a function $f(x,y,z)$ over S: $\int\int\int f dzdydx$.
2) Suppose the densityof S is given by a function $\delta(x,y,z)$. What is the function f to integrate if one want to compute: a. The volume of S.
b. The Mass of S.
c. The moment of inertia around y axis.
d. The moment of inertia line defined by $x = 0,\ z = 1$.
e. The x coordinate $\bar{x}$ of the center of mass.
f. The gravitational attraction exerted by S on a untit mass placed at origin.

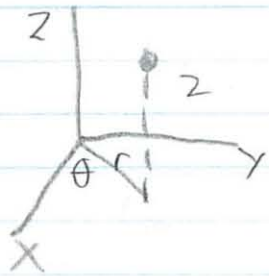
Ex3. Set up limits of integration for $\int\int\int_S f dV$, where S is
a) Cylinder or radius a and height 2 (Cylindrical).
b) Part of the unit sphere (around origin) in first octant (Cylindrical, Spherical)
c) Part of unit sphere (around origin) above plane $z = 1/2$ (Rectangular,Cylindrical, Spherical)
d) Part of unit sphere (around origin) above cone $z = \sqrt{x^2+z^2}$ (Rectangular,Cylindrical, Spherical)
e) Ball of radius a around $Q = (0,0,a)$ (Cylindrical, Spherical).
f) Part of that ball above plane $z = a$ (Cylindrical, Spherical).
g) Intersection of unit sphere around origin and unit sphere around (0,0,1) (Cylindrical).

Recitation
Review for Test 3

4/21

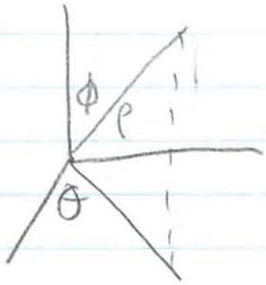
Lecture Coordinate systems

Cylindrical



$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
$\theta \in [0, 2\pi]$

Spherical



$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$
ρ = distance from center

$r = \rho\sin\phi$
$\theta \in [0, \pi]$

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$dM = \delta\,dV$$

Gravitational attraction of P_2 over P_1

$$\vec{F} = \frac{Gm_1m_2}{d^2}\frac{\overrightarrow{P_1P_2}}{|P_1P_2|} = \frac{Gm_1m_2}{d^3}\overrightarrow{P_1P_2}$$

ex 1 $f(x,y,z) = x + \sqrt{x^2+y^2} + \underbrace{x^2+y^2}_{\downarrow} + z^2$

Questions on sheet

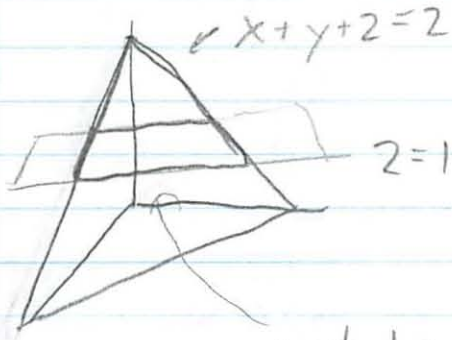
a) Cylindrical coords

$$f(r,\theta,z) = r\cos\theta + r + r^2 + z^2$$

b) spherical coords

$$f(\rho,\theta,\phi) = \rho\sin\phi\cos\theta + \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2}$$
$$+ (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi)^2$$
$$= \rho\sin\phi\cos\theta + \rho\sin\phi + (\rho\sin\phi^2) + (\rho\cos\phi)^2$$
$$= \rho\sin\phi\cos\theta + \rho\sin\phi + \rho^2$$

ex2

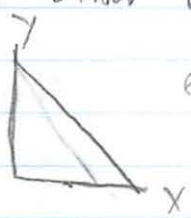


want top triangle

a) set up limit over that volume

$$\int_x \int_y \int_z f \, dz \, dy \, dx$$

Draw shadow of region in x-y plane

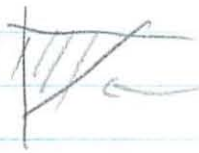


project volume onto xy plane (get rid of z)

eliminates z in $\begin{cases} x+y+z=2 \\ z=1 \end{cases}$

Shadow region bounded by $x+y+1=2$

$x+y=1$



← the more automated way to do it

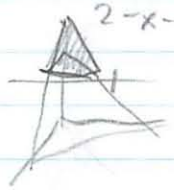
$$\int_0^1 \int_0^{1-x} \int_1^{2-x-y}$$

x y z

← solve for z

$z = 2-x-y$

$2-x-y$



b) If you want Volume function f you integrate $= 1$

Mass $\rightarrow f = \delta(x,y,z)$

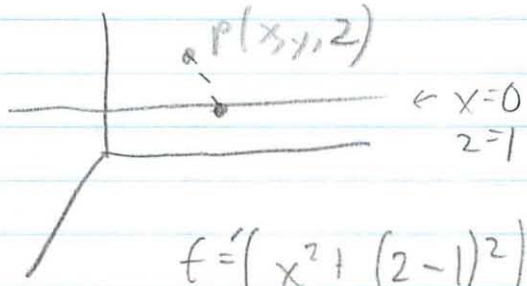
c) Moment of inertia solid around y axis
- know formulas from lecture

$$MI = \iiint \underset{\text{from } y \text{ axis}}{\text{distance}}^2 \, \delta \, dV$$

distance of $p = (x, y, z)$ from y-axis $= \sqrt{x^2+y^2}$

$f = (x^2+y^2)\delta$

d) distance of $P(x,y,z)$ from line $= \sqrt{x^2 + (z-1)^2}$



$f = \left(x^2 + (z-1)^2\right)\delta$

e) $\bar{x} = \frac{1}{M} \iiint \underbrace{x\delta}_{f}\, dV$

f) Gravitational attraction

$$\vec{F} = \frac{G m_1 m_2}{d^2} \frac{\overrightarrow{P_1 P_2}}{|P_1 P_2|} = \frac{G m_1 m_2}{d^3} \overrightarrow{P_1 P_2}$$

x component of gravitational attraction

x component of $F = \vec{F} \cdot \hat{i} = \frac{G m_1 m_2}{d^3} \overrightarrow{P_1 P_2} \, \hat{i}$

$$f = \frac{G m_1 m_2}{d^3} \overrightarrow{P_1 P_2} \, \hat{i}$$

$m_1 = 1 \Rightarrow$ unit mass

m_2 = mass given by small cube $= dm = \delta \, dV$

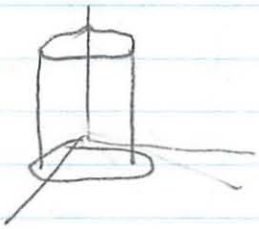
$$f = \frac{G \cdot 1 \cdot \delta}{(x^2 + y^2 + z^2)^{3/2}} x$$

$\overrightarrow{P_1 P_2} = \langle x, y, z \rangle$

$P_1 P_2 \cdot \hat{i} = x$

Then integrate, if required

3.

