18.03 EXAM III

Tuesday, November 22, 2011

Name: Michael Plasmeier

Recitation Instructor (Circle One): L. Chumakova / N. Olver / R. Rosales / V. Shende / J. van Ekeren / Z. Yun

Recitation Hour: R10

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 4 questions and a 55 minute time limit on this exam. Good luck.

Question	Score	Maximum
1	7.5	15
2	3+0	10
3	5.5	8
4	4	8
Total	20	41

 Name: Plasmeier

1. Using the Laplace transform solve the following initial value problems:

a) (5 pts)

distribute!

$$y'' + 6y' + 9y = 0 \qquad y(0) = 1, y'(0) = 3.$$

$(s^2+6s+9)\mathcal{L}(x) - s(1) - 3 - 6 \cdot 1 = \mathcal{L}(0) = 0$

$\mathcal{L}(x) = \dfrac{s+4}{s^2+6s+9} = \dfrac{s+4}{(s+3)^2} = \dfrac{s}{(s+3)^2} + \dfrac{4}{(s+3)^2}$

Partial fractions

$\dfrac{s+4}{(s+3)^2} = \dfrac{Ax+B}{(s+3)^2}$

3

$\mathcal{L}(e^{at}f(t)) = F(s-a)$ $\quad e^{-3t}\mathcal{L}\left(\frac{4}{s^2}\right)$

$e^{-3t}\mathcal{L}\left(\frac{s}{s^2}\right) =$ $\quad e^{-3t} t \cdot 4$

need to shift both s vars. in num/denom.

$e^{-3t}\mathcal{L}\left(\frac{1}{s}\right)$

$e^{-3t} \cdot 1$ $\quad \mathcal{L}(t) = \frac{1}{s^2}$

$= (4t+1)e^{-3t}$

b) (5 pts)

$$y'' + 6y' + 9y = e^{-3t} \qquad y(0) = 1, y'(0) = 3.$$

$(s^2+6s+9)\mathcal{L}(x) - s(1) - 3 - 1 = \dfrac{1}{s+4}$

for e^{-3t}??

$\mathcal{L}(x) = \dfrac{\frac{1}{s+4} + s + 4}{s^2+6s+9}$

split!

Yikes!

$x = \dfrac{s+4}{(s+4)(s+3)(s+3)}$

$= \dfrac{1}{(s+3)^2}$

$= te^{-3t}$

2.5

 Name: Plasmeier

c) (5 pts)

$$y'' + 6y' + 9y = \delta(t-3) \qquad y(0) = 1, y'(0) = 3.$$

$$(s^2+6s+9)\mathcal{L}(x) - s(1) - 3 - 1 = \mathcal{L}(\delta(t-3))$$

$$\mathcal{L}(x) = \frac{\mathcal{L}(\delta(t-3)) + s + 4}{s^2+6s+9}$$ ✓

addition not mult. so not convolution this way...

$$x = \mathcal{L}^{-1}(\mathcal{L}(\delta(t-3))) + \mathcal{L}^{-1}\left(\frac{s+4}{s^2+6s+9}\right)$$

(convolution)

$$x = \begin{cases} \text{part a answer} & t > 3 \\ 0 & t < 3 \end{cases}$$

$$= \begin{cases} (4t+1)e^{-3t} & t > 3 \\ 0 & t < 0 \end{cases}$$

2

 Name: Plasveier

2. a) (4 pts) Consider the differential equation

correction

$$y'' - 2y' + 10 = f(t)$$

did not study → correction as a fn of t

with rest conditions $y(0) = y'(0) = 0$. Find the weight function $w(t)$ corresponding to this equation.

$$w = \mathcal{L}^{-1}\left(\frac{1}{p(s)}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2-2s+10}\right)$$

$$\frac{1}{s^2-2s+10} = \frac{A}{1+\sqrt{18}i} + \frac{B}{1-\sqrt{18}i}$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\frac{2 \pm \sqrt{4-4\cdot 1\cdot 10}}{2\cdot 1} = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm \sqrt{18}i$$

1

b) (3 pts) Use your answer from (a) to express the solution $y(t)$ as an integral involving $f(t)$, valid for any input function $f(t)$. spent ended

$$y(t) = \int_0^t f(v)\, g(t-v)\, dv$$

2

c) (3 pts) Express the convolution $\delta_\pi * u_{2\pi}$ as a single function of t. Here $\delta_\pi(t) = \delta(t-\pi)$ and $u_{2\pi}(t) = u(t-2\pi)$ where $u(t)$ is the unit step function.

$$\delta(t-\pi) * \delta(t-2\pi)$$

0

$$\int_0^t \delta(t-\pi)\, g(t-t-\pi)\, d(t-\pi)$$

X

no clue

 Name: Plaswier

3. Consider a square wave with period 2 defined on $[-1, 1]$ by

$$f(t) = \begin{cases} 2 & \text{for } 0 < t < 1, \\ -2 & \text{for } -1 < t < 0. \end{cases} \quad L=1$$

The Fourier series of $f(t)$ is

$$\frac{8}{\pi} \sum_{n>0 \text{ odd}} \overbrace{\frac{1}{n}}^{b_n} \sin(n\pi t).$$

a) (5 pts) Find a particular solution $y(t)$ (as an infinite sum) to the differential equation

$$y'' + 4y = f(t).$$

$$y(t) = \sum b_n \sin \frac{n\pi t}{L}$$

4.5

$$\sum \left(\frac{-n^2\pi^2}{L^2} + 4\right) b_n \sin \frac{n\pi t}{L} = \frac{8}{\pi} \sum_{\substack{n>0 \\ \text{odd}}} \frac{1}{n} \sin(n\pi t)$$

$$\left(\frac{n^2\pi^2}{L^2} + 4\right) b_n = \frac{8}{\pi}\frac{1}{n}$$

$$b_n = \frac{\frac{8}{\pi}\frac{1}{n}}{n^2\pi^2 + 4} = \frac{8}{\pi n(n^2\pi^2 + 4)}$$

4 changes to k^2

$$y(t) = \sum_{n>0 \text{ odd}}^{\infty} \frac{8}{\pi n(n^2\pi^2 + 4)} \sin(n\pi t)$$

b) (3 pts) The solution to part (a) is a periodic function. Consider the modified equation $y'' + k^2 y = f(t)$ where k is some real number. For what values of k does the solution $y(t)$ exhibit resonance?

1

When is $\sqrt{k^2}$ a solution above

No, when denom. is zero

Ie when does $k = \frac{8}{\pi n(n^2\pi^2 + 4)}$ for any odd integer n will produce resonance

~~Solve for n~~

$\pi n(n^2\pi^2 + 4) k = 8$

$\pi n^3 + 4\pi n = \frac{8}{k}$

Complicated

Solve for n - where n = odd integer

So $n=1$ $\frac{8}{(\pi 3 + 4)} = k$

$n=2$ $\frac{8}{(8\pi 3 + 4)} = k$

etc

4. Consider the function $u(x,t)$ defined on $0 \le x \le 1$, $t \ge 0$ describing the temperature of a rod of length 1 with insulated ends. It satisfies the heat equation

Cosine

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$$

with boundary conditions

$$\frac{\partial}{\partial x}u(0,t) = \frac{\partial}{\partial x}u(1,t) = 0.$$ insulated

Recall that each function

$$u(x,t) = e^{-3n^2\pi^2 t}\cos(n\pi x),$$

for any non-negative integer n is a solution to this differential equation.

$\Rightarrow n=1$ $\Rightarrow u(x,t) = e^{-3\pi^2 t}\cos(\pi x)$

a) (2 pts) Find $u(x,t)$ if the initial temperature profile is $u(x,0) = \boxed{\cos(\pi x)}$.

$$U(x,t) = \frac{a_0}{2} + \sum a_n e^{\frac{-\pi^2 n^2 k t}{L^2}} \cos\frac{n\pi x}{L}$$

$k=3$ $L=1$?

Need a_0, a_n

$$a_0 = \int_{-L}^{L} \cos(\pi x)\,dx$$

$$= \sin(x\pi)\Big|_0^{L=1} = \sin(\pi) - \sin(-\pi)$$ $\ne 0$

$$= \cancel{0-0=0}$$

1

$$a_n = \frac{1}{\pi}\int_{-L}^{L} \underbrace{\underbrace{\cos(\pi x)}_{\text{even}}\underbrace{\cos(n\pi x)}_{\text{even}}}_{\text{even}} = \frac{2}{\pi}\int_0^L \cos(\pi x)\cos(n\pi x)$$

$u=\cos$ $dv=\cos$ $du=-\sin$ $v=\sin$

$$= \frac{2}{\pi}\cdot \underbrace{\cos(\pi x)\sin(n\pi x)}_{0} - \underbrace{\int_0^L \sin(n\pi x)\cos(\pi x)}_{\text{will be 0 (since 2 x that)}}$$

? Umm integrated wrong? yes.

$$\cancel{U(x,t) = \frac{0}{2} + \sum 0 \cdot e^{-\pi^2 n^2 3t}\cos n\pi x}$$

1

 Name: Plasmeier

b) (6 pts) Repeat part (a) but with the initial temperature profile

$$u(x,0) = 2x.$$

$$u(x,t) = \frac{a_0}{2} + \sum a_n e^{\frac{-\pi^2 n^2 k t}{L^2}} \cos\frac{n\pi x}{L}$$

$$a_0 = \int_{-L}^{L} 2x\,dx = \frac{2x^2}{2}\Big|_{-L}^{L=1} = (1)^2 - (-1)^2 = 0$$

$$\cancel{a_n = \frac{1}{1}\int_{-L}^{L} 2x\cos(n\pi x)}$$

$U = 2x \quad dv = \cos n\pi x$

$du = 2 \quad v = \frac{\sin n\pi x}{n}$

$\int v\,dv = vu - \int v\,du$

$\frac{2}{L}\int_0^L 2x\cos(n\pi x)\,dx$

OR

$\frac{1}{L}\int_{-L}^{L} |2x|\cos(n\pi x)\,dx$

because need an even extension.

$$\cancel{= \frac{2x\sin(n\pi x)}{n} - \int_{-1}^{1} \underbrace{\underbrace{\frac{\sin n\pi x}{n}}_{odd} \cdot \underbrace{\cos(n\pi x)}_{even}}_{odd}}$$

$\int_{-1}^{1} odd = 0$

$$\cancel{= \frac{2x\sin(n\pi x)}{n} - 0}$$

$$\cancel{= \frac{2\sin(n\pi 1)}{n} - \frac{2\sin(n\pi(-1))}{n}}$$

$+\pi \quad 2\pi$

$$= 0$$

Umm... - did it wrong again

3

$$\cancel{u(x,t) = \frac{0}{2} + \sum 0 \cdot e^{-\pi^2 n^2 3t} \cos n\pi x}$$

Solutions My Exam 3

 Name: ______________

1. Using the Laplace transform solve the following initial value problems:

a) (5 pts)

$$y'' + 6y' + 9y = 0 \qquad y(0) = 1, y'(0) = 3.$$

$\mathcal{L}(y) = Y$, $\mathcal{L}(y') = sY - 1$, $\mathcal{L}(y'') = s^2Y - s - 3$

$\mathcal{L}(LHS) = (s^2 + 6s + 9)Y - s - 9$, $\mathcal{L}(RHS) = 0$

So $Y = \dfrac{s+9}{(s+3)^2} = \dfrac{1}{s+3} + \dfrac{6}{(s+3)^2}$, $\mathcal{L}^{-1}\left(\dfrac{1}{s+3}\right) = e^{-3t}$

$\mathcal{L}^{-1}\left(\dfrac{1}{(s+3)^2}\right) = te^{-3t}$.

So $\boxed{y(t) = e^{-3t} + 6te^{-3t}}$.

b) (5 pts)

$$y'' + 6y' + 9y = e^{-3t} \qquad y(0) = 1, y'(0) = 3.$$

$\mathcal{L}(RHS) = \dfrac{1}{s+3}$. So $Y = \dfrac{1}{(s+3)^3} + \underbrace{\dfrac{6}{(s+3)^2} + \dfrac{1}{s+3}}_{\text{From (a)}}$.

$\mathcal{L}^{-1}\left(\dfrac{1}{(s+3)^3}\right) = \dfrac{1}{2}t^2e^{-3t}$.

So $\boxed{y(t) = \frac{1}{2}t^2e^{-3t} + 6te^{-3t} + e^{-3t}}$.

 Name: ____________________

c) (5 pts)

$$y'' + 6y' + 9y = \delta(t-3) \qquad y(0) = 1, y'(0) = 3.$$

$\mathcal{L}(RHS) = e^{-3s}$.

So

$$Y = \frac{e^{-3s}}{(s+3)^2} + \underbrace{\frac{1}{s+3} + \frac{6}{(s+3)^2}}_{\text{from (a)}}.$$

$$\mathcal{L}^{-1}\left(\frac{e^{-3s}}{(s+3)^2}\right) = u(t-3)\left[\mathcal{L}^{-1}\left(\frac{1}{(s+3)^2}\right)\right](t-3).$$

$$= u(t-3)(t-3)e^{-3(t-3)}.$$

So $\boxed{y(t) = e^{-3t} + 6te^{-3t} + u(t-3)(t-3)e^{-3(t-3)}}$

2. a) (4 pts) Consider the differential equation

$$y'' - 2y' + 10 = f(t)$$

with rest conditions $y(0) = y'(0) = 0$. Find the weight function $w(t)$ corresponding to this equation.

Char. polyn: $r^2 - 2r + 10 = (r-1)^2 + 9$

So

$$w(t) = \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2+9}\right)$$

Use shift rule & $\mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right) = \frac{1}{3}\sin 3t$.

$$\boxed{w(t) = \frac{1}{3}e^t \sin 3t}$$

b) (3 pts) Use your answer from (a) to express the solution $y(t)$ as an integral involving $f(t)$, valid for any input function $f(t)$.

$$y(t) = w(t) * f(t).$$

$$\boxed{= \int_0^t \left(\frac{1}{3}e^u \sin 3u\right) f(t-u)\,du}$$

c) (3 pts) Express the convolution $\delta_\pi * u_{2\pi}$ as a single function of t. Here $\delta_\pi(t) = \delta(t-\pi)$ and $u_{2\pi}(t) = u(t-2\pi)$ where $u(t)$ is the unit step function.

$$[\delta_\pi * u_{2\pi}](t) = \int_0^t \delta_\pi(s)\, u_{2\pi}(t-s)\,ds = \int_0^t \delta(s-\pi)\, u(t-s-2\pi)\,ds$$

$$= \int_0^{t-2\pi} \delta(s-\pi)\,ds = \begin{cases} 1 & \text{if } \pi < t-2\pi \\ 0 & \text{if } \pi > t-2\pi. \end{cases}$$

So this $\boxed{= u(t-3\pi)}$

 Name: ____________________

3. Consider a square wave with period 2 defined on $[-1, 1]$ by

$$f(t) = \begin{cases} 2 & \text{for } 0 < t < 1, \\ -2 & \text{for } -1 < t < 0. \end{cases}$$

The Fourier series of $f(t)$ is

$$\frac{8}{\pi} \sum_{n>0 \text{ odd}} \frac{1}{n} \sin(n\pi t).$$

a) (5 pts) Find a particular solution $y(t)$ (as an infinite sum) to the differential equation

$$y'' + 4y = f(t).$$

$y'' + 4y = \sin nt \rightsquigarrow y = \frac{1}{4 - n^2} \sin nt$ (unless $n^2 = 4$)

So

$$\boxed{y(t) = \frac{8}{\pi} \sum_{\substack{n>0 \\ \text{odd}}} \frac{1}{n} \cdot \frac{1}{4 - (n\pi)^2} \sin(n\pi t)}$$

b) (3 pts) The solution to part (a) is a periodic function. Consider the modified equation $y'' + k^2 y = f(t)$ where k is some real number. For what values of k does the solution $y(t)$ exhibit resonance?

Resonance occurs when $\frac{1}{k^2 - (n\pi)^2}$, as above, is undefined for some term in the series. i.e., when $\boxed{k = n\pi \text{ for } n \text{ an odd integer.}}$

 Name: ______________

4. Consider the function $u(x,t)$ defined on $0 \leq x \leq 1$, $t \geq 0$ describing the temperature of a rod of length 1 with insulated ends. It satisfies the heat equation

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$$

with boundary conditions

$$\frac{\partial}{\partial x}u(0,t) = \frac{\partial}{\partial x}u(1,t) = 0.$$

Recall that each function

$$u(x,t) = e^{-3n^2\pi^2 t}\cos(n\pi x),$$

for any non-negative integer n is a solution to this differential equation.

a) (2 pts) Find $u(x,t)$ if the initial temperature profile is $u(x,0) = \cos(\pi x)$.

$t=0$ in $u(x,t)$ above : $u(x,0) = \cos(n\pi x)$.

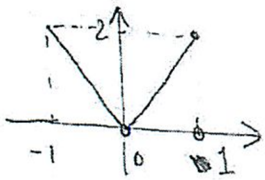
Put $n=1$, find

$$\boxed{u(x,t) = e^{-3\pi^2 t}\cos(\pi x)}.$$

b) (6 pts) Repeat part (a) but with the initial temperature profile

$$u(x,0) = 2x.$$

Fourier cosine series for $2x$ on interval $0 \le x \le 1$:



extend to even function. $L = 1$.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{1}\int_0^1 (2x)\cos\frac{n\pi x}{1}\,dx$$

$$= 4\int_0^1 x\cos(n\pi x)\,dx.$$

$n=0$: $a_0 = 4\int_0^1 x\,dx = \left[2x^2\right]_0^1 = 2.$

$n>0$: $\int \underbrace{x}_{u}\, \underbrace{\cos kx\,dx}_{dv} = \frac{x}{k}\sin kx - \int \frac{1}{k}\sin kx\,dx = \frac{x}{k}\sin kx + \frac{1}{k^2}\cos kx + C$

Integ. by parts. $\int u\,dv = uv - \int v\,du$. $du = dx$ $v = \frac{1}{k}\sin kx$

So $a_n = 4\left[\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{(n\pi)^2}\cos(n\pi x)\right]_0^1$

$\sin(n\pi x) = 0$ for $x = 0, 1$ any n.

$$= \frac{4}{(n\pi)^2}\left[\cos(n\pi x)\right]_0^1 = \frac{4}{(n\pi)^2}\left[(-1)^n - 1\right].$$

$$= \begin{cases} -\dfrac{8}{(n\pi)^2} & \text{if } n \text{ odd.} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

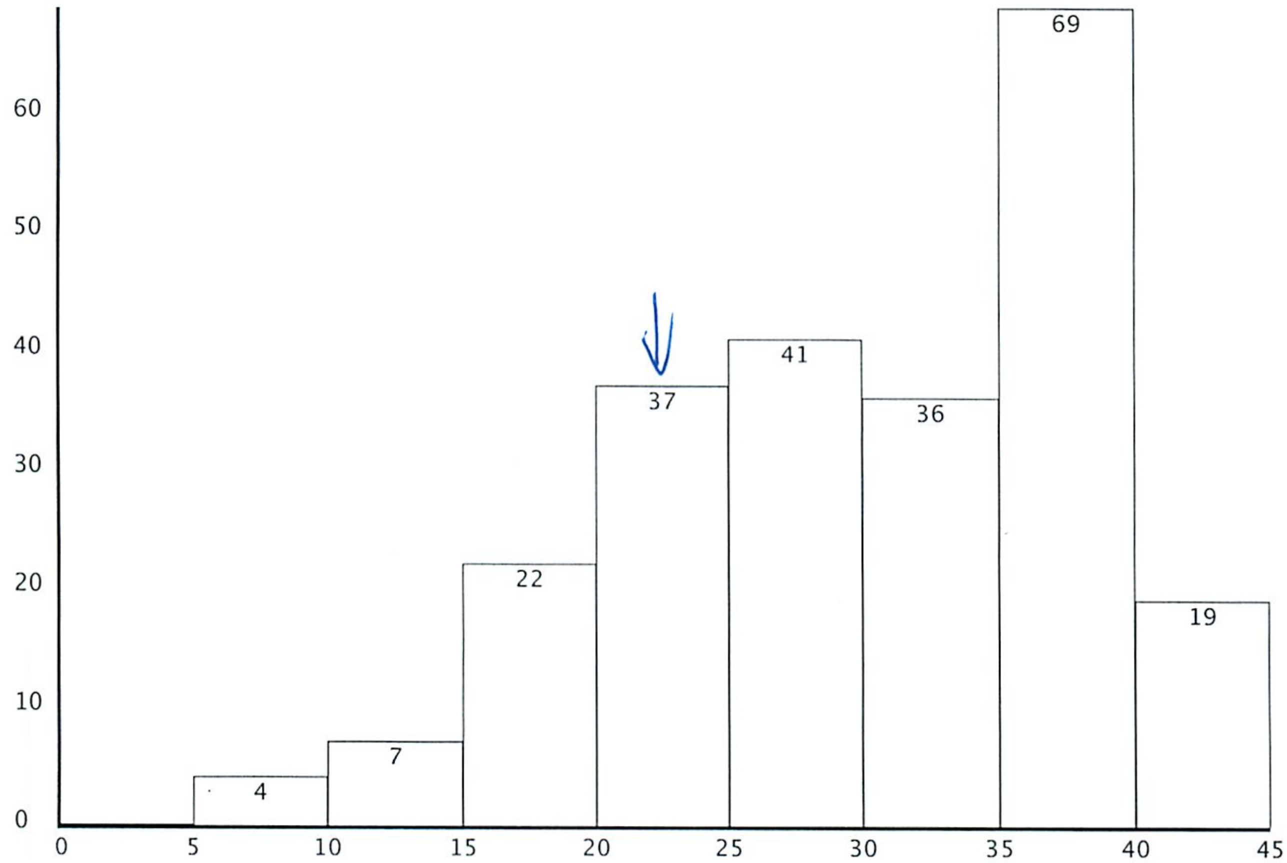
So $u(x,0) = -\frac{8}{\pi^2}\sum_{n\text{ odd}} \frac{1}{n^2}\cos n\pi x.$ $\therefore$

$$\boxed{u(x,t) = -\frac{8}{\pi^2}\sum_{n\text{ odd}} e^{-3n^2\pi^2 t}\cdot\frac{1}{n^2}\cos(n\pi x).}$$

18.03 Differential Equations

Dashboard	Students	Assignments

Grading Summary for Exam 3



Number of Scores:	235
Average:	29.58
Standard Deviation:	8.50

18.03 Lecture 29

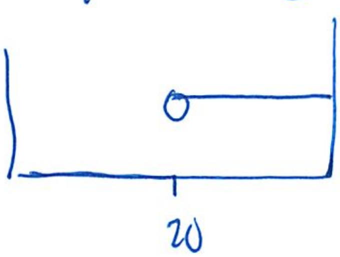
11/28

Starting Linear Systems

New P-Set out
- 2nd to last
- kinda mini

Tests graded
- entering into Stellar
- Exam 1 
- Exam 2 
- Exam 3 

early returns:



Last Unit: Systems of Equations

many dependent variables.

One independent variable - usually t: time

②

ex position of particles in space at time t

$$\left.\begin{matrix} x(t) \\ y(t) \\ z(t) \end{matrix}\right\} 3D$$

If wanted position under influence of gravitational force

$$\vec{F}(t, \vec{s}(t)) = m\,\underline{s}''(t)$$

position vector
underline ~~...~~ = bold

$$\underline{s}(t) = (x(t), y(t), z(t))$$

$$\underline{s}''(t) = (x''(t), y''(t), z''(t))$$ ← just do calculus on each component

$\underline{s}(t)$ is really 3 equations
- one for each component

$$F_1(t, \underline{s}(t)) = m\,x''(t)$$

where

$$F(t, \underline{s}(t)) = (F_1(t, \underline{s}(t)), F_2(t, \underline{s}(t)), F_3(t, \underline{s}(t)))$$

So

$$F_1(t, \underline{s}(t)) = m x''(t)$$

$$F_2(t, \underline{s}(t)) = m y''(t)$$

$$F_3(t, \underline{s}(t)) = m z''(t)$$

3

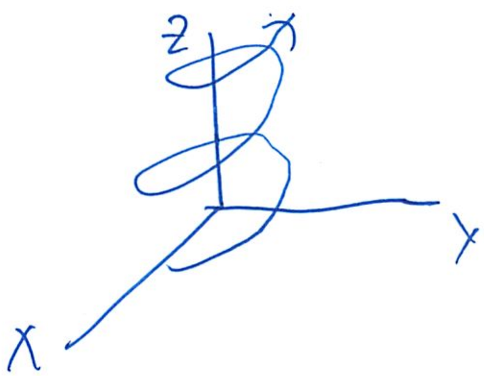
Solution might look like

$$(x(t), y(t), z(t)) = (\cos t, \sin t, t)$$

~~parametric eq~~

~~So circle~~

Parametric curve in $\mathbb{R}^3$



So circle helix

We'll mostly do systems w/ 2 dep. variables

So have parametric eq in a plane

Higher order ODE

- charc eq
- factor
- get roots
- etc

But! ↓

(4)

Systems: only deal w/ 1st order ODEs.

└ Since we introduce dummy variables like in our gravitation problem:

$$x(t) := x_1(t)$$

$$x'(t) = x_1'(t) := x_2(t)$$

So

$$x''(t) = x_1''(t) = x_2'(t)$$

So have good machine in linear algebra

└ just run machine

easiest unit in 18.03

$$x_1'(t) = x_2(t)$$

$$m x_2'(t) = F_1(t, x_1(t), y_1(t), z_1(t))$$

introduce as many variables as you need

+4 more for y_1', y_2', z_1', z_2'

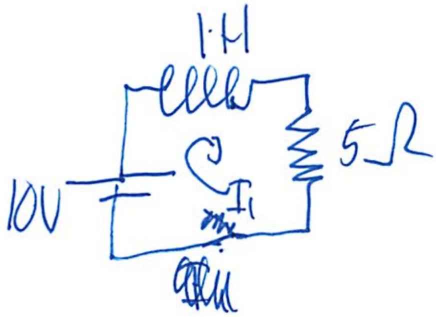
So have 6 total eq

- but all w/ 1st order or lower

(5)

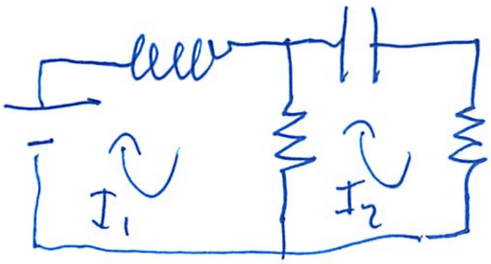
example Circuit

w/ 2 closed loops



Could use Kircoffs law (sum of voltage drops = 0)

But add another loop



LH Loop

$$1 \cdot \frac{dI_1}{dt} + 5(I_1 - I_2) - 10 = 0$$

?note direction

We got all this from the book

RH Loop

$$4Q_2 + 2I_2 + 5(I_2 - I_1) = 0$$

Differentiate to turn into statement about current

$$4 I_2 + 2 I_2' + 5(I_2' - I_1') = 0$$

So we have a system now

$$\frac{dI_1}{dt} = 5 I_2 - 5 I_1 + 10$$

$$\frac{7 I_2}{dt} = 5 I_1' - 4 I_2$$

rewrite if don't like I'

$$= 5(5 I_2 - 5 I_1 + 10) - 4 I_2$$

$$= 21 I_2 - 25 I_1 + 50$$

2 eq'ns

Could draw a picture

But how to solve exactly?

- will eventually do w/ linear algebra
  Lon Wed

For now, solve by eliminating a variable

6b

Study Linear Algebra – multiply matricies

– find inverse

– " determinates

– Igen values + vectors

Backward way: eliminating a variable

- make into single 2nd order eq

- (not what we did earlier ~~that~~ today)

Can eliminate I_2

Solve for I_2 using top eq'n

$$I_2 = \frac{I_1' + 5I_1 - 10}{5} = \frac{I_1'}{5} + I_1 - 2$$

Two ways to express I_2'

1. From bottom pair of eqns
2. From taking deriv of I_2

   L of both sides w/ respect to t

7

1, $I_2' = (21 I_2 - 25 I_1 + 50)/7$

2, $I_2' = \frac{I_1''}{5} + I_1'$

Can set them equal to each other

$$\frac{I_1''}{5} + I_1' = \frac{21}{7} I_2 - \frac{25}{7} I_1 + \frac{50}{7}$$

Sub in $I_2 = \frac{I_1'}{5} + I_1 - 2$

(algebra)

$$\frac{7}{5} I_1'' + \frac{14}{5} I_1' + 4 I_1 = 8$$

↑ Some 2nd order ODE

Could solve using lucky guess method

Constant - so all derivs of 8

So 1 sol + homogeneous solution

Get $I_1(t)$, $I_2(t)$ as sol's

Can find $I_1(t)$

Then plug in to find $I_2(t)$

8

In general we'll deal w/ systems

$$x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1(t)$$

$$\vdots$$

$$x_n' = a_{n1}x_1 + \cdots + a_{nn}x_n + f_n(t)$$

some extra fn of t

x_i = functions of t

a_{ij} = constants

Could you write this in more efficent way?