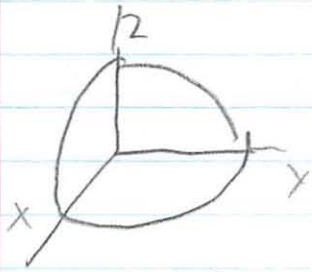


$$\int_0^{2\pi} \int_0^{a} \int_0^{2} f \, dV$$

$\theta \quad r \quad z$

always do this order! (I thought it was other way)

b)



Cylindrical coords
One 1/8 of a circle "unit sphere"

Cylindrical $\int_0^{\pi/2}_{\theta} \int_0^{1}_{r} \int_0^{r}_{z}$ → $x^2+y^2+z^2=1$
$r^2+z^2=1$

↑ from the shadow

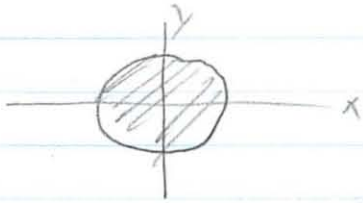
$\int_{\theta=0}^{\pi/2} \int_0^{1} \int_0^{1-r^2}$

spherical $\int_0^{\pi/2}_{\theta} \int_0^{\pi/2}_{\phi} \int_0^{1 \text{ or } r}_{\rho}$ ← better

c)

rectangular
1. find shadow region
2. eliminate z $\begin{cases} x^2+y^2+z^2=1 \\ z=\frac{1}{2} \end{cases}$ → $x^2+y^2=\frac{3}{4}$

Shadow is disc of radius $\frac{\sqrt{3}}{2}$



$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int_{-\sqrt{\frac{3}{2}-x^2}}^{\sqrt{\frac{3}{2}-x^2}} \int_{\frac{1}{2}}^{\sqrt{1-x^2-y^2}}$$

x y z

each time you fix something

cylindrical

- very similar in that need to find shadow region

$$\int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}} \int_{\frac{1}{2}}^{\sqrt{1-r^2}}$$

θ r z

Spherical

$$\int_0^{2\pi} \int_0 \int$$

θ ϕ ρ

$\phi_{max} =$

$\sin(\phi_{max}) = \frac{\sqrt{3}}{2}$

$\phi_{max} = \frac{\pi}{3}$

$$\int_0^{2\pi} \int_0^{\pi/3} \int_{\uparrow}^{1}$$

$z = \frac{1}{2}$

$\rho \cos\phi = \frac{1}{2}$

$\rho = \frac{1}{2\cos\phi}$

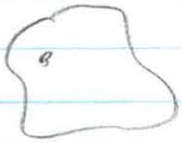
$$\int_0^{2\pi} \int_0^{\pi/3} \int_{\frac{1}{2\cos\phi}}^{1}$$

Seems calser than I thought

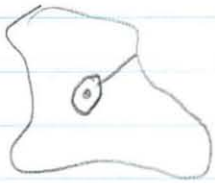
Test Review

0+1

4/21



$$\oint \vec{F} \cdot \vec{n}\, ds = \iint_R \operatorname{div} \vec{F}\, dA$$



$$\oint_C = \oint_{C'} + \cancel{\iint_R \operatorname{div} F\, dA}$$

$$\iint_R \operatorname{div} F\, dA = 0 \quad \text{if } \operatorname{div} = 0$$

---

parallel = tangent there

$$\int \vec{F} \cdot dr = 0$$

$$\int_C \vec{F} \cdot \vec{n}\, ds = \iint_C = \text{lenght} = \sqrt{2}$$

$$\vec{F} \cdot \vec{n} = \|\vec{F}\| \|\vec{n}\|$$

18.02 Exam 3 Thurs. Apr. 21, 2010 11:05-11:55

Directions:

1. There are 3 sheets, printed on both sides: eight problems in all.
2. Do all the work on these sheets; use the blank part below if truly necessary. Write down enough to show you are not guessing.
3. No books, notes, calculators, use of cell-phones, etc.
4. Please don't start until the signal is given; stop at the end when asked to; don't talk until your paper is handed in.
5. When the exam starts, read through the exam and start with what you are surest of.
6. Fill out the information below now.

Name Michael Plasmeier e-mail@mit.edu theplaz

Recitation teacher Oliver Rec. hour 12

$$\int_0^{\pi/2} \sin^n \theta \, d\theta = \int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{(n-1)!!}{n!!} A, \quad \text{where } A = \begin{cases} \pi/2, & n \text{ even} \\ 1, & n \text{ odd} \end{cases} \quad \text{and } n!! = n(n-2)(n-4)\ldots$$

Mean 76
passing 60

pg.1 16
pg.2 11
pg.3 6
pg.4 8
pg.5 11
Total 52

16

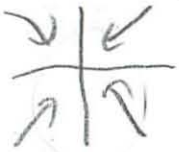
$\nwarrow\nearrow$ out $\frac{x\hat{\imath}+y\hat{\jmath}}{\sqrt{x^2+y^2}}$ in $\frac{-x\hat{\imath}-y\hat{\jmath}}{\sqrt{x^2+y^2}}$

Problem 1. (10 pts: 4; 3,3)

a) Write down in xy-coordinates the vector field $\mathbf{F}$ whose vector at (x, y) points *towards* the origin and has unit length.

did not memorize

4



$F = \text{dir} \cdot \text{lenght}$

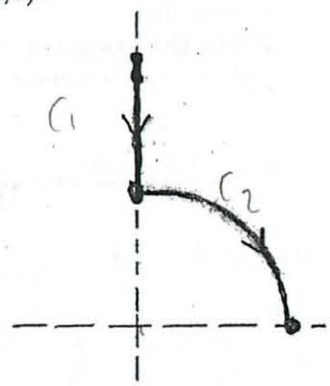
$\frac{-x\hat{\imath}-y}{\sqrt{x^2+y^2}}$ ✓

$\frac{-x}{-x} \cdot 1$

b) (No calculation required or expected): For the oriented path C consisting of the line segment from (0,2) to (0,1), followed by the quarter-circle (center (0,0), radius 1) from (0,1) to (1,0):

(i) find the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ on the path C

(ii) find the flux of $\mathbf{F}$ across C.