Yes - as a vector

$$\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

(9)

$\underline{X}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \underline{X} + \underline{0}$ zero vector ← homogenous eq if $f(t)=0$ (like it is here)

have 2 components

$x_1' = -3x_1 + x_2 + 0$

$x_2' = x_1 - 3x_2 + 0$

interesting fn of t we are allowed

~~nhomogenea~~

Can use above system in game theory modeling

On Wed: solve system by staring at Matrix

└ Using Igan values + matrices

But now must rewrite into system of eq'ns

Eliminate variables

$x_2 = x_1' + 3x_1$

$x_1'' + 6x_1' + 8x_1 = 0$

$r^2 + 6r + 8 = (r+4)(r+2)$

(10)

$$\underbrace{x_1 = c_1 e^{-4t} + c_2 e^{-2t}}$$

$-2t > -4t$

So this is dominate term

Now can plug in for x_2

$$x_2 = x_1' + 3x_1$$

$$= -4c_1 e^{-4t} - 2c_2 e^{-2t} + 3c_1 e^{-4t} + 3c_2 e^{-2t}$$

$$= -c_1 e^{-4t} + \underbrace{c_2 e^{-2t}}_{\text{dominant term}}$$

Could say this is nuclear weapons stockpile over time

x_1, x_2 are two countries' stockpiles

Can rewrite as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

Igen vectors

Igen values

(11)

Applet: Linear Phase Portraits: Matrix Entry

On left - top values of A always 0, 1

bottom " are point you pic

Phase shifts on the red line

Lecture 28

First-order systems : many dependent variables, one independent variable (usually t : time)

Example : Position of particle in space given by three coordinates

$x(t)$, $y(t)$, $z(t)$. Might write them as vector : $\underline{s}(t) = (x(t), y(t), z(t))$

If want to position under influence of gravitational force F

$$\vec{F}(t, \underline{s}(t)) = m\,\underline{s}''(t) \qquad \text{(this is three equations)}$$

Write each of these out further: $F_i(t, x(t), y(t), z(t))$ $\longrightarrow$

$$F_1(t, \underline{s}(t)) = m x''(t)$$
$$F_2(t, \underline{s}(t)) = m y''(t)$$
$$F_3(t, \underline{s}(t)) = m z''(t)$$

Picture of soln : maybe $(x, y, z) = (\cos t, \sin t, t)$ → then plot on 3-D space as helix.

Immediate reduction : Always introduce more variables to get (larger) system ~~number~~ of equations with first derivative as highest derivative appearing.

Call these first-order systems. They'll be our most important examples.

For example : In above gravitational force problem.

$$x(t) = x_1(t)$$
$$x_1'(t) = x_2(t)$$
$$x_1''(t) = x_2'(t) = F_1(t, \underline{s}(t))$$

so can write above equations :

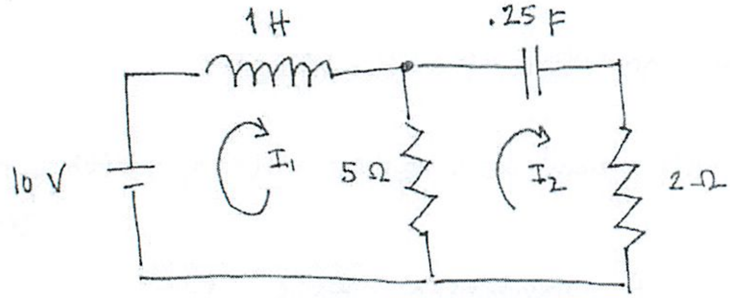
$$x_1'(t) = x_2(t)$$
$$x_2'(t) = F_1(t, x_1(t), y_1(t), z_1(t))$$

etc. for y, z

so now have system of 6 equations, but of first order!

If we can solve first-order systems, we can solve any system.

Given circuit (as pictured in 5.1.4 in book)



I_1, I_2 currents in each loop.

Kirchhoff's Law: sum of voltage drops across circuits' closed loops is 0.

1. $\dfrac{dI_1}{dt} + 5(I_1 - I_2) - 10 = 0$ ← Left hand loop

$4Q_2 + 2I_2 + 5(I_2 - I_1) = 0$ ← Right-hand loop.

Then since $\dfrac{dQ_2}{dt} = I_2(t)$ after differentiating: $4I_2 + 2I_2' + 5(I_2' - I_1') = 0$

$4I_2 + 7I_2' - 5I_1' = 0$

So $\dfrac{dI_1}{dt} = 5I_2 - 5I_1 + 10.$ (*)

$7\dfrac{dI_2}{dt} = 5I_1' - 4I_2 = 5(5I_2 - 5I_1 + 10) - 4I_2$

$= 21I_2 - 25I_1 + 50$ (**)

First order system. How to solve?

Can eliminate $I_2 = I_1 + \frac{1}{5}\dfrac{dI_1}{dt} - 2$ from (*)

then $I_2' = I_1' + \frac{1}{5}I_1''$

now plug these answers in to (**)

then plug back in to solve for I_2.

$7I_1' + \frac{7}{5}I_1'' = 21(I_1 + \frac{1}{5}I_1' - 2) - 25I_1 + 50$

$\frac{7}{5}I_1'' + \frac{14}{5}I_1' + 4I_1 = 8.$ Solve for I_1.

The previous example can be solved by earlier methods (either guessing or Laplace transform. Either way, using characteristic equation)

Then two solutions $I_1(t)$, $I_2(t)$.

Could plot answers on (I_1, I_2) plane, to see trajectories of solution.

Re-do this in a simpler example. First terminology:

Write linear systems in form

$$x_1'(t) = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1(t)$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$x_n'(t) = a_{n1}x_1 + \cdots + a_{nn}x_n + f_n(t)$$

can be re-expressed as

$$\underline{x}'(t) = \underset{\sim}{A}\,\underline{x}(t) + \underline{f}(t)$$

matrix multiplication.

where $\underline{x}'(t) = (x_1'(t), \dots, x_n'(t))$

as column vector

$\underline{x}(t) = (x_1(t), \dots, x_n(t))$

$\underline{f}(t) = (f_1(t), \dots, f_n(t))$.

Ex: $x_1' = -3x_1 + x_2$ (simple game theory model for arms race)

$x_2' = x_1 - 3x_2$

then rewrite: $\underline{x}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \underline{x} + 0$

A: coefficient matrix.

if $\underline{f}(t) = (0, \dots, 0)$ then we say the f.o. system is "homogeneous"

Again eliminate $x_2 = x_1' + 3x_1$

$\Rightarrow \quad x_2' = x_1'' + 3x_1' = x_1 - 3(x_1' + 3x_1)$

$\Rightarrow \quad x_1'' + 6x_1' + 8x_1 = 0$

roots of char equ: $-4, -2$ $\longrightarrow$ $x_1(t) = c_1 e^{-4t} + c_2 e^{-2t}$

Since $x_2(t) = x_1' + 3x_1$, then $x_2(t) = -4c_1e^{-4t} - 2c_2e^{-2t}$

$$+ 3c_1e^{-4t} + 3c_2e^{-2t}$$

$$= -c_1e^{-4t} + c_2e^{-2t}$$

so, again borrowing matrix notation: $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-4t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2t}$

e^{-2t} is dominant term here, so trajectory of solution should tend to origin asymptotic to the line through $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

BIG Q: How to relate $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (and $-4, -2$) appearing in solution) to $\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$.

Note: This is really "wrong direction" to solve these problems. We'll want methods that will work for all systems using techniques of linear algebra.

Other benefits of systems: return of numeric, qualitative methods

---

Start with $x_1'' + 6x_1' + 8x_1 = 0$ then convert to system.

$$x_1' = y, \quad x_1'' = y' = -6x_1' - 8x_1 = -6y - 8x_1$$

Write in vector form:

$$\begin{bmatrix} x_1' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}}_{\text{Companion matrix}} \begin{bmatrix} x_1 \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

since equation was homogeneous.

Note: Not the same matrix we obtained before. What is relationship?

Companion matrix. Top row $[0 \ 1]$

Bottom row det'd by coeffs. of characteristic equation.

18.03 Recitation

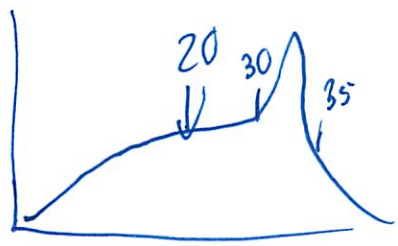
11/29

Quiz 3 back

Got 20/41

average = 30

st dev = 8.51



4. $\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$

ignore boundry conditions
given solutions

$u(x,t) = e^{-3n^2\pi^2 t} \cos(n\pi x)$

a whole lot of physics encoded in boundry notation and what it means

(2)

Find $u(x,t)$ given $u(x,t) = \cos(\pi x)$

So $n=1$

So $u(x,t) = e^{-3\pi^2 t} \cos(nx)$

Next q^u. Now $u(x,0) = 2x$

$$u(x,t) \overset{\text{general form}}{=} \frac{a_0}{2} + \sum_{n \geq 1} a_n e^{-3\pi^2 n^2 t} \cos(n\pi x)$$

$$2x = u(x,0) = \frac{a_0}{2} \sum a_n \cos(n\pi x)$$

Now problem reduced to Fourier expansion of $2x$

$$a_n = \frac{1}{1} \int_{-1}^{1} 2x \cos(n\pi x)\, dx$$

figure that out

(not going to do)

(3)

3. $f(t) = \frac{8}{\pi} \sum_{\substack{n>0 \\ \text{odd}}} \frac{1}{n} \sin(n\pi t)$

Sole diff eq $y'' + 4y = f(t)$

Write y. expand sin fns.

$y = \sum \phi_n \sin(n\pi t)$

only odd

$y = y_e + y_o$

$y_e'' + y_e = 0$

$y_o'' + y_o = f(t)$

$y'' = \sum n^2 \pi^2 \phi \sin(n\pi t)$

$b_n (4 - n^2 \pi^2) = \begin{cases} \frac{8}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

Solve for b_n

Get answer

(4)

b) When does it exhibit resonance

$$y'' + k^2 y = f(t)$$

$$\pi, 2\pi, 3\pi$$

resonance at $k = $

2. $y'' + 2y' + 10y = f(t)$

Find weight function

$$w = \mathcal{L}^{-1}\left(\frac{1}{s^2+2s+10}\right)$$

Now can do partial fraction or rearrange like this

$$= \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2+9}\right)$$

$$= \frac{1}{3} e^{t} \sin 3t$$

(5)

b) For arbitary $f(t)$

$$y(t) = (y * f)(t)$$

write as integral

$$= \int_0^t w(z)\, f(t-z)\, dz$$

c) $\delta(t-\pi) * u(t-2\pi)$

$$= \mathcal{L}^{-1}\left(\mathcal{L}(\delta(t-\pi)) \cdot \mathcal{L}(u(t-2\pi))\right)$$

$$= \mathcal{L}^{-1}\left(e^{-\pi s} \cdot \frac{e^{-2\pi s}}{s}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{e^{-3\pi s}}{s}\right)$$

$$= u(t-3\pi)$$

(6)

1. $x'' + 6y' + 9y = f(t) \quad y(0) = 1 \quad y'(0) = 3$

$$(s^2 + 6s + 9)\,\mathcal{L}(y) = \mathcal{L}(f(t)) + 6y(0) + s\,y(0) + y'(0)$$

$$(s^2 + 6s + 9)\,\mathcal{L}(y) = \mathcal{L}(f(t)) + s + 9$$

$$\mathcal{L}(y) = \frac{\mathcal{L}(f(t))}{(s+3)^2} + \frac{1}{s+3} + \frac{6}{(s+3)^2}$$

$$y = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(f)}{(s+3)^2}\right) + e^{-3t} + 6te^{-3t}$$

I need to compute in each of the 2 cases

b) $$\mathcal{L}^{-1}\left(\frac{\mathcal{L}(e^{-3t})}{(s+3)^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s+3)^3}\right) = \frac{t^2}{2}e^{-3t}$$

(7)

c) $\mathcal{L}^{-1}\left(\dfrac{\mathcal{L}[\delta(t-3)]}{(s+3)^2}\right) = \mathcal{L}^{-1}\left(\dfrac{e^{-3s}}{(s+3)^2}\right)$

$= \mathcal{U}(t-3)\, e^{-3(t-3)}\, (t-3)$

Matrices

Matrix = m∘n box of #s

indicating a linear map from m dimension space to n dimension space

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_n \end{pmatrix}$$

8

Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 0 \\ 3\cdot 1 + 4\cdot 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

One Matrix Special Name: Identity Matrix

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{Identity matrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

If M is a matrix

$M^{-1}M = MM^{-1} = \text{Identity}$

But does not always exist

- Some special cases where it does not

(9) System of Eqns

$$X_1' = a x_1 + b x_2$$

$$X_2' = C x_1 + d x_2$$

Convient to write

$$\underline{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$D\underline{x} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{M} \underline{X}$$

So $(D - M)\underline{X} = 0$

$$\underline{X} = e^{mt} \begin{pmatrix} a \\ b \end{pmatrix}$$

↑ what does this mean when M = matrix

$$= \sum_{n=0}^{\infty} \frac{(mt)^n}{n!}$$

Meaningful when multiply maticies

10

Multiplying Matricies

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} =$$

$$= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

But computing this for all M - not easy!
What we actually do

$$M = Z^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Z$$

↑ some other matrix

↑ usually possible

$$M^2 = Z^{-1} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} Z Z^{-1} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} Z$$

$$= Z^{-1} \begin{pmatrix} \lambda_1^2 & \\ & \lambda_2^2 \end{pmatrix} Z$$

(11)

Can do again + again

$$M^n = Z^{-1}\begin{pmatrix}\lambda_1^n & \\ & \lambda_2^n\end{pmatrix} Z$$

Can compute exponential

$$e^{Mt} = Z^{-1}\begin{pmatrix}e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t}\end{pmatrix} Z$$

"decoupling the system)

Write in that way w/ Iegen values

- rest of the semester

$$\underline{X} = Z^{-1}\begin{pmatrix}e^{\lambda_1 t} & \\ & e^{\lambda_2 t}\end{pmatrix} Z \begin{pmatrix}a \\ b\end{pmatrix}$$ general sol'n

$$\begin{pmatrix}x_1 \\ x_2\end{pmatrix} = Z^{-1}\begin{pmatrix}e^{\lambda_1 t} & \\ & e^{\lambda_2 t}\end{pmatrix} Z \begin{pmatrix}a \\ b\end{pmatrix}$$

specific sol'n

12

How to find the Z by example
- too long a story for today
- will say something about it

$$M = Z^{-1}\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} Z$$

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$Mv = Z^{-1}\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} Z v$$

Now move Z to other side

Not caring

How to find inverse

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Should check validity of this
By multiplying both sides by each other
Should get identity matrix

13

$$\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \overset{\text{should}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

18.03 OH

11/30

$p(D).\ y = f(t)$

$w(t) = \mathcal{L}^{-1}\left(\frac{1}{p(s)}\right) \leftarrow$ defined

$y(t) = (w * f)(t) \leftarrow$ true at rest conditions

$\mathcal{L}(y).\ p(s) = \mathcal{L}(f(t))$

$\mathcal{L}(y) = \mathcal{L}\left(\mathcal{L}^{-1}\left(\frac{1}{p(s)}\right)\right) \cdot \mathcal{L}(f(t)) = \mathcal{L}(w(t))\,\mathcal{L}(f(t))$

$\cancel{\mathcal{L}(y)}$

$= \mathcal{L}(w(t) * f(t))$

$y = w * f$

Rewriting of what always been doing

Makes it easy to solve for different GAs

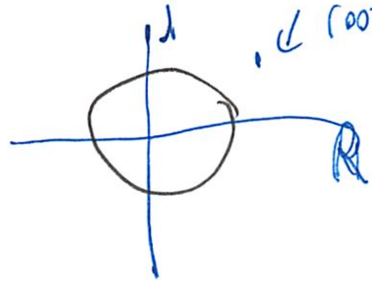
(when rest conditions!)