3 i) Only C_1 matters, because $C_2 \perp$ so it $= 0$

$\int M\,dx + N\,dy$ $\quad \int \frac{-y}{\sqrt{0^2+y^2}}\,dy$ ✓

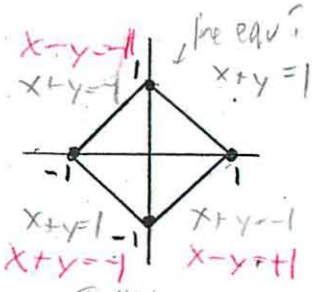
$dx \to 0$, $x = 0$ $\quad \int_1^2 1\,dy = 1$

2 ii) Flux $= \int M\,dy - N\,dx$ $\quad \int_0^{\pi/2} \vec{F} \cdot dr$ $\quad r = 1$

Only C_2 matters because $\perp$ $\quad \frac{1}{4} \cdot 2\pi r = \frac{\pi r}{2}$ $\quad$ stupid

Problem 2. (10) Use Green's theorem to evaluate $\oint_C (y^3 + 2y)\,dx + x(3y^2 - 1)\,dy$ over the positively oriented square C having vertices at the four points ± 1 on the x- and y-axes.

7



think got c wrong

$M = y^3 + 2y$ $\quad N = 3xy^2 - x$

$N_x = 3y^2 - 1$ $\quad M_y = 3y^2 + 2$

$\iint_R 3y^2 - 1 - (3y^2 + 2)\,dA$

oh it was $N_x - M_y$

- and flux is $M_x + N_y$
- confused them

$\int_{x=-1}^{1} \int_{y=-1+x}^{1-x} 6y^2 - 1\,dy\,dx$

$y = -1 - x$

← they just multiplied

$-3 \cdot$ area

$-3 \cdot (\sqrt{2})^2$

know lenght of side

-6

$\frac{6y^3}{3} - y \Big|_{-1-x}^{1-x}$

$(1-x)^2(1-x)$

$(1^2 - 2x + x^2)(1-x)$

$1 - 2x + x^2 + x - 2x^2 - x^3$

$-x^3 + 3x^2 - 3x + 1$

$2(1-x)^3 - (1-x) - [2(-1-x)^3 - (-1-x)]$

$2(-x^3 + 3y^2 - 3x + 1) - 1 + x - 2(-1-x)^3 - 1 - x$

Problem 3. (20 pts: 5,10,5) Let $\mathbf{F} = ay(x-y)\,\mathbf{i} + (x^2 - bxy + 3y^2)\,\mathbf{j}$, where a and b are constants.

a) Prove that if $\mathbf{F}$ is conservative, then $a = 2,\ b = 4$.

$N_x = M_y$

$M = 2xy - 2y^2$ $\qquad$ $N = x^2 - 4xy + 3y^2$

$M_y = 2x - 4y$ $\qquad$ $N_x = 2x - 4y$

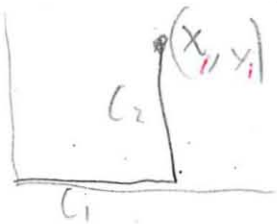
$2x - 4y = 2x - 4y$

$N_x = M_y$ so conservative

b) Let $a = 2,\ b = 4$.
Using a systematic method (show work), find a function $f(x,y)$ such that $\mathbf{F} = \nabla f$.

integrate along path

stupid thought mistake



$\int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy$ ← flipped

$\int_0$ $\quad$ $\int_0$ $\quad$ $x = x$ $\quad$ $\int_0$

$y = 0$

work $\int M\,dx + N\,dy$

flux $\int M\,dy - N\,dx$

$\int_0^x \cancel{x^2 - 4xy + 3y^2\,dx} + \int_0^y \cancel{2xy - 2y^2}\,dy$

$x = 0$ $\quad$ $C_1 = 0$ $\quad$ $x^2 - 4xy + 3y^2$

2 mistakes
- flipping work + flux
- and $y=0$ $dy=0$

$\int_0^x 3y^2\,dx + \int_0^y 2xy - 2y^2\,dy$

$3xy^2\Big|_0^x + \frac{2xy^2}{2} - \frac{2y^3}{3}\Big|_0^y$

$3xy^2 + xy^2 - \frac{2}{3}y^3$

$4xy^2 - \frac{2}{3}y^3$

$x_1^2 y_1^2 - 2x_1 y_1^2 + y_1^3$

should have #
- but don't have x, y pts!
so ok

X − 5

c) Continue to use $a = 2,\ b = 4$, and using the earlier parts of this problem:
evaluate $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ along the path C given by: $x = \tan t,\ y = \sin 2t$, as t runs from 0 to $\pi/4$.
(This is easy, but you must show the work.)

Fundemental Theorm of calculus line integrals

$t = 0 \to (0,0)$

$t = \pi/4 \to (1,1)$

well gradiant field so any path is the same if gradiant field

$\int_C F \cdot dr = f(x,y)\Big|_{P_1}^{P_2}$

Could use path above

at $\frac{\pi}{4}$ $\quad$ $x = \tan\frac{\pi}{4} = 1$

$y = \sin 2\frac{\pi}{4} = 0$

X −4

$4(1)(0)^2 - \frac{2}{3}(0)^3 = 0$

$4x^2 - 2y^2 x + y^3 \Big|_{0,0}^{1,1} = 0$

6/20 ran out of time

Problem 4 (20 pts: 5, 10, 5, 0)

Verify Green's theorem in the tangential (work) form for the vector field $\mathbf{F} = -y^3\,\mathbf{i} + x^3\,\mathbf{j}$ and the closed curve C consisting of the two segments of the x- and y-axes connecting the origin with the points $(a, 0)$ and $(0, a)$, plus the quarter-circle given by $x = a\cos\theta$, $y = a\sin\theta$, $0 \le \theta \le \pi/2$, $a > 0$.

That is, showing work:

(a) apply Green's theorem;

(b) evaluate the line integral (over all of C);

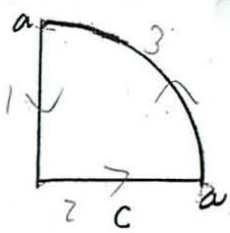
(c) evaluate the double integral;

(d) spend the next twenty minutes trying to make them agree.