2

Pole Diagrams

$$\frac{1}{p(s)}$$

plot in $\mathbb{C}$ the roots of $p(s)$



$$\mathcal{L}^{-1}\left(\frac{1}{(s-k)^n}\right) = \frac{t}{\dots} e^{kt}$$

→ ones furthest at dominates

$$\parallel \qquad k = a+bi$$

$$\frac{t}{\dots}\, e^{at} \cdot e^{ibt}$$

largest

of absolute value 1
since complex #
radius = 1 since on circle
is complex # sin__ + cos__

$$e^{ibt} = i\sin bt + \cos bt$$

$$1 = \sqrt{\sin^2(bt) + \cos^2(bt)}$$ always true

how sin, cos work

18.03 Lecture 30 11/30

Solving w/ eigenvalues

Last time started studying systems of ODE
- can always reduce to 1st order systems

Solve systems of the form

$$\underline{x}' = A\underline{x} + \underline{f}(t)$$

↑ underline = vector

(many eq masquerading as one)

$\boxed{n = \text{dimension}}$ $\boxed{n \text{ eqs}}$

A = matrix w/ constant coefficents

Sols: $\underset{\uparrow \text{vector}}{\underline{x}} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

Can plot as a parametric curve in $\mathbb{R}^n$

Focus on 2×2 systems (n=2)

Drew trajectories using mathlet

(2)

Today Linear Algebra Methods

before convert to 2nd order diff eq

easy since 2 × 2 - could elim a variable

hard when 17 × 17 w/o linear algebra

- Easier to solve on Computer

Nuclear Proliferation

$$x_1' = -3x_1 + x_2$$
$$x_2' = x_1 - 3x_2$$

← depend proportional

inverse proportion

↑ predator - prey methods

Solution from last time

$$\underline{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \cdot 1 \cdot e^{-4t} + c_2 \cdot 1 \cdot e^{-2t} \\ c_1 \cdot -1 \cdot e^{-4t} + c_2 \cdot 1 \cdot e^{-2t} \end{bmatrix}$$

(3)

Guess $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$

λ: real constant (later complex)

v_1, v_2: real constants

Need v_1, v_2, λ

$v_1 = 1$ $\quad \lambda = -4$

$v_2 = -1$

$$\underline{x}' = \begin{bmatrix} (v_1 e^{\lambda t})' \\ (v_2 e^{\lambda t})' \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$$

$$\underline{x}' = A\underline{x}$$

$$\hookrightarrow A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$$

Substitute in guess

$$\lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t} = A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$$

Cancel scalars

(4)

$$\lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

can't cancel

ie. $\lambda v_1 = -3v_1 + v_2$

$\lambda v_2 = v_1 - 3v_2$

Now need to solve eq

But ~~Blee~~ 2 eq 3 unknowns! Hmm

Not linear - even scarier

So cheat and say not many values of λ where this work

- 2 values of λ will work

So declare λ fixed constant

Solve v_1, v_2

$(-3-\lambda)v_1 + v_2 = 0$

$1 \cdot v_1 + (-3-\lambda)v_2 = 0$

⑤

Q: When does this have non-zero solution

ie sol'n $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Now we can do linear algebra

<u>Linear algebra</u>: This system has non-zero vector

- Sol if and only if the determinate of the associated matrix is 0

<u>Associated matrix</u>

- we are trying to solve a system

$$\begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix}$$ pull out coefficients

Want $\det(\uparrow) = 0$

Multiply cross terms + subtract in right order

$$(-3-\lambda)^2 - (1 \cdot 1) = \lambda^2 + 6\lambda + 8$$

(6)

Same as

$$x_1'' + 6x_1' + 8x_1 = 0 \text{ then solving for } x_2$$

That darn charastic eq'n is back!!

We will have ~~det~~ det $= 0$ if $\lambda = -2, -4$

Back to linear algebra statement

└ formal way of solving systems of eq'ns

Det $= 0$ when we have redundent info

will have more choices among variables

Lets try plugging in values for λ in to eq'n

$$\begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix}$$

if $\lambda = -2$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \text{ multiply } -1 \qquad \leftarrow 2 \text{ eqn equivilent}$$

↑ contains redundent info

7

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ ← can solve top ← row or bottom row

Now can solve for vales of v
Solve either eq'n

$-V_1 + V_2 = 0$

$V_1 - V_2 = 0$ ← picked

Can we find vector that solves this eq'n

Lots! Any multiple of ~~$V_1 = V_2$~~

$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will do

$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ works

$\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ " $V_1 = V_2$

So far guessed $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} e^{\lambda t}$

Found $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$

8

For $\lambda = -4$. Can plug in $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Any multiple $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ will do

Gives us other solution

$$c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

Superposition - Sum is also a sol

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

Same as before

<u>Summary</u> In general guess $\underline{x} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$

$$\underline{x}' = A x$$

$\hookrightarrow \lambda \underline{v} = A \underline{v} \rightarrow A\underline{v} - \lambda \underline{v} = 0$

9

Like to say

$$(A - \lambda)\underline{v} = 0$$

But need

$$(A - \lambda I)\,\underline{v} = 0$$

where I = identity matrix = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So $\lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

So can immeditly jump to

$$\boxed{\det(A - \lambda I) = 0}$$

Works for any system of eq

just need det() = 0 — any dimensions

Terminology

$\det(A - \lambda I)$ ← is a polynomial in λ

"charastic polynomial"

matches what we called before

(10)

In 2x2 case

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - \underbrace{(a+d)}_{\text{trace}(A)}\lambda + \underbrace{(ad-bc)}_{\det(A)}$$

↳ 1st non trivial case

Repeated roots - complicated - do later

~~Complex roots~~

System w/ $\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$

↓ can convert to second-order ODE

$$x'' + 6x' + 8x = 0$$

last time: can convert to 1st order w/ dummy variables

$$x = x_1$$

$$x_2 = x_1'$$

(11)

Get a system

$x_1' = x_2$

$x_2' = -6x_1' - 8x_1$

can sub in

$= -6x_2 - 8x_1$

Made matrix for system

$$\begin{array}{ccl} \begin{pmatrix} 0 & 1 \\ -8 & -6 \end{pmatrix} & & \begin{array}{l} \leftarrow x_1' \\ \leftarrow x_2' \end{array} \\ x_1 \quad x_2 & & \end{array}$$

But this does not look like my original matrix

L since we substituted in for dummy variables

two related by change in variables

$x_1' = -3x_1 + x_2$

$x_2' = x_1 - 3x_2$

Converted to 2nd order ODE in x_1

$x_1'' + 6x_1' + 8x_1 = 0$

So

$$X_2 = x_1' + 3x_1$$

X_2 variable upstairs + downstairs different!

It depends what you care about
- Original ~~about~~ about nuclear prolif.
- Swapped not about "

Relate two w/ change of coords

Notice trace and determinate = !!!

So same charastic eqn

So both able to solve diff eq

Complex Eigenvalues

Just solve complex eqn

Take Re(), Im() parts

└ a bit diff than before

example ↓

(13)

~~Ann~~ $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = A$

$\det(A - \lambda I) = \lambda^2 - 4\lambda + 5$

$= (2-\lambda)^2 + 1$

Completing the square gives us complex roots

$\lambda = 2 \pm i$

Now exact same thing - but w/ complex values

$\begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix} \to \overset{\text{pick}}{\lambda = 2 - i} \to \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$

Are these redundant info?

~~not~~ bottom $\neq -1 \cdot$ top

But if

bottom $= i \cdot$ top

$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(14)

$iv_1 - v_2 = 0$

$v_1 + iv_2 = 0$

Sol'n complex multiple of $\begin{bmatrix} 1 \\ i \end{bmatrix}$

Sol'n $x(t) = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(2-i)t}$

Only need to check 1 complex root - other is the same

$$= \begin{bmatrix} 1 \\ i \end{bmatrix} e^{2t} \left(\cos(-t) + i \sin(-t)\right)$$

$$= \begin{bmatrix} 1 \\ i \end{bmatrix} e^{2t} \left(\cos t - i \sin t\right)$$

$$= \begin{bmatrix} e^{2t}\cos t - e^{2t}\sin t \, i \\ i e^{2t}\cos t + e^{2t}\sin t \end{bmatrix}$$

Now take Re(), Im() part of each component

$$\mathrm{Re}() = \begin{bmatrix} e^{2t}\cos t \\ e^{2t}\sin t \end{bmatrix}$$

$$\mathrm{Im}() = \begin{bmatrix} -e^{2t}\sin t \\ e^{2t}\cos t \end{bmatrix}$$

15

Put it together

$$= e^{2t}\begin{bmatrix}\cos t\\ \sin t\end{bmatrix} + e^{2t}\begin{bmatrix}-\sin t\\ \cos t\end{bmatrix}$$

General sol

$$= C_1 e^{2t}\begin{bmatrix}\cos t\\ \sin t\end{bmatrix} + C_2 e^{2t}\begin{bmatrix}-\sin t\\ \cos t\end{bmatrix}$$

Lecture 36 — 11/30

Last time, looking at systems: (first-order systems in particular)

$$\underline{x}' = A\underline{x} + \underline{f}(t)$$

Solutions $\underline{x} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ are pictured as parametric curves in $\mathbb{R}^n$.

so for two dependent variables, $x_1(t), x_2(t)$, we can picture their trajectory in plane $\longrightarrow$ (point (x_1, x_2) at each time t)

(See Mathlet Linear Phase Portraits - Matrix Entry)

Solved simple 2×2 system by converting back to second order ODE.

Today - use linear algebra to obtain solution. (maybe start to think about phase portraits)

System last class:

$$x_1' = -3x_1 + x_2$$
$$x_2' = x_1 - 3x_2$$

or $\underline{x}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \underline{x}$

$\underline{\text{sol'n}}$: convert to $x_1'' + 6x_1' + 8x_1$, solve for x_1, sub for x_2, get $\underline{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$

$\underline{\text{Idea}}$: Guess $\underline{x} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$. Try to find choices of λ, v_1, v_2 that give sol'n.

Then $\underline{x}' = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$ comment about component-wise differentiation.

so system becomes ~~####~~ $\lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t} = A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$

so want $\lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. In our case, this gives system:

$\lambda v_1 = -3v_1 + v_2$

$\lambda v_2 = v_1 - 3v_2$

Seems scary : 3 vars λ, v_1, v_2 / 2 equations.

Think of λ as fixed constant momentarily

solve for v_1, v_2

$(-3-\lambda)v_1 + v_2 = 0$

$v_1 + (-3-\lambda)v_2 = 0$

or $\begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

Linear algebra fact: This has non-trivial (i.e. more than just $\underline{v} = \underline{0}$) solution if and only if $\det \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} = 0$.

($\Rightarrow$) Proof by contrapositive. If $\det(A) \neq 0$, then A invertible

so $\underline{A}\,\underline{x} = \underline{b}$ has unique solution $\underline{x} = \underline{A}^{-1} \cdot \underline{b}$

(in our case $\underline{b} = \underline{0}$ so $\underline{A}^{-1} \cdot \underline{b} = \underline{0}$)

($\Leftarrow$) Follows from fact that linearly independent vectors, when placed in matrix, have non-zero determinant.

$$\det \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} = (-3-\lambda)^2 - 1\cdot 1 = \lambda^2 + 6\lambda + 8$$

want this to be 0.

characteristic poly. of second order ODE.

$\lambda = -2, -4$.

Then to find v_1, v_2, we substitute both choices of $\lambda = -2, -4$ and solve:

$\lambda = -2$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then any scalar multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ will do.

$\lambda = -4$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow$ any multiple of $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ will do.

comment about systems of equations.

General rule: Given $\underline{x}' = \underline{A}\,\underline{x}$, guess $\underline{x} = \underline{v}\, e^{\lambda t}$

$\Rightarrow \quad \lambda \underline{v} = A \underline{v}$ (think of $\lambda \underline{v}$ as $\lambda \cdot I_n \cdot v$)

then we can write $(A - \lambda I_n)\underline{v} = 0$ so want λ s.t. $\det(A - \lambda I_n) = 0$

In 2x2 case: $A - \lambda I_n = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$

with $\det(A - \lambda I_n) = \lambda^2 - \underbrace{(a+d)}_{\substack{Tr(A) \\ \text{trace of matrix}}}\lambda + \underbrace{(ad-bc)}_{Det(A)}$ ~ det. of matrix.

Comments: Two ways of getting back and forth between second-order equations and linear first-order systems.

What is relationship between their companion matrices?

What to do with complex eigenvalues?

Solve complex equation. Take real and imaginary parts.

ex. $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \longrightarrow$ char. eqn: $\overbrace{(2-\lambda)^2 + 1}^{\lambda^2 - 4\lambda + 5} = 0$

with roots $2 \pm i$

substitute $2 + i$ into $\begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix} \longrightarrow \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

then solve: $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ any (complex) multiple of $\begin{bmatrix} 1 \\ i \end{bmatrix}$ will do.

gives cx. soln:

$$\underline{x}(t) = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(2-i)t}$$

$$= \begin{bmatrix} 1 \\ i \end{bmatrix} e^{2t} \cdot (\cos t - i \sin t)$$

$$= \begin{bmatrix} \cos t - i \sin t \\ i \cos t + \sin t \end{bmatrix} e^{2t}$$

and taking real and imag. parts gives the desired pair of solution.

Reading

12/1

Paramatizing

Parabola

$y = x^2$

Same as

$x = t$ $\quad$ t = free parameter

$y = t^2$

Circle

$x^2 + y^2 = 1$

$(\cos t, \sin t)$ for $0 \leq t < 2\pi$

; So run t from $0 \to 2\pi$ an watch?

Lets you diff / int a curve termize

; Still don't get how to convert...

System of Linear Eq'ns

$$x_1' = a x_1 + b x_2$$
$$x_2' = c x_1 + d x_2$$

Write as matrix eqn

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\underline{x}' = \underline{M}\ \underline{x}$$

Leads to that ans has exponentials

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t}$$

(think in later class) if # e^{Mt}
more news on how to exponentiate a Matrix later on

$$\left(\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t} \right)' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t}$$

Take derivitive
Cancel

(2)

$$\lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Something is a sol. when this is solved

Rewrite

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

So for $\rightarrow$ when can this be 0?

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

expand out

$$= \begin{pmatrix} wv_1 + xv_2 \\ yv_1 + zv_2 \end{pmatrix}$$

$$= \begin{pmatrix} w \\ y \end{pmatrix} v_1 + \begin{pmatrix} x \\ z \end{pmatrix} v_2$$

So must have

$$\begin{pmatrix} w \\ y \end{pmatrix} = \frac{-v_2}{v_1} \begin{pmatrix} x \\ z \end{pmatrix}$$ one column is multiple of other one

(otherwise no sol to $\begin{pmatrix} w & x \\ y & z \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$)

③

How to detect when one column equals to other one

$$\begin{pmatrix} w & \overset{\text{constant}}{c}w \\ y & cy \end{pmatrix}$$

Conversly

$$wz = xy$$

$$z = \left(\frac{x}{w}\right) y$$

$$x = \left(\frac{x}{w}\right) w$$

Brings us to charaistic eq'n

Has non-zero sol'n for

$$\underbrace{(a-\lambda)(d-\lambda) = bc}_{\text{charastice eq'n}}$$

$$\downarrow$$

$$\underbrace{\lambda^2 - (a+d)\lambda + ad - bc = 0}_{\text{charast'ic polynomical}}$$

trace determine

9

Now solve charastic eqn'
for which λs do we have a non-trivial sol'n.

Find roots λ_1, λ_2

Then study

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

Try to solve it

· Looks like 2 equations here

$$(a-\lambda_1)c_1 + b\,c_2 = 0$$
$$c\,c_1 + (d-\lambda_1)\,c_2 = 0$$

So rows are multiples of each other
~~so~~ ~~as~~ like how cols are multiples of each other

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = k_1 \begin{pmatrix} -b \\ a-\lambda_1 \end{pmatrix}$$

(5)

Can now write general sol'n to this equation

$$\lambda_i \begin{pmatrix} -b \\ a-\lambda_i \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -b \\ a-\lambda_i \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t} =$$

$$= \underbrace{k_1}_{\text{arbitary constants}} \begin{pmatrix} -b \\ a-\lambda_1 \end{pmatrix} e^{\lambda_1 t} + k_2 \begin{pmatrix} -b \\ a-\lambda_1 \end{pmatrix} e^{\lambda_2 t}$$

λ_1, λ_2 roots of char. polynomial

k_1 ; k_2 arbitary constants

Now w/ #s

$$x_1' = x_1 + 2x_2$$

$$x_2' = 2x_1 + x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(6)

Try $\left(\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda x}\right)' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda x}$

take deriv - pulls down λ

- cancel $e^{\lambda x}$

$$\lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda x}}$$

Move λ over

Set = to 0

$$0 = \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Expand out

det: $(1-\lambda)(1-\lambda) - 4 = 0$

$\lambda^2 - 2\lambda - 3 = 0$

7

which can factor

$$(\lambda+1)(\lambda-3)=0$$

roots: $\lambda = -1, 3$

Do for $\lambda=-1$ |

$$\begin{pmatrix} 1--1 & 2 \\ 2 & 1--1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ can put in <u>any</u> constant here

$$\boxed{\begin{gathered} 2c_1 + 2c_2 = 0 \\ c_1 = -c_2 \end{gathered}}$$

$\lambda=3$ |

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(8)

Now write general form of solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

Get rid of vectors

$$x_1 = k_1 e^{-t} + k_2 e^{3t}$$

$$x_2 = -k_2 e^{-t} + k_2 e^{3t}$$

Now try solving some

- same as before
- but roots are worse

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

for M =

(1) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (4) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(2) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (5) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

(3) $\begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix}$

(9) — This is the Eigenvalue + Eigenvector method

— There is no difference b/w [ ] and ( )

— Repeated roots

not as much of an issue as

non-diagonability — can't solve (4)

will see more details later

So (1)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\|$$

$$\left(\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t} \right)' = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t}$$

$$\lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$0 = \begin{pmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Just following steps - don't know what means/why

det $(1-\lambda)(4-\lambda) - 6 = 0$

(10)

$$x^2 - 5x + 4 - 6 = 0$$

$$x^2 - 5x - 2 = 0$$

$$\frac{5 \pm \sqrt{25 - 4 \cdot 1 \cdot -2}}{2}$$

$$\frac{5 \pm \sqrt{33}}{2}$$ (??)

roots) $\frac{5}{2} \pm \frac{\sqrt{33}}{2}$

Goal change variables so get something unrelated and solve them in that way

$$\boxed{\lambda = \frac{5}{2} + \frac{\sqrt{33}}{2}}$$

$$\begin{pmatrix} 1 - \frac{5}{2} + \frac{\sqrt{33}}{2} & 2 \\ 3 & 4 - \frac{5}{2} + \frac{\sqrt{33}}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$c_1\left(1 - \frac{5}{2} + \frac{\sqrt{33}}{2}\right) + 2c_2 = 0$$

$$3c_1 + c_2\left(4 - \frac{5}{2} + \frac{\sqrt{33}}{2}\right) = 0$$

¿then what?
Solve for c_1, c_2

(1)

$$\lambda = \frac{5}{2} - \frac{\sqrt{33}}{2}$$

etc

Skip now

(2)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

etc ⋮

$$0 = \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$(1-\lambda)(1-\lambda) = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

roots

$$(\lambda - 1)(\lambda - 1)$$

$$\lambda = 1, 1$$

¿repeated roots?

not a problem ~~[illegible]~~
just contine

(12)

$\underline{\lambda = 1}$

$$\begin{pmatrix} 1-1 & 0 \\ 0 & 1-1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$0c_1 + 0c_2 = 0$$

$$0c_1 + 0c_2 = 0$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = k \,(\text{anything})$$

¿how to say?

$$X = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{1t}$$ (V) Answer

(3)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$0 = \begin{pmatrix} 1-\lambda & 3 \\ 4 & 1-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$(1-\lambda)(1-\lambda) - 12 = 0$$

$$\lambda^2 - 2\lambda + 1 - 12 = 0$$

$$\lambda^2 - 2\lambda - 11 = 0$$

(13)

That's not going to factor nicely

$$\frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot -11}}{2}$$

$$\frac{2 \pm \sqrt{48}}{2}$$

? reduces - but to what?

Oh forgot

Key Eq'n

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = k_1 \begin{pmatrix} -b \\ a - \lambda_1 \end{pmatrix}$$

Finding matrix which can't be inverted
 └ don't try to invert the matrix

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t}$$

Key Eq'n

$$= k_1 \begin{pmatrix} -b \\ a - \lambda_1 \end{pmatrix} e^{\lambda_1 t} + k_2 \begin{pmatrix} -b \\ a - \lambda_2 \end{pmatrix} e^{\lambda_2 t}$$

(14)

(4) Not going to explain yet in general

If get confused w/ matrices can fall back on doing it by hand

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_1' = x_1 + x_2$$
$$x_2' = x_2$$

Plug in $\hookrightarrow x_2 = ce^t$

$$x_1' - x_1 = ce^t$$

Solve as before - from 1st 2 months

18.03 FALL 2011 – Problem Set 9

Due FRIDAY 12/02/11, high noon in 2-106

To encourage you to keep up with homework as it appears in lecture, both Part I and Part II problems are listed with the accompanying lecture in which the material will be covered.

Part I (14 points)

Lecture 29. Mon. Nov. 28: Linear systems and matrices
READ: EP 5.1–5.3, Notes LS.1 HW: Notes 4A-2,4,5

Lecture 30. Wed. Nov. 30: Eigenvalues and eigenvectors
READ: EP 5.4, Notes LS.2 HW: Notes 4B-1,3,4,6 (all parts in each)

Lecture 31. Fri. Dec. 2: Complex and repeated eigenvalues
READ: EP 5.4, 5.6, Notes LS.3 HW: To be assigned on the next pset

Part II (19 points)

0. (3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-102," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

1. (Monday, 16 pts)

a) Find the general solution to each of the following differential equations:

i) $x'' - 3x' + x = 0$

ii) $x'' + 2x' + 3x = 0$

b) Find the companion matrix to both (i) and (ii) in the form

$$\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}.$$

c) Go to the mathlet `Linear Phase Portraits:Matrix Entry`. Enter the values for c and d you found in part b.(i) on the right of the diagram. Draw a picture of several trajectories in the phase plane, with arrows pointing in the direction of increasing time. Describe the asymptotic behavior of trajectories as $t \to \infty$.

d) Use the formula for the general solution to (i) you found in part a. to find the vector form of the general solution,

$$\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}$$

expressed using two arbitrary constants. What is the dominant term? Explain how this is consistent with what you saw in the applet. With the help of the formula, say exactly what the asymptotic directions of the trajectories are.

e) Repeat parts (c) and (d) using the differential equation in (ii). (For part (c), the description of the behavior as $t \to \infty$ can be more qualitative.)

f) In these companion matrix examples, whenever a trajectory crosses the x axis it seems to do so perpendicularly. That is, its tangent vector is vertical. Explain why.

Michael Plasmeier 18.03 P-Set 4 11/30

25/33

Part 1

Lecture 29 Linear Systems + Matricies

4A-2 If $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ $B = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$

Show $AB \neq BA$

AB

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \times \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1\cdot 0 + 2\cdot 2 & 1\cdot -1 + 2\cdot 1 \\ 3\cdot 0 + -1\cdot 2 & 3\cdot -1 + -1\cdot 1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & +1 \\ -2 & -4 \end{pmatrix}$$ ✓ Matlab

BA

$$\begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0\cdot 1 + -1\cdot 3 & 0\cdot 2 + -1\cdot -1 \\ 2\cdot 1 + 1\cdot 3 & 2\cdot 2 + 1\cdot -1 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 1 \\ 5 & 3 \end{pmatrix}$$ ✓ Matlab

✓

$\boxed{\neq \text{ Not equal}}$

②

4A-4 | Verify formula in LS.1 for inverse 2x2 matrix

Inverse $AA^{-1} = I$ $A^{-1}A = I$

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

determinant

So $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$|A| = ad - bc$

So

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\frac{1}{ad-bc} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ad + b-c & a-b + ba \\ cd + d-c & c-b + da \end{pmatrix}\frac{1}{ad-bc}$$

$$\begin{pmatrix} ad - bc & -ab + ba \\ cd - cd & ad - bc \end{pmatrix}\frac{1}{ad-bc}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ ✓ verified ✓

(3)

4A-5 | $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ Find A^3

A^2 $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 0\cdot 0 + 1\cdot 1 & 0\cdot 1 + 1\cdot 1 \\ 1\cdot 0 + 1\cdot 1 & 1\cdot 1 + 1\cdot 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ ← solutions online wrong! checked w/ Matlab

A^3 $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 1\cdot 0 + 1\cdot 1 & 1\cdot 1 + 1\cdot 1 \\ 1\cdot 0 + 2\cdot 1 & 1\cdot 1 + 2\cdot 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ ✓ Matlab but <u>not</u> solutions

(4)

Lecture 30 Eigenvalues + Eigenvectors

4B-1 Write the following eq'n as equiv 1st order system

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + tx^2 = 0$$

$$x'' + 5x' + tx^2 = 0$$

Substitutions

$$x_1 = x$$

$$x_2 = x' = x_1'$$

$$x_3 = x'' = x_2' = x_1''$$

$$x_4 = x''' = x_3' = x_2'' = x_1'''$$

$$x_3 + 5x_2 + t x_1^2 = 0$$

(X) Not what the book had in mind
answered wrong question

? Seems like elimination method
(but we don't have 2 equations

(5)

Book Sol $x' = y$

$y' = -tx^2 - 5y$

?? I don't get how we got that

Call x' as y

Then write y' ?

So basically $x' = x_2$

$x_3 = -tx_1^2 - 5x_2$ ← rearrange of what I had

? But why did we write it as that form?

b) $y'' - x^2 y' + (1-x^2) y = \sin x$

? So

$y_3 - x^2 y_2 + (1-x^2) y_1 = \sin x$

So $y_3 = \sin x + x^2 y_2 + (1-x^2) y_1$

Is that the form they want??

pretty much

$y' = z$

$z' = (x^2-1)y + x^2 z + \sin x$

I guess you could look at it as x, y, z paramitized dimensions. Is that the point?

(6)

4B-3) Write out $x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x \quad x = \begin{pmatrix} x \\ y \end{pmatrix}$

as system of two 1st order diff eq'ns.

(what did in recitation on 12/1)

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x' = 1x + 1y$$

$$y' = 4x + 1y$$

ok

a) Eliminate y so single 2nd order ODE

↳ so 5.2 Elimination Method

So solve second eq for x

$$x = \frac{y' - y}{4}$$

Differentiate

$$x' = \frac{1}{4}y'' - \frac{1}{4}y'$$

Sub in x, x' into 1st eqn

$$\left(\frac{1}{4}y'' - \frac{1}{4}y'\right) = \left(\frac{1}{4}y' - \frac{1}{4}y\right) + y$$

7

Simplify

$$\frac{1}{4}y'' - \frac{1}{4}y' - \frac{1}{4}y' + \frac{1}{4}y - y = 0$$

$$\frac{1}{4}y'' - \frac{2}{4}y' - \frac{3}{4}y = 0$$

Find charastic eqn

Multiply by 4 first

$$y'' - 2y' - 3y = 0$$

$$(y-3)(y+1) = 0$$

So general sol is

$$y(t) = Ae^{3t} + Be^{-t}$$

Now need $x(t)$

$$y'(t) = 3Ae^{3t} + -1Be^{-t}$$

$$x(t) = \frac{1}{4}\left(3Ae^{3t} - Be^{-t}\right) - \frac{1}{4}\left(Ae^{3t} + Be^{-t}\right)$$

$$x(t) = \frac{1}{2}Ae^{3t} - \frac{1}{2}Be^{-t}$$

?. -1

Done

Solutions converted to system - didn't find final answer

Oh weren't supposed to - got carried away

(8) Opps didn't follow directions

b) Write as equiv 1st order eq'n

$$y_1 = y$$

$$y_2 = y_1' = y'$$

$$y_3 = y_2' = y_1'' = y''$$

$$y_3 = 2y_2 + 3y_1$$

Write as

$$x_1' = x_2$$

$$x_2' = 2x_2 + 3x_1$$

← ¿ why just write as ← here whole system

Oh well write as

Relationship b/w variables

$$x_1 = y$$

$$x_2 = x + y$$ ← ¿ where did we get that ¿ ¿

(-1) x

(9)

4B-4) For the system

$$x' = 4x - y$$
$$y' = 2x + y$$

a) Using matrix notation, verify 2 are solutions

$$x = e^{3t} \qquad x = e^{2t}$$
$$y = e^{3t} \qquad y = e^{2t}$$

(This was in text book 5.3 example 6)

$$\frac{dx}{dt} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} x$$

$$x = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \qquad y = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

So x)

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

$$= \begin{bmatrix} 4e^{3t} - e^{3t} \\ 2e^{3t} + e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ 3e^{3t} \end{bmatrix} = x' \quad ✓$$

verifys it, right?

10

y

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} 4e^{2t} - e^{2t} \\ 2e^{2t} + e^{2t} \end{bmatrix} = \begin{bmatrix} 3e^{2t} \\ 3e^{2t} \end{bmatrix}$$ ⊗ No ;

Look at solutions

ah we actually had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$

where did we get this ?