(In calculating the line integral you will need the formulas on the cover sheet.)

hopefully not confused

correction parametrizing w/ θ



$\int\int_{C_3} M\,dy - N\,dx$ ← flux form

$dx = -a\sin\theta$

$dy = a\cos\theta$

$-a^3\sin^3\theta\, a\sin\theta - a^3\cos^3\theta\, a\cos\theta$

curl not calculated

~~Not theorem~~ Green not stated

$\int_0^{\pi/2}\int_0^a \left(-a^4\sin^4\theta - a^4\cos^4\theta\right) r\,dr\,d\theta$ $a = r$

why double integral?

$-a^4\frac{r^2}{2}\left(\sin^4\theta - \cos^4\theta\right)\Big|_0^a$

$-\frac{a^6}{2}\left(\sin^4\theta + \cos^4\theta\right)$

$-\frac{a^6}{2}\int_0^{\pi/2} \sin^4\theta + \cos^4\theta$

$\int \frac{(n-1)!!}{n!!} A$ 1 same

$2\,\frac{(4-1)!!}{4!!}\cdot\frac{\pi}{2}$

$2\,\frac{3(3-2)}{4(4-2)(4-4)}\cdot\frac{\pi}{2}$ ← ?

$-\frac{a^6}{2}\cdot\frac{4-6}{16-8}\cdot\frac{\pi}{2}$ $\frac{-a^6 3\pi}{8}$

Now need to do each seg + Green's theorem

$\int M\,dy$ $\int -N\,dx$

$\int_0^a -y^3\,dy$ $\int_0^a x^3\,dy$

$-\frac{a^4}{4}$ $\frac{a^4}{4}$

0 both incorrect

$\int N_x + M_y$

$\int 3x^2 - 3y^2$

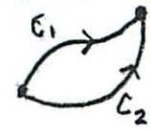
? ran out of time

seperate sheet

Problem 5. (10) Let $f(x, y)$ be a function with continuous second derivatives that satisfies the equation

$$f_{xx} + f_{yy} = 0.$$

Show that the field $\mathbf{F} = \nabla f$ has the same *flux* across any two differentiable paths C_1 and C_2 which start at the same point P_1, end at the same point P_2, and don't intersect.



because it is a gradient field

Curl $\vec{F} = 0$

separate sheet

2

$\cancel{N_x + M_y} = 0$

$\iint_R 0 = 0$

$N_x = f_{yx}$

$M_y = f_{xy}$

if $f_{yx} + f_{xy} = 0$ then $f_{xx} + f_{yy} = 0$

Create closed surface

Use Green's Theorem

$\iint M_x + N_y$

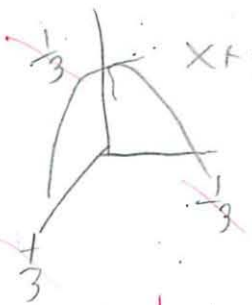
Problem 6. (10 pts.: 8,2)

6

Let D be the domain of finite volume in the first octant cut off by the plane $x + y + z = 1$.

a) Set up an iterated triple integral in rectangular coordinates $\int\int\int f(x, y, z)\, dz\, dy\, dx$ for the total mass of D, if the density function is given by $\underline{\delta = z}$. (Give the integrand and limits.)

b) Then take only the first step in evaluating it, i.e., reduce it to a double iterated integral in x and y.



Oh that was wrong

was all 1

Mass $\iiint \delta\, dV$

$\int_0^{1/3}\int_0^{\frac{1}{3}(1-z)}\int_0^{\frac{1}{3}(1-z-y)} z\, dx\, dy\, dz$ opps

$\int_0^1\int_0^{1-x}\int_0^{1-x-y} z\, dz\, dy\, dx$

set x constant

$y = 1 - z$

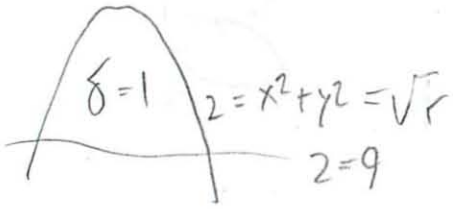
$\int_0^{\frac{1}{3}-\frac{1}{3}z-\frac{1}{3}y} z\, dx$

$zx \Big|_0^{1/3-\frac{1}{3}z-\frac{1}{3}y}$

$\int_0^{1/3}\int_0^{\frac{1}{3}(1-z)} z\left(\frac{1}{3} - \frac{1}{3}z - \frac{1}{3}y\right)$

$\frac{1}{3}z - \frac{1}{3}z^2 - \frac{1}{3}yz \; dy\, dz$

Problem 7. (10) Let V be the finite domain in xyz-space lying between the paraboloid $z = x^2 + y^2$ and the plane $z = 9$. Taking density $\delta = 1$, set up an iterated triple integral in cylindrical coordinates (provide limits and integrand, but do not evaluate) giving its moment of inertia about the z-axis.



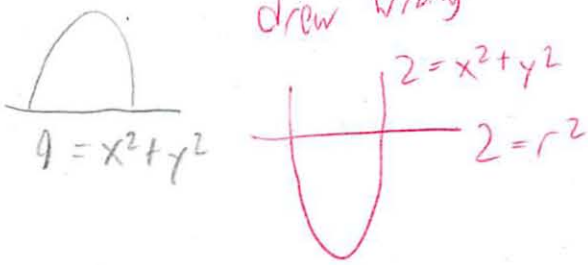
mass $\int_0^{2\pi} \int_0^{\sqrt{9}=3} \int_{9}^{x^2+y^2=\sqrt{r}\ r^2} dz\, r\, dr\, d\theta$

← did wrong order

just pase

$\int_{r^2}^{9}$

drew wrong



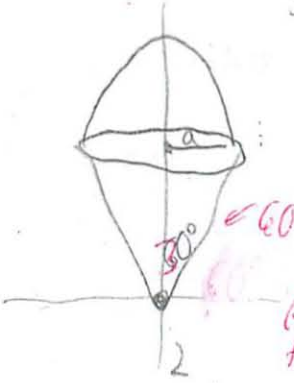
if want distance take $\sqrt{\ }$

$x^2+y^2=r^2$

not $\sqrt{r}$

Problem 8. (10) Let D be the "filled ice-cream-cone" in 3-space whose top surface is formed by a portion of the sphere of radius a centered at the origin, and whose bottom surface is a portion of the circular cone having as axis the z-axis, and whose vertex angle is 60°. Set up a triple iterated integral in spherical coordinates which gives the volume of D (provide limits and integrand, but do not evaluate).

$\iiint \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$

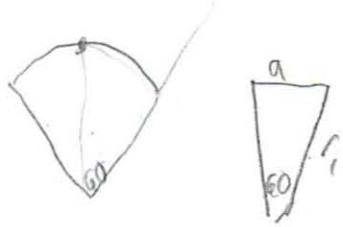


← 60° was for full thing

$2\int_0^{2\pi} \int_0^{\cancel{60^\circ}\ 30^\circ} \int_0^{\cancel{\sqrt{2}a}\ a} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$

$\theta \quad \varphi \quad \rho \quad 0$ ← can just say

6



$\frac{a}{\frac{\sqrt{2}a}{2}} = \frac{2a}{\sqrt{2}a} = \frac{2\sqrt{2}}{2}$

$\tan 60 = \frac{a}{r}$

$\tan 60 = \sqrt{2}$

$\sin 60 = \frac{a}{r}$

$r \sin 60 = a$

$\sin 60° = \frac{\sqrt{2}}{2}$

4)

a) $\oint -y^3\,dx + x^3\,dy = \iint_A 3x^2 + 3y^2\,dA$

think I screwed it up again

~~Flux $\iint M\,dy - N\,dx$ ← what I wrote~~

~~Work $\iint M\,dx + N\,dy$ ← what it asked for~~

$\oint F\uparrow dx + F\uparrow dy$

$\iint M\,dx + N\,dy$

$F\uparrow_x$

tangent

normal

Work	$\oint M\,dx + N\,dy \overset{GT}{=} \iint (N_x - M_y)\,dA$	Both Always work
Flux	$\oint M\,dy - N\,dx \overset{GT}{=} \iint (M_x + N_y)\,dA$	Grad field or not.

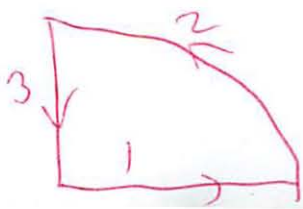
the fundemental problem I had on the test was screwing up these ∇

∇

$\int M\,dx + N\,dy = \iint (N_x - M_y)\,dA$

$M = F\uparrow$ one equation

b) Now break into parts



$C_1 = y = 0 \quad dy = 0 \quad S = 0$

$C_3 = x = 0 \quad dx = 0 \quad S = 0$

→

$C_2 = \quad x = a\cos\theta \qquad dx = -a\sin\theta$

$\qquad\quad y = a\sin\theta \qquad dy = a\cos\theta$

✓ had this

$$\int_{C2} = \int_0^{\pi/2} -a^3\sin^3\theta\,(-a\sin\theta) + a^3\cos^3\theta\,(a\cos\theta)\,d\theta$$

not double integral just along line

$$a^4 \int_0^{\pi/2} (\sin^4\theta + \cos^4\theta)\,d\theta$$

$$a^4 \left(\frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2} + \frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2}\right)$$

$$\frac{3\pi}{8}a^4$$

What did I screw up here

remember !! = every other one

$n = 4$

$n-1 = 3$

$(n-1)!! = (4-1)\cdot(4-1-2) = 3\cdot 1$

$n!! = 4\cdot 2$

this was close but I screwed it up

No it was actually right
except I did not do 16-8
really stupid!

c) Here double integral

$$= \iint_R 3x^2 + 3y^2 \, dA$$

(but w/ these #

$$\int_0^{\pi/2} \int_0^a 3r^2 \cdot r \, dr \, d\theta$$

- use polar

$$= \frac{3r^4}{4} \Big|_0^a \cdot \frac{\pi}{2}$$

$$\frac{3\pi}{8} a^4$$

---

5. $\nabla f = \langle f_x, f_y \rangle$

Did not know these rules

flux of $\vec{\nabla} f$ over C

$$= \int_C M\,dy - N\,dx$$

$M = f_x \quad N = f_y$

$$\int M\,dy - N\,dx$$

div

$$\iint_R f_{xx} + f_{yy} \, dA$$

$$\int_{C_1' + C_2} = -\int_{C_1} + \int_{C_2} = 0$$

$$\int_{C_1} = \int_{C_2}$$

yeah that it is a - proof

18.02 Exam 3 Solns Spring 2010

1 a) $\vec{F} = -\dfrac{\langle x, y\rangle}{\sqrt{x^2+y^2}}$

b) $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \underset{\text{on } C_1}{1} + \underset{\text{on } C_2}{0} = 1$

flux over $C = \underset{\text{on } C_1}{0} + \underset{\text{on } C_2}{\frac{\pi}{2}\cdot 1} = \frac{\pi}{2}$

2 $\displaystyle\oint_C (y^3+2y)\,dx + x(3y^2-1)\,dy$

$= \displaystyle\iint_R (N_x - M_y)\,dA$

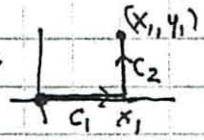
$= \displaystyle\iint_R (3y^2-1)-(3y^2+2)\,dA$

$= \displaystyle\iint_R -3\,dA = -3\cdot\underbrace{(\sqrt{2})^2}_{\text{area}} = -6$

3 $\dfrac{\partial M}{\partial y} = a(x-2y)$ $\quad \dfrac{\partial N}{\partial x} = 2x - by$

a) $a(x-2y) = 2x - by \Rightarrow \begin{array}{l} a=2 \\ b=2a=4 \end{array}$

b) Method 1:



$f(x_1,y_1) = \displaystyle\int_{C_1+C_2} 2y(x-y)\,dx + (x^2-4xy+3y^2)\,dy$

$\displaystyle\int_{C_1} = 0+0$ (since $y=0$, $dy=0$)

$\displaystyle\int_{C_2} = \underset{(dx=0)}{0} + \int_0^{y_1}(x_1^2-4x_1y+3y^2)\,dy$

$= x_1^2y_1 - 2x_1y_1^2 + y_1^3$

Method 2:

$f_x = 2y(x-y)$ $\quad f_y = x^2-4yx+g(y)$

$f = yx^2 - 2y^2x + g(y)$ $\quad = x^2-4xy+3y^2$

$\therefore g' = 3y^2,\ g = y^3$

$f = yx^2 - 2y^2x + y^3$ (+c unnecessary)

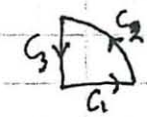
c) $t=0 \Rightarrow (x,y)=(0,0)$ $\quad$ use