Oh I copied it wrong!

y retry

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} 4e^{2t} - 2e^{2t} \\ 2e^{2t} + 2e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 4e^{2t} \end{bmatrix}$$ ✓ That works verified ✓

(11.)

b) Verify they form a fundamental set of sols - ie are linearly indep

L are linear ind if determinate $\neq 0$

$$\begin{vmatrix} e^{3t} & e^{2t} \\ e^{3t} & 2e^{2t} \end{vmatrix} = e^{3t} \cdot 2e^{2t} - e^{2t} e^{3t}$$

?; why can we write it like that?; ?; Wronskian?

$$= e^{5t}$$

$$\neq 0$$ ✓ works

c) Write the general sol'n to the system in terms of two arbitrary constants c_1, c_2

General sol

$$y = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$$

$$= \begin{bmatrix} c_1 e^{3t} + c_2 e^{2t} \\ c_1 e^{3t} + c_2 2 e^{2t} \end{bmatrix}$$

$$x = c_1 e^{3t} + c_2 e^{2t}$$

$$y = c_1 e^{3t} + 2c_2 e^{2t}$$

✓ understand this

12

4B-6 Solve the system $x' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x$ in 2 ways

(did in recitation)

a) Substitution

$$x' = x + y$$
$$y' = y$$

Hmm want to get it all in terms of 1 variable

Oh just write

$$y = c_1 e^t$$

$$x' - x = c_1 e^t$$

Solve like before

$$x = c_1 t e^t + c_2 e^t$$
$$y = c_1 e^t$$

separate parametrized!

b) More complicated substitution

$$y = x' - x$$
$$(x-1) = 0$$

$$(x'-x)' = x' - x$$

(13)

$$x'' - x' = x' - x$$

$$x'' - 2x' + x = 0$$

$$(x-1)(x-1) = 0$$

$$x = C_1 e^t + C_2 t e^t$$

$$x = C_1 e^t$$

(14)

Part 2

0. No one yet

1. Find the general sol to the following diff eq

$$x'' - 3x' + x = 0$$

$$(x^2 - 3x + 1)$$

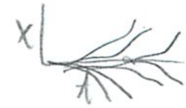
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{3 \pm \sqrt{9 - 4\cdot 1\cdot 1}}{2\cdot 1}$$

$$\frac{3 \pm \sqrt{5}}{2}$$

$$x = Ae^{\frac{3+\sqrt{5}}{2}t} + Be^{\frac{3-\sqrt{5}}{2}t}$$

(1) WA



(15)

ii) $x'' + 2x' + 3x = 0$

$$(x^2 + 2x + 3)$$

$$\frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 3}}{2}$$

$$-1 \pm \sqrt{-2}$$

$$-1 \pm \sqrt{2}i$$

$$x = Ae^{(-1+\sqrt{2}i)t} + Be^{(-1-\sqrt{2}i)t}$$

had to look up - should know!

$$= Ae^{-1t} e^{\sqrt{2}it} + Be^{-1t} e^{-\sqrt{2}it}$$

$$= Ae^{-t} \cos(\sqrt{2}t) + i\sin(\sqrt{2}t) + Be^{-t} \cos(-\sqrt{2}t) + i\sin(-\sqrt{2}t)$$

And on p 131

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} = C_1 e^{(a+bi)x} + C_2 e^{(a-bi)x}$$

$$= C_1 e^{ax}(\cos bx + i \sin bx) + C_2 e^{ax}(\cos bx - i \sin bx)$$

$$= (C_1 + C_2) e^{ax} \cos bx + i (C_1 - C_2) e^{ax} \sin bx$$

C_1, C_2 can be complex

Can choose $C_1 = -\frac{1}{2}i \quad C_2 = \frac{1}{2}i$

never really understood

So $e^{ax}(c_1 \cos bx + c_2 \sin bx)$

16

$$x(t) = A e^{-t} \sin(\sqrt{2}t) + B e^{-t} \cos(\sqrt{2}t)$$

17

b) Find the companion matrix in the form

$$\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}$$

So this is form

$x_1' = x_2$

$x_2' = -6x_2 - 8x_1$ from $x'' + 6x' + 8x = 0$

So

i) $x_2' = 3x_2 - x_1$

$$\begin{pmatrix} 0 & 1 \\ 3 & -1 \end{pmatrix}$$

e X

-1

ii) $x_2' = -2x_2 - 3x_1$

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

X -1

(18)

c) Mathlet: Linear Phase Portraits: Matrix Entry

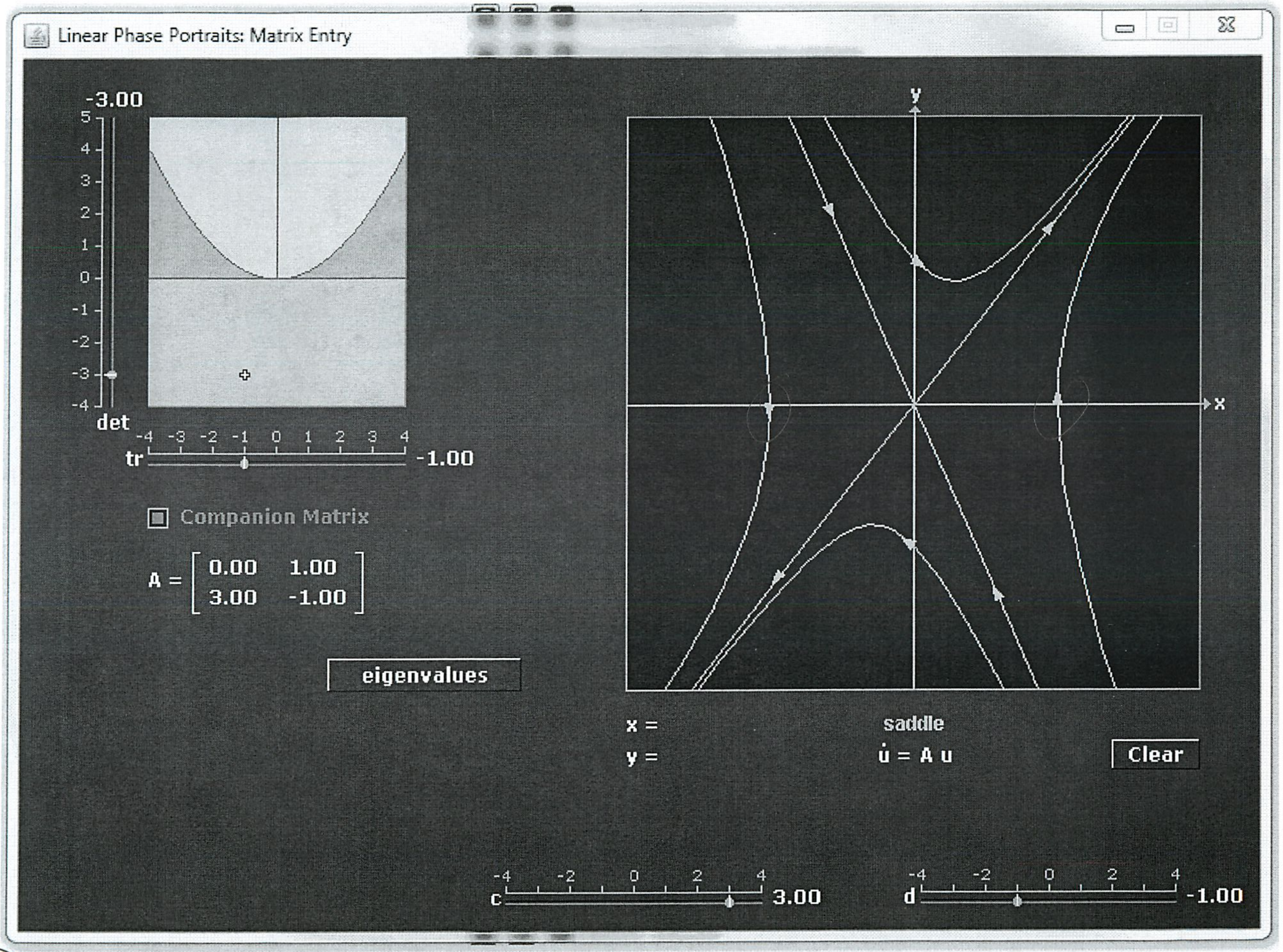
enter $\begin{pmatrix} 0 & 1 \\ 3 & -1 \end{pmatrix}$ ← why is 0 1 special again

See printed results

wrong graph!

-2

follows pointing to ? time



6

(20)

d) Use the formula from (a) for (i) to find the vector form of the general solution

$$\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}$$

expressed using two arbitrary constants,

$$x(t) = Ae^{\frac{3+\sqrt{5}}{2}t} + Be^{\frac{3-\sqrt{5}}{2}t}$$

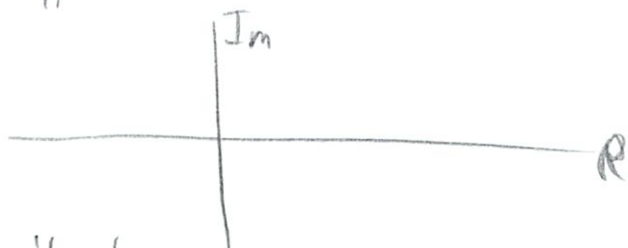
$$x'(t) = \frac{3+\sqrt{5}}{2}Ae^{\frac{3+\sqrt{5}}{2}t} + \frac{3-\sqrt{5}}{2}Be^{3-\sqrt{5}t}$$

& that in vector form

$$\begin{bmatrix} Ae^{\frac{3+\sqrt{5}}{2}t} + Be^{\frac{3-\sqrt{5}}{2}t} \\ \frac{3+\sqrt{5}}{2}Ae^{\frac{3+\sqrt{5}}{2}t} + \frac{3-\sqrt{5}}{2}Be^{\frac{3-\sqrt{5}}{2}t} \end{bmatrix}$$

What is the dominate term?

; Plot it



We don't have values to do that

(21)

But larger term dominates

$L\, x'(t)$

Explain how matches applet

? what are we looking for in applet?

W/ help from formula say exactly what the asymptotic directions of trajectory are

? how do we answer that?

From graph line $(-3,-4) \to (3,4)$

$$\frac{\Delta y}{\Delta x} = \frac{4--4}{3--3} = \frac{8}{6} = \frac{4}{3}$$

$$y = \frac{4}{3}x + 0$$

(22)

Other line $(-1.73, 4)$ $(1.73, -4)$

$$\frac{\Delta y}{\Delta x} = \frac{-4-4}{1.73 - -1.73} = 0$$

?? not right

¿ How find analytically?

x (2)

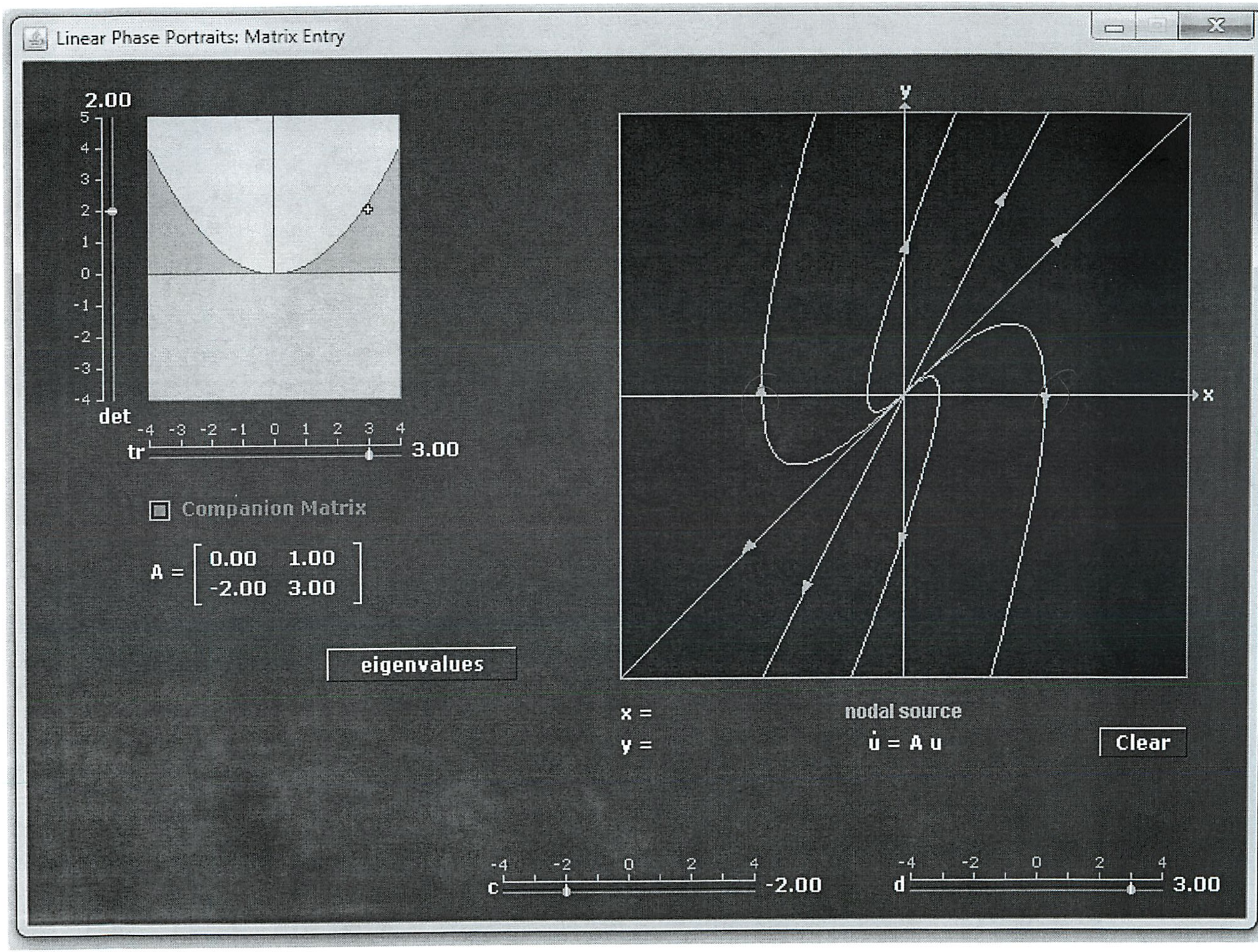
(23)

e) do for $x'' + 2x' + 3x = 0$

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

c) See next sheet

24


Linear Phase Portraits: Matrix Entry
2.00
det
tr
3.00
Companion Matrix
A = [0.00 1.00; -2.00 3.00]
eigenvalues
y
x
x =
y =
nodal source
u̇ = A u
Clear
c
-2.00
d
3.00

(25)

d) $x(t) = A e^{-t} \sin(\sqrt{2}t) + B e^{-t} \cos(\sqrt{2}t)$

$$x'(t) = e^{-t}\left(A\left(\sqrt{2}\cos(\sqrt{2}t) - \sin(\sqrt{2}t)\right) - B\left(\sqrt{2}\sin(\sqrt{2}t) - \cos(\sqrt{2}t)\right)\right)$$

Oh can take lim like this?

$$\lim_{t\to\infty} \left(x(t) \right)$$

$= 0$ says WA

$$\lim_{t\to\infty} \left(x'(t) \right) = 0$$

Not very interesting

don't know plot or dominant

(26)

f) In these companion matrix examples, whenever a trajectory crosses the x axis it seems to do so perpendicularly. That is its tangent vector is vertical. Why?

Oh - I did not notice that before.
Don't know why though....