$t=\pi/4 \Rightarrow (x,y)=(1,1)$ $\quad$ F.T. Calc for line integrals

$yx^2 - 2y^2x + y^3\Big]_{(0,0)}^{(1,1)} = 0-0 = 0$

4 a) $\displaystyle\oint_C -y^3\,dx + x^3\,dy = \iint_R 3x^2+3y^2\,dA$

b) On C_1: $y=0, dy=0$ $\quad \int_{C_1} = 0$



C_3: $x=0, dx=0$ $\quad \int_{C_3} = 0$

C_2: $x = a\cos\theta$ $\quad y = a\sin\theta$

$dx = -a\sin\theta$ $\quad dy = a\cos\theta$

$\displaystyle\int_{C_2} = \int_0^{\pi/2} -a^3\sin^3\theta(-a\sin\theta) + a^3\cos^3\theta\cdot(a\cos\theta)\,d\theta$

$= a^4\displaystyle\int_0^{\pi/2}(\sin^4\theta+\cos^4\theta)\,d\theta =$

$= a^4\left(\frac{3\cdot1}{4\cdot2}\cdot\frac{\pi}{2} + \frac{3\cdot1}{4\cdot2}\cdot\frac{\pi}{2}\right) = \frac{3\pi a^4}{8}$

c) $\displaystyle\iint_R 3x^2+3y^2\,dA = \int_0^{\pi/2}\int_0^a 3r^2\cdot r\,dr\,d\theta$

(use polar coords) $= \dfrac{3r^4}{4}\Big]_0^a\cdot\pi/2$

$= \dfrac{3\pi a^4}{8}$

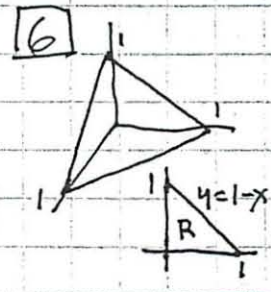
5 $\vec{\nabla} f = \langle f_x, f_y\rangle$

Flux of $\vec{\nabla} f$ over C (any C) $= \displaystyle\int_C M\,dy - N\,dx$ $\quad M = f_x,\ N = f_y$

$\displaystyle\oint_{C_1'+C_2} M\,dy - N\,dx = \iint_R \underbrace{f_{xx}+f_{yy}}_{\text{div}(\vec{\nabla} f)}\,dA = 0$

$\displaystyle\int_{C_1'+C_2} = -\int_{C_1} + \int_{C_2} = 0,\ \therefore \int_{C_1} = \int_{C_2}$

6

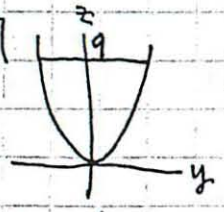


$\displaystyle\int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz\,dy\,dx$

Inner: $\frac{1}{2}z^2\Big]_0^{1-x-y}$

$\therefore \displaystyle\int_0^1\int_0^{1-x}\frac{1}{2}(1-x-y)^2\,dy\,dx$

$\boxed{7}$



$z = x^2 + y^2$

$z = r^2$

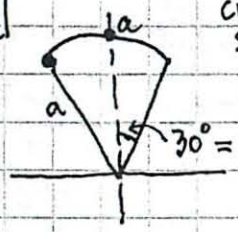
(cross-section)

$$\int_0^{2\pi}\int_0^{3}\int_{r^2}^{9} r^2 \cdot dz \cdot \underbrace{r\,dr\,d\theta}$$

moment of inertia about z-axis

dA in polar coords

$\boxed{8}$



cross-section

eqn cone: $\varphi = \pi/6$

eqn cap: $\rho = a$

$$\text{Vol} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{a} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

integrand = dV in spherical coords

Triple Integrals

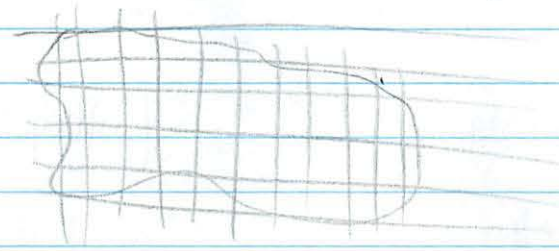
From Book Chap 20.5

4/20

Same pattern as double integral (20.2)

Review

- which was over a region



Add the blocks

$$\iint_R f(x,y)\,dA$$

Must be bounded by a rectangle
- Not off to ∞

Add up the area of each rectangle

The function is the height
- volume = height • area of base

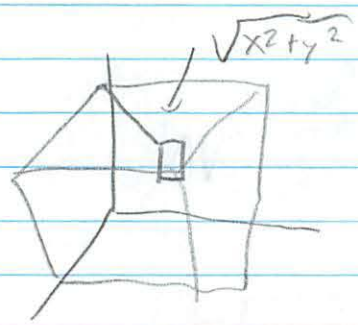
end review

$f(x,y,z)$ defined in 3D region R

Divide R into many small rectangular boxes along coords
- add volumes

Take limit as diagnoal of boxes $\to 0$

$$\Delta V = \Delta x \Delta y \Delta z$$
$$dx\, dy\, dz$$



function must be contineous, etc, etc

Region R is thought of as a solid body $\delta = \delta(x, y, z)$

$$M = \iiint_R \underbrace{\delta\, dV}_{\text{element mass}}$$

Moments

$$M_{yz} = \iiint x \delta\, dV$$
$$M_{xz} = \iiint y \delta\, dV$$
$$M_{xy} = \iiint z \delta\, dV$$

Moments of inertia

$$I_x = \iiint_R (y^2+z^2)\delta\, dV$$
$$I_y = \iiint_R (x^2+z^2)\delta\, dV$$
$$I_z = \iiint_R (x^2+y^2)\delta\, dV$$

$$\bar{x} = \frac{M_{yz}}{M} \quad \bar{y} = \frac{M_{xz}}{M} \quad \bar{z} = \frac{M_{xy}}{M}$$

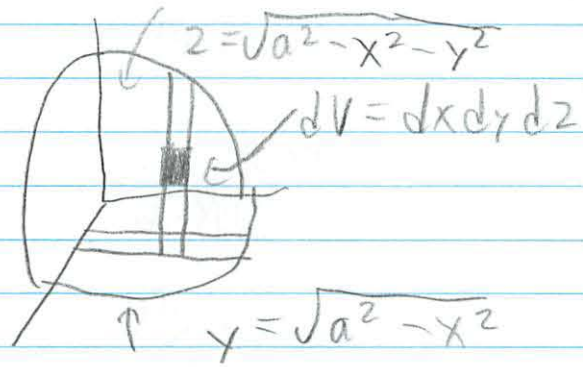
center of mass if δ constant

Evaluate w/ iteration

$$a \le x \le b$$
$$y_1(x) \le y \le y_2(x)$$
$$z_1(x,y) \le z \le z_2(x,y)$$

$$\int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

Can choose any order you want however



Lecture 28

Vector Fields in 3-space, flux surface S

4/23

<u>Integration shortcut</u>

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x)\, g(y) \; dy \; dx$$

↑ product
each function uses 1 variable
limits are constants
- not functions
- all of them

Can do 2 integrals <u>seperately</u>

$$= \int_{a_1}^{b_1} f(x)\,dx \cdot \int_{a_2}^{b_2} g(y)\,dy$$

Same for triple integrals $\iiint$

<u>Now extend vector calculus to space</u>

- already did tripple integrals
- Green's theorm
- flux
- work

Need vector fields in 3-space

$$\vec{F} = M\hat{i} + N\hat{j} + P\hat{k} \quad (\text{or } p, q, r)$$

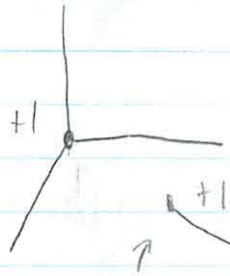
↑ ↑ ↑

are each functions of 3 variables

$$F = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$$

Examples

gravitational fields
electrostatics
electromagnetic
velocity
fluid flow



they repel each other
which way does vector go?

point (x,y,z)

$$F = \frac{\langle x,y,z \rangle}{\rho}$$ ← unit vector (direction)

$$\vec{F} = \underbrace{\frac{\langle x,y,z \rangle}{\rho}}_{\text{direction}} \cdot \underbrace{\frac{k}{\rho^2}}_{\text{magnitude}} = \frac{k\langle x,y,z \rangle}{(x^2+y^2+z^2)^{3/2}}$$

---

in 3 space flux + work are 2 totally different concepts
each needs like 3 - 4 lectures

flux → flow

2D

(+) is left to right as you travel

3D flux across surface

Draw a little piece of territory on surface

If tangent = no flux
so only interested in component $\perp$
to surface

But what direction is positive?
- you must specify
- not automatic for 3D

Count up all the little surface areas

Flux through a little piece

$\vec{F} \cdot \hat{n}$

↑ component of vector in the direction you are interested in) dot product

$(\vec{F} \cdot \hat{n})\, dS$

↑ surface, not area called - but same thing

$= \vec{F} \cdot d\vec{S}$

↑ $d\vec{S} = \hat{n}\, dS$

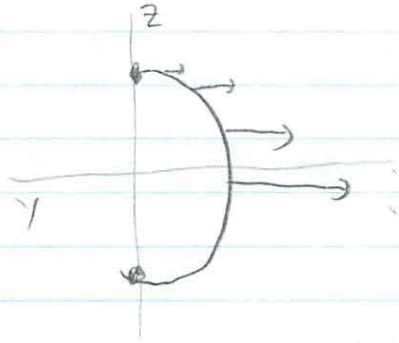
Flux through Surface

$= \iint_S \vec{F} \cdot d\vec{S}$ surface integral for flux of $\vec{F}$

The General formulas are useless
Need examples

$\vec{F} = y\hat{j}$ Si right hemisphere
of radius a
center at origin

Draw useful 2D pictures
Not complex 3D pictures



Must calculate
<u>not</u> intuition

(<u>not</u> direction of normal vector)

We need $\vec{F}, \hat{n}, d\vec{S}$

$\vec{F} = y\hat{j}$ ← now

$\hat{n} = \dfrac{\langle x, y, z\rangle}{a}$ ← same for all spheres
picked radially out (easier)

$dS =$ ← "is a formula for
but its stupid" - Mattuck

Remember $dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$
Very small thickness $d\rho$

$$dS = a^2 \sin\varphi \, d\varphi \, d\theta$$

2 coord systems are fighting

$$\vec{F}\cdot\hat{n} = \frac{y^2}{a}$$

Convert y to spherical coordinates

$$y = r\sin\theta$$

$$r = a\sin\varphi$$

$$\vec{F}\cdot\hat{n} = \frac{y^2}{a} = \frac{(a\sin\varphi\sin\theta)^2}{a}$$

$$\text{flux} = \iint (\vec{F}\cdot\hat{n})\,dS$$

$$= \int\int \underbrace{a\sin^2\varphi\sin^2\theta}_{\vec{F}\cdot\hat{n}} \cdot \underbrace{a^2\sin\varphi\,d\varphi\,d\theta}_{dS}$$

$$= a^3 \int_0^{\pi}\int_0^{\pi} \sin^3\varphi\sin^2\theta\,d\varphi\,d\theta$$

θ φ

now we can use our shortcut
but we could make it even easier
if have $\frac{1}{4}$ hemisphere (in 1st octant)

$$= 4a^3 \int_0^{\pi/2}\int_0^{\pi/2} \sin^3\varphi\sin^2\theta\,d\varphi\,d\theta$$

$$\text{Flux} = 4a^3 \int_0^{\pi/2} \sin^3 \varphi \, d\varphi \cdot \int_0^{\pi/2} \sin^2 \theta \, d\theta$$

$$= 4a^3 \cdot \frac{2}{3 \cdot 1} \cdot \frac{1}{2} \frac{\pi}{2} \cdot$$

$$= \frac{2}{3} \pi a^3$$

$$= \underbrace{2\pi a^2}_{\text{area hemisphere}} \cdot \underbrace{\frac{a}{3}}_{\text{velocity}}$$

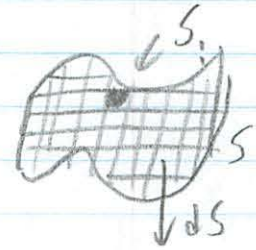
Recitation

4/26

New material

Surface Integrals

Def $\iint_S f(P)\,dS = \lim \sum f(P_i)\,dS_i$



* Flux of 3D field across S

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S}$$

$d\vec{S} = \hat{n}\,dS$

$\hat{n}$ = unit vector $\perp$ to surface

dS = area

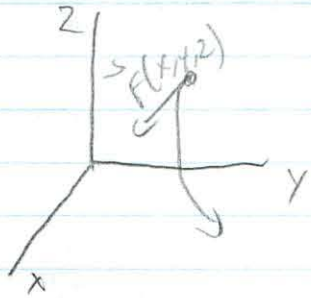
Today <u>no</u> parametrization

Field in 3D

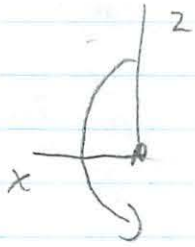
→ <u>Ex 1</u> Give flux of field $\vec{F}(x,y,z)$ which is velocity field of rigid rotation of angular velocity ω

a) axis = y axis around

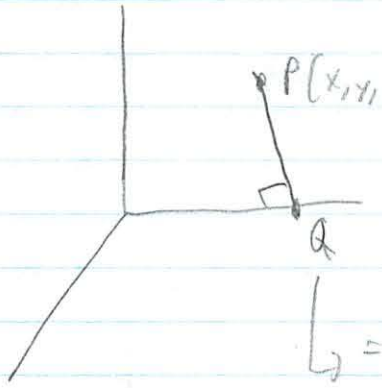
b) around a line through origin direction $\vec{u}$