✓

Homework 9 Part II

$\boxed{1}$

(a)

(i) $x'' - 3x' + x = 0$

$r^2 - 3r + 1 = 0 \qquad r = \frac{3 \pm \sqrt{5}}{2}.$

So $x(t) = Ae^{\frac{3+\sqrt{5}}{2}t} + Be^{\frac{3-\sqrt{5}}{2}t} \qquad (A, B \in \mathbb{R}).$

(ii) $x'' + 2x' + 3x = 0$

$r^2 + 2r + 3 = 0 \qquad r = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm \sqrt{2}\, i.$

so $x(t) = e^{-t}(A \cos \sqrt{2}t + B \sin \sqrt{2}t)$

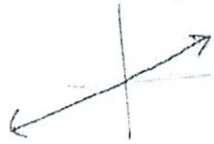
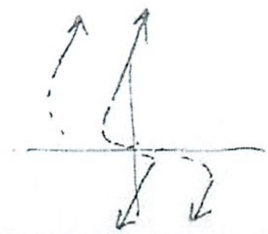
(b)

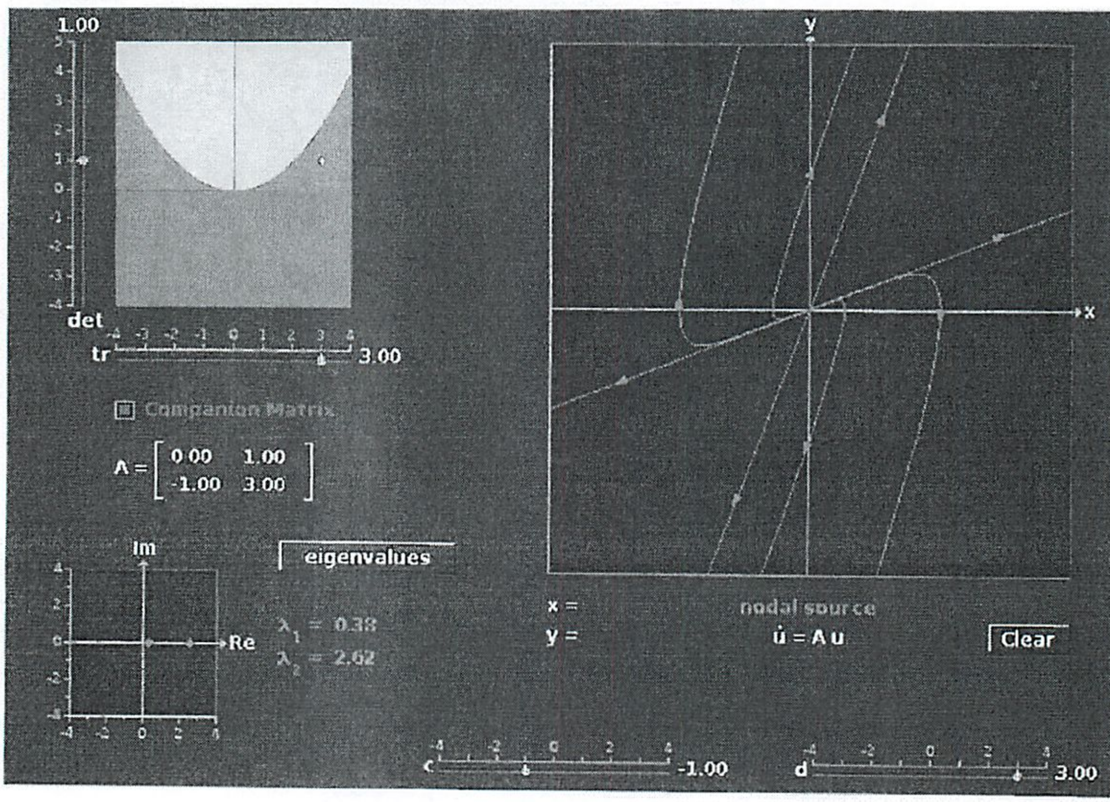
(i) $y = x', \quad y' = 3x' - x = -x + 3y.$

$$\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}$$

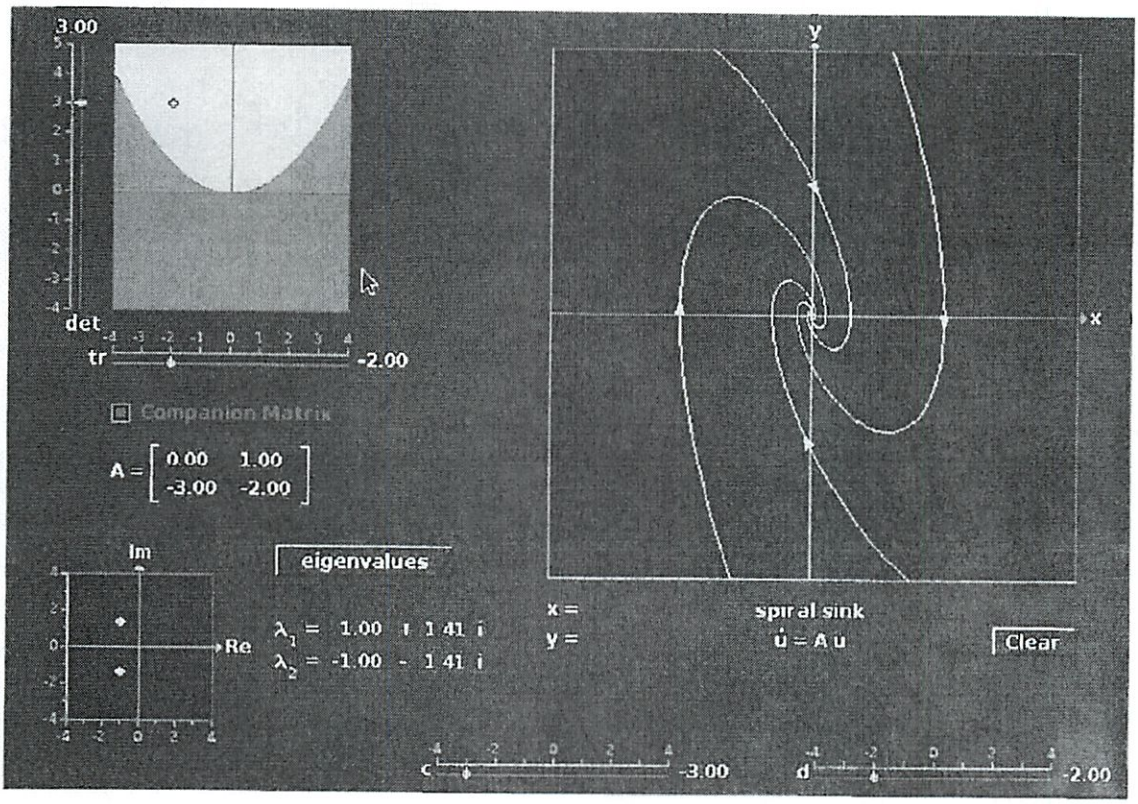
(ii) $y = x', \quad y' = -2x' - 3x = -3x - 2y.$

$$\begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}$$

(c) As $t \to \infty$, all trajectories go to infinity. It appears that some special ones go like:  & all the rest end up going like: 



(e) Solutions spiral around the origin.
They move inwards.



(d)

$$x = Ae^{\frac{3+\sqrt{5}}{2}t} + Be^{\frac{3-\sqrt{5}}{2}t}$$

So $$x' = A\frac{3+\sqrt{5}}{2}e^{\frac{3+\sqrt{5}}{2}t} + B\frac{3-\sqrt{5}}{2}e^{\frac{3-\sqrt{5}}{2}t}.$$

& $$\begin{pmatrix} x \\ x' \end{pmatrix} = A\begin{pmatrix} 1 \\ \frac{3+\sqrt{5}}{2} \end{pmatrix}e^{\frac{3+\sqrt{5}}{2}t} + B\begin{pmatrix} 1 \\ \frac{3-\sqrt{5}}{2} \end{pmatrix}e^{\frac{3-\sqrt{5}}{2}t}.$$

$\frac{3+\sqrt{5}}{2} > \frac{3-\sqrt{5}}{2}$, so as $t \to \infty$ the 1st term dominates.

The trajectories end up going out in the directions

$$\pm\begin{pmatrix} 1 \\ \frac{3+\sqrt{5}}{2} \end{pmatrix}$$

(e) $$x = e^{-t}(A\cos\sqrt{2}t + B\sin\sqrt{2}t)$$

So $$x' = e^{-t}\left(-A\cos\sqrt{2}t - B\sin\sqrt{2}t - \sqrt{2}A\sin\sqrt{2}t + \sqrt{2}B\cos\sqrt{2}t\right)$$

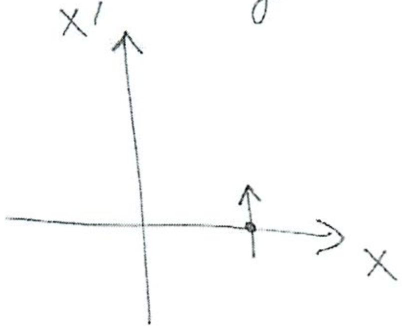
$$= e^{-t}\left([\sqrt{2}B - A]\cos\sqrt{2}t + [-\sqrt{2}A - B]\sin\sqrt{2}t\right)$$

& $$\begin{pmatrix} x \\ x' \end{pmatrix} = A\begin{pmatrix} e^{-t}\cos\sqrt{2}t \\ -e^{-t}\cos\sqrt{2}t - \sqrt{2}e^{-t}\sin\sqrt{2}t \end{pmatrix} + B\begin{pmatrix} e^{-t}\sin\sqrt{2}t \\ \sqrt{2}e^{-t}\cos\sqrt{2}t - e^{-t}\sin\sqrt{2}t \end{pmatrix}.$$

Neither term is dominant. Both are decaying sinusoids, so the decaying spiral makes sense.

There aren't asymptotic directions in case (ii).

(f) Crossing x-axis means $x' = 0$.



But if $x'(t) = 0$ that means that (at that time t), x is not changing.

"x not changing" graphically means the curve crosses the x-axis vertically.

18.03 Lecture 31

12/2

Complex Eigenvalue/vectors

higher order ODE + systems are diff animals

↳ sometimes related

Today: Systems

$$\underline{x}' = A\underline{x}$$

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$$

$$\begin{matrix} x_1'(t) = \\ x_2'(t) = \end{matrix} \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$x_1'(t) = 3x_1(t) - 2x_2(t)$$
$$x_2'(t) = 4x_1(t) - x_2(t)$$

$$x_1' = 3x_1 - 2x_2$$
$$x_2' = 4x_1 - x_2$$

Could be model of two restaurants competing on same block

②

Always start w/ a simple guess

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{bmatrix}$$

Substitute guess

$$(A - \lambda I)\, \underline{v} = \underline{0}$$

← can immediately skip here

$$\underbrace{\begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}}_{A - \lambda I} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(after canceling $e^{\lambda t}$ on both sides)

$$\lambda I = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

This only has boring sol'n ($\underline{v} = 0$)

✱ → Unless information $(A - \lambda I)$ is redundant in rows

(forgot before)

Short hand way to check when differ by constant

∟ Compute determinant

2 variables, 1 eq

∟ so 1 degree of freedom → so many interesting sols

(3)

Can skip to here - remember <u>short cut</u>

$\det(A - \lambda I) = \lambda^2 - \text{trace}(A)\,\lambda + \det(A)$

in a 2x2 $\qquad$ $a+d$ $\qquad$ $ad-bc$

$= \lambda^2 - 2\lambda + 5$

example

must be 0 to get interesting sols - so what are the roots?

(oh, makes more sense now!)

$\lambda^2 - 2\lambda + 5 = 0$

for $\lambda = 1 \pm \frac{\sqrt{-16}}{2}$

$= 1 \pm 2i$

Only sols so far is when λ is complex diff eq

$X(t) = \underline{V} \cdot e^{(1 \pm 2i)t}$

proposed

To finish: solve for v

if $\underline{\lambda = 1 - 2i}$

$$\begin{bmatrix} 2+2i & -2 \\ 4 & -2+2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(4)

Write out lines

$(2+2i)v_1 - 2v_2 = 0$

Try to find simpelest solution

Pick $v_1 = 1$

Solve for v_2

$(2+2i) = 2v_2$

$v_2 = 1+i$

So get $v = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$ as one choice

(forgot this)

So found one solution

$\underline{x}(t) = \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{(1-2i)t}$

Thats our complex sol'n

Now need real sol

claim $Re(x(t))$, $Im(x(t))$ are sols to real ODE

Why true?

(5)

$\downarrow$ are real

$\mathrm{Re}(x(t)) = \underline{u}(t)$

$\mathrm{Im}(x(t)) = \underline{v}(t)$

So $\underline{x}(t) = u(t) + v(t)\,i$

$\downarrow$ derivs of components & pieces

Then $x'(t) = u'(t) + v'(t)\,i$

$u'(t) + i\,v(t) = A\left(u(t) + i(v(t))\right)$

$\uparrow$ only = if Re, Im parts equal

So

$u'(t) = A\underline{u}$

$v'(t) = A\underline{v}$

So need Re(), Im() parts of above

$$\underline{x}(t) = e^{t}\begin{bmatrix} 1(\cos(-2t) + i\sin(-\overset{\text{- tricky}}{2t})) \\ (1+i)((\cos(-2t)) + i\sin(-2t)) \end{bmatrix}$$

Take Re(), Im() part of top & bottom

$$\mathrm{Re}(x(t)) = e^{t}\begin{bmatrix} \cos(2t) \\ \cos 2t + \sin 2t \end{bmatrix}$$

$\uparrow$ must take the cross terms

consider even & odd-ness

$i \cdot i = \text{real}$

$\mathrm{Im}(x(t)) = \text{up to us}$

(6)

Hardest part is Factoring and finding respective Eigenvalues

Prof: Mattuck's notes are better than book here

λ = eigenvalues

$\underline{v}$ = eigenvectors

Last bothersome case: Repeated Eigenvalues

det $(A - \lambda I)$ with root α that occurs more than once

Like a cubic eq'n $(\lambda-1)^2 (\lambda+3)$

(like a repeated root)

Somewhere add a t, right?

Yup

Need each eigenvalue to make an eigenvector — usually linearly ind

If two same - we still need to make it linearly ind

(7)

When does this repeated eigenvalue have the right #
of linearly ind. eigenvectors?

linear ind - not multiples of each other
can check w/ wronskian

Case of multiplicity 2:

Distinguish 2 cases

complete case
have enough
(have 2 lin. ind. eigenvectors)

incomplete case
don't have enough

For 2x2 matricies A

α is a repeated complete eigen value
(repeated root and two linearly dep eigen vectors)

Only when $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$

ODE $\quad x_1' = \alpha x_1$

$\quad x_2' = \alpha x_2$

Shouldn't use systems method
Boring - need 3x3 case

(8)

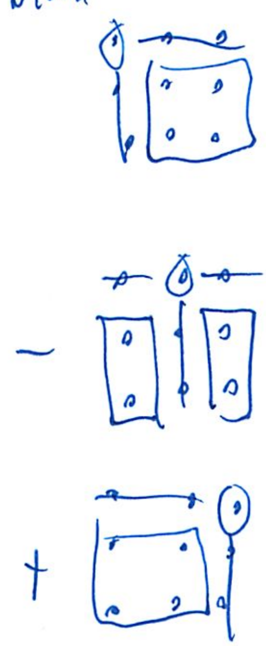
3x3 case

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{pmatrix}$$

$\det(A - \lambda I)$ = can expand by minors

$$-2-\lambda \cdot \det\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}$$

block



$$-1 \cdot \det\begin{bmatrix} 1 & 1 \\ 1 & -2-\lambda \end{bmatrix}$$

remove middle

$$+1 \cdot \det\begin{bmatrix} 1 & -2-\lambda \\ 1 & 1 \end{bmatrix}$$

along top

$$= \lambda(\lambda+3)^2$$

So $\boxed{\lambda = 0}$ eigenvalue

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

Solve for v_1, v_2, v_3

(9) 2 lin ind eq w/ 3 variables

(two fixed fill in last one)

↳ 1 degree of freedom

Can't invert matrix

Use need to use old fasion Gaussian elimination ← (need to read up on)

Some row operations to make it look like identity

⋮

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

→ when in doubt on test try all combos 1, -1, 0

$\lambda = -3$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

→ complete since 1 diff eq

3 variables →

2 degrees of freedom

(two variables free - can do anything - balance out w/ last one)

So tons of choices for

$V_1 + V_2 + V_3 = 0$

two possible $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

10

So end solution

$$x(t) = C_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{0t} + C_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-3t} + C_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-3t}$$

Interesting Theorm:

A is symetric real ~~matr~~ square matrix $(A^T = A)$
└ about diagonal

Then all of its eigenvalues are real + complete

Basis of google ~~p~~ page rank
└ largest Eigenvales = highest page rank

Incomplete case: next time

Lecture 31

12/2

Example : $\underline{x}' = A\underline{x}$ with $A = \begin{pmatrix} 2 & 1 \\ -4 & -1 \end{pmatrix}$

Guess $\underline{x} = \underline{v}e^{\lambda t} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{bmatrix}$. $\Rightarrow$ $\underbrace{\begin{pmatrix} 2-\lambda & 1 \\ -4 & -1-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{(A-\lambda I)\,\underline{v} = \underline{0}}$

Normally, 2 eqns + 2 unknowns $\rightarrow$ 1 soln.