What is magnitude of field? $|\vec{F}(x,y,z)|$



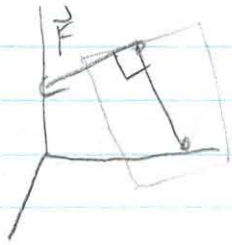
velocity = radius $\cdot\, w$
= distance to y axis $\cdot\, w$



Projection of point P onto y-axis
$\hookrightarrow = (0, y, 0)$

distance $\sqrt{x^2 + z^2}$

$$|\vec{F}| = w\,|\vec{PQ}| = w\sqrt{x^2+z^2}$$



$\vec{F} \perp |\underline{PQ}|$

$\overline{PQ} = \langle x, 0, z \rangle$

$\vec{F} = w\langle z, 0, -x \rangle$

↑ to get right magnitude, adjust

Check sign is right

$$|\vec{F}| = w\,|\vec{PQ}|$$

$\vec{F} \perp \vec{U}$
$\vec{F} \perp \vec{PQ}$

↑ take cross product to get something in right direction

$\vec{F} = \_\_ \vec{U} \times \vec{OP}$

↑ what should coefficient be?
just w
for amplitude

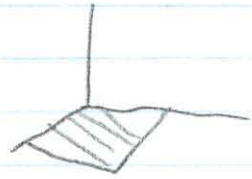
$$|\vec{U} \times \vec{OP}| = |\vec{U}| \times |\vec{QP}| = |\vec{QP}|$$

In fact $\vec{F} = w\,\vec{U} \times \vec{OP}$
because $\vec{OP} = \vec{OQ} + \vec{OP}$
and $\vec{U} \times \vec{OQ} = \langle 0,0,0 \rangle$

Surface Integrals w/o Parametrization

ex2 What is the volume of flux $\phi = \iint_S \vec{F} \cdot d\vec{S}$

S = horizontal rectangle w/ coord
(0,0,0) (0,1,0) (2,0,0) (2,1,0)



must define:
oriented upward

a) $\vec{F} = xy\,\hat{i} + z\,\hat{j}$
b) $F(x,y,z) = \hat{i}x$
c) $F(x,y,z) = xy\,\hat{k}$

Use some geometric property to simplify

Perpendicular to surface is vertical

$d\vec{S} = \hat{k}\, dS$

a) $\iint_S \vec{F} \cdot d\vec{S} = \iint_S 0\, dS = 0$

Horizontal - no partical going across

b) $\iint_S \vec{F} \cdot d\vec{S} = \iint_S dS = \text{Area} = 2 \cdot 1 = 2$

$\parallel$

Same direction

$\hat{k} \cdot \hat{k}\, dS = dS$

c) $\iint_S \vec{F} \cdot d\vec{S} = \iint_S xy\, dS = \iint xy\, dx\, dy$

$\parallel$

$xy\hat{k} \cdot \hat{k}\, dS$

$= xy\, dS$

the small area is this

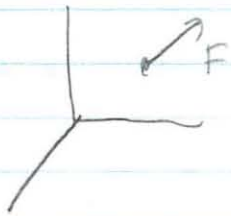
$= \int_{x=0}^{2} \int_{y=0}^{1} dy\, dx$

$= 1$

ex 3 More w/ symmetrics

Compute $\iint_S \vec{F} \cdot d\vec{S}$

a) $\vec{F}(x,y,z) = \langle x, y, z \rangle$

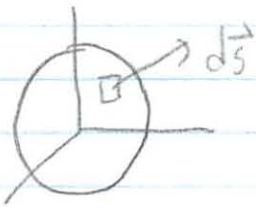


Surface = sphere radius a at origin

b) $\vec{F}(x,y,z) = \langle 0, 0, z \rangle$
S = sphere

c) $\vec{F}(x,y,z) = \langle 0, 0, z \rangle$
S = upper half of sphere

a)



direction of vector $\perp$ surface $= \vec{F}$

$$\iint \vec{F} \cdot d\vec{S}$$
$$= |\vec{F}||d\vec{S}| = \sqrt{x^2+y^2+z^2}\, dS = a\, dS$$

because $\vec{F} \parallel d\vec{S}$

parallel to

$$\iint_S a\, dS$$

$= a \cdot \text{Area}$
$= 4\pi a^3$

b) New Field is only part of previous one

Symmetry of field

$\langle x, y, z \rangle = \langle x, 0, 0 \rangle + \langle 0, y, 0 \rangle + \langle 0, 0, z \rangle$

$\iint_S \langle x, y, z \rangle \, d\vec{S} =$

$\iint \langle x, 0, 0 \rangle d\vec{S} + \iint \langle 0, y, 0 \rangle d\vec{S} + \iint \langle 0, 0, z \rangle d\vec{S}$

they should be all $\doteq$

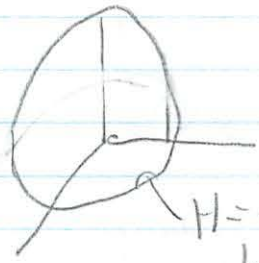
By symmetry

$\iint \langle x, 0, 0 \rangle d\vec{S} = \iint \langle 0, y, 0 \rangle d\vec{S} = \iint \langle 0, 0, z \rangle d\vec{S}$

Hence each $= \frac{1}{3}$ total

$\iint_S \langle 0, 0, z \rangle dS = \frac{1}{3} 4\pi a^3$

c) Now symmetry of the surface



$\iint_H \langle 0, 0, z \rangle \cdot dS$

H = upper half sphere

$= \frac{1}{2} \iint_S \langle 0, 0, z \rangle d\vec{S}$

disk

Bottom surface of sphere does not count
- problem does not care

$$\iint_H \langle 0,0,z \rangle \, d\vec{S} = \frac{1}{2} \iint_S \langle 0,0,z \rangle \, dS$$