(unless eqns are redundant = linearly dependent)

Find λ to make them redundant : Happens exactly when $\det(A-\lambda I) = 0$.

Shortcut : $\lambda^2 - tr(A)\lambda + \det(A) = \lambda^2 - \lambda + 2$

roots : $+\frac{1}{2} \pm \frac{\sqrt{7}\,i}{2} = \lambda$ So far : $\underline{x} = \underline{v}\, e^{(+\frac{1}{2} \pm \frac{\sqrt{7}}{2} i)t}$

Solve for $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

As solution to complex ODE.

$$\begin{pmatrix} 2-(+\frac{1}{2} - \frac{\sqrt{7}}{2} i) & 1 \\ -4 & -1-(\frac{1}{2} - \frac{\sqrt{7}}{2} i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Top row : $(\frac{3}{2} + \frac{\sqrt{7}}{2} i)v_1 + v_2 = 0$

Pick $v_1 = 1$, $v_2 = -(\frac{3}{2} + \frac{\sqrt{7}}{2} i)$

Get cx. soln $\underline{x}(t) = \begin{bmatrix} 1 \\ -\frac{3}{2} - \frac{\sqrt{7}}{2} i \end{bmatrix} \underbrace{e^{(\frac{1}{2} + \frac{\sqrt{7}}{2} i)t}}_{e^{\frac{1}{2}t}(\cos\frac{\sqrt{7}}{2}t + i\sin\frac{\sqrt{7}}{2}t)}$

Take $Re(x(t))$, $Im(x(t))$. This gives solutions to real ODE.

Why? If $\underline{x}(t) = \underline{u}(t) + i\underline{v}(t)$ with $\underline{u}, \underline{v}$ real vector-valued functions then $\underline{x}' = \underline{u}' + i\underline{v}'$ so $\underline{x}' = A\underline{x}$ becomes

$\underline{u}' + i\underline{v}' = A(\underline{u} + i\underline{v})$. Comparing real and imaginary parts,

we see $\underline{u}' = A\underline{u}$ and $\underline{v}' = A\underline{v}$ as desired.

To finish our example, $Re(\underline{x}(t)) = \begin{bmatrix} e^{\frac{1}{2}t}\cos\frac{\sqrt{7}}{2}t \\ -\frac{3}{2}e^{\frac{1}{2}t}\cos\frac{\sqrt{7}}{2}t \\ +\frac{\sqrt{7}}{2}e^{\frac{1}{2}t}\sin\frac{\sqrt{7}}{2}t \end{bmatrix}$, similarly for Im.

Gives our two independent solns.

Repeated eigenvalues:

Suppose we compute $\det(A - \lambda I) = (\lambda - \alpha_1)(\lambda - \alpha_2)\cdots(\lambda - \alpha_n)$

if $\alpha_i = \alpha_j$, we say this evalue is repeated. E.g.

$\det(A - \lambda I) = (\lambda - 1)^2(\lambda + 3)$.

Then need to find two independent solns to system w/ e-value 1.

Always get at least 1 by plugging e-value into matrix and solving.

Two cases: (1) Complete case: find two independent vectors $\underline{v}, \underline{w}$ both having same repeated e-value.

Then solution $x(t) = \underline{v}e^t + \underline{w}e^t + \underline{u}e^{-3t}$ in our example above.

The following result says that the 2×2 case is boring.

Thm : Given 2×2 matrix A, the number α is a repeated, complete e-value $\iff A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$

(so the corresponding ODE: $x_1' = \alpha x_1$, $x_2' = \alpha x_2$ Solution: $\begin{bmatrix} c_1 e^{\alpha t} \\ c_2 e^{\alpha t} \end{bmatrix}$)

not very interesting system since variables don't interact.)

pf : $\Leftarrow$ is easy: $A - \lambda I = \begin{pmatrix} \alpha-\lambda & 0 \\ 0 & \alpha-\lambda \end{pmatrix}$

sub. in $\lambda = \alpha$: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

so any vector $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ solves system.

$\Rightarrow$ See Mattuck's Notes.

Pick two independent ones. Complete.

Try a 3×3 example: $A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

Then $\det(A - \lambda I) = \cdots = -\lambda(\lambda+3)^2$.

So eigenvalues are $\lambda = 0, -3$.

$\lambda = 0$ solved just as before: (plug in 0, solve system) $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

turns out that $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is solution (or any multiple of it)

In general, various ways of solving this system.

If we plug in $\lambda = -3$, then get

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In luck, since have 3 vars, but only 1 equation (2 deg. of freedom to produce desired 2 independent solns.)

Only condition is $v_1 + v_2 + v_3 = 0$.

so pick $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, for example.

Then soln:

$$c_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \underbrace{e^{0t}}_{=1} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-3t}.$$

How did we pick this example? Bit of theory:

Thm If A is symmetric real (square) matrix (i.e. $A^T = A$) then all its eigenvalues are real and complete.

Case 2: Incomplete (repeated eigenvalue has only one lin. independent eigenvector)

In case of double root, experience tells us to guess:

$$\underline{x}(t) = (\underline{v}_1 + t\underline{v}_2)e^{\alpha t} \quad \text{if } \lambda = \alpha \text{ is repeated.}$$

Here choose $\underline{v}_2$: ~~eigenvalue~~ eigenvector for α. Then

$$\underline{x}'(t) = \alpha \underline{v}_1 e^{\alpha t} + \underline{v}_2 e^{\alpha t} + \alpha t \underline{v}_2 e^{\alpha t}$$

$$A\underline{x} = A\underline{v}_1 e^{\alpha t} + tA\underline{v}_2 e^{\alpha t}$$

these cancel since $\underline{v}_2$ is eigenvector for α

so $A\underline{v}_2 = \alpha \underline{v}_2$.

then it remains to solve:

$$(A - \alpha I)\,\underline{v}_1 = \underline{v}_2$$

<u>Note</u>: can't invert the matrix on LHS of equation, since α is eigenvalue so $\det(A - \alpha I) = 0$. Solve by elimination.

18.03 Lecture 32

12/5

Repeated Eigenvalues

Today: finish repeated root case

L w/ fun extras

Wed : Modeling + pictures

Fri : Non-linear cases

L overview, few specifics

End of new material

Systems minor shorter - but = weight on exam

12 qu - 2hrs but 3 hrs to do it

L 3 of each of the 4 units

Next week : Review

Friday: Repeated Eigenvalues cases

L ~~can~~ to solve system compute $\det(A - \lambda I)$

but may get $(\lambda - \lambda_1)^2 (\lambda - \lambda_2)$

repeated

(say in a 3×3 example)

(2)

Can we find enough linearly ind Eigen vectors for each Eigenvalue

↑ we can find at least 1

- but we need enough (= to multiplicity)

If enough → complete ← did 1 example on Fri

If not " → incomplete

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$\det(A - \lambda I) = -\lambda(\lambda+3)^2$

Set $\lambda = -3$; $A - \lambda I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

So in solving $(A - \lambda I)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Found only condition

$\underbrace{v_1 + v_2}_{\text{2 degrees of freedom}} + v_3 = 0$

Yes. we can find 2 ind vectors $\underline{u}, \underline{v}$ whose components $\Sigma = 0$ since two degrees of freedom

Picked $v = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $u = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Theorm: Symmetric maticies are always complete

w/ real Eigenvalues

Symmetric around main diagonal $\begin{pmatrix} 1 & 4 & 6 \\ 4 & 2 & 5 \\ 6 & 5 & 3 \end{pmatrix}$

(3)

Incomplete ex $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$x_1' = x_1 + x_2$$
$$x_2' = x_2$$

$\det(A - \lambda I) =$

Can either

(1) Do. multiple in λ $\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix}$, take det

(2) Remember shortcut $\lambda^2 \underbrace{-2\lambda}_{\text{trace}} \underbrace{+1}_{\text{det}}$

Now need sol'ns to

$$\underbrace{(A - 1I)}_{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

subtract the identity

top row $0 \cdot v_1 + 1 \cdot v_2 = 0$

So $v_2 = 0$

$0 \cdot v_1 + 0 \cdot v_2 = 0$

So any multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an Eigenvector.

But that's it.

4

So we have 1 solution

$$x(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{1t}$$

Could check, by plugging back into $x' = Ax$

But dim = 2, so expect 2 sols

2 inde Eigen vectors

Must do something different

Rember 2nd order case

Could Guess

$$\underline{x}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \underbrace{t}_{\text{add in } t \text{ here}} e^{1t}$$

Check if sol by comparing the two sides

$$\underline{x}'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left(te^{t} + e^{t} \right)$$

Then $\underline{Ax}$ = actual multiplication

$$A\underline{x} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} te^{t}$$

some stuff cancles

$$= 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} te^{t}$$

5

~~\#~~ Does it match?

No - this doesn't work

Didn't give any flexibility

So more sophisticated

$$\underline{x}(t) = \left(\underline{u} + \begin{pmatrix}1\\0\end{pmatrix} t\right) e^{t}$$

to be defined

Solve for a v that works

· have been able to cancle some other vital factors

$$\underline{x}'(t) = \underline{u}\, e^{t} + \begin{pmatrix}1\\0\end{pmatrix}(1+t)\, e^{t}$$

$$A\underline{x} = A\underline{u}\, e^{t} + \begin{pmatrix}1\\0\end{pmatrix} t e^{t}$$

Now compare the 2 sides

What can we cancle $te^{t} = \begin{pmatrix}1\\0\end{pmatrix} te^{t}$

and e^{t} on both sides

So left with

$$1 \cdot \underline{u} + \begin{pmatrix}1\\0\end{pmatrix} = A\underline{u}$$

⑥

So sol'n it

$$(A - 1 I)\underline{u} = \begin{pmatrix}1\\0\end{pmatrix}$$

Now need to solve for a u that does that

$$(A - I)\underline{u} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\cdot\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

So pick

$$\begin{pmatrix}U_1\\U_2\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

but could pick any # on top

↳ like $\begin{pmatrix}78\\1\end{pmatrix}$

So now we have a second solution!

$$\underline{x}(t) = C_1\begin{bmatrix}1\\0\end{bmatrix}e^t \quad \leftarrow \text{general sol'n}$$

$$+ C_2\left(\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}t\right)e^t$$

Some other ways to write

⑦

In general ~~Guess X is arbitrary vector~~

$$\text{Guess: } \underline{x}(t) = (\underline{u} + \underline{v}\,t)e^{\lambda t}$$

$\underline{u}$: arbitrary vector

$\underline{v}$: eigenvector already found

$$\text{Solve } (A-\lambda I)\,\underline{u} = \underline{v}$$

Does it bother you that we are guessing?
Is there a more systematic method?
(Theory is in notes)

But we could also consider systems which are still linear but A can have entries that are all functions of t)

ie 2×2 case

$$x_1' = a(t)\,x_1 + b(t)\,x_2$$
$$x_2' = c(t)\,x_1 + d(t)\,x_2$$

$\rightarrow$ ie $A = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$ are fns of t

Everything still true when items are fns of t

⑧

So cool systematic answer: Fundamental matrix

Fundamental matrix F (Mattuck calls X)

matrix whose column vectors are lin. ind' soln's to $\underline{x}' = A\underline{x}$

ex (from very begining)
$A \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$ $\underline{x}' = A\underline{x}$

Soln $\underline{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$

two diff sols

So good choice of F is made of the 2 vectors

$$F = \begin{pmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & e^{-4t} \end{pmatrix}$$

So what is that good for?

All sols: $\underline{x} = F \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ ⊛

or F satisfies $F' = AF$

respects columns when on right

fancy notation

9

Can use this identity to solve IVPs

if $\underline{x}(t_0) = \underline{x}_0$

then $\underline{X}_0 = F(t_0)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ using (*)

i.e. $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = F(t_0)^{-1}\underline{x}_0$

<u>Example</u> (from above)

Let $t_0 = 0$

$F(t_0) = F(0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Take inverse w/ determinite + negate + switch (Remember $I^{-1} = I$)

$F(t_0)^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

(don't need to worry if $F(t_0)$ is invertable since columns vectors are always linearly ind for all t)

10

Then Sol to IVP

$$\underline{X}(t) = F(t)\, F(0)^{-1}\, \underline{x_0}$$

$$x(t) = \begin{pmatrix} e^{+2t} & e^{-4t} \\ e^{-2t} & e^{-4t} \end{pmatrix} \cdot -\frac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix}$$

do matrix multiplication

plug in initial values

So gives us a sol'n

But how do you guess te^t in systems?
(last 2 min - will be opaque)

Exponential $\quad e^x = \underbrace{\sum_{n=0}^{\infty} \frac{x^n}{n!}}_{\text{Taylor Series}} = 1 + x + \frac{x^2}{2!} + \cdots$

So make exponential matrix that looks like e^λ

$$e^a = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$A^3 = A \cdot A \cdot A$

(h)

So can add the $n \times n$ matricies

Entries are each ∞ series which converge
L since very fast converging series

---

e^{At} is similar. Multiply by scalar t

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

Theorm: $e^{At} = F(t) \cdot F(0)^{-1}$ for any Fund. Matrix F

Fundemental maxtrix

So $e^{At}\underline{x_0}$ is a soln to $\underline{x}' = A\underline{x}$

Lecture 32 More repeated eigenvalues

① 12/5

Last time: Talking about repeated eigenvalues.

Compute $\det(A - \lambda I) = (\lambda - \alpha_1)^2 (\lambda - \alpha_2)$ (say in 3×3 system)

Q: Can we find enough independent eigenvectors for each eigenvalue?

Here enough means equal to their multiplicity. complete v. incomplete.

On Friday, saw an example of 3×3 matrix which was "complete"

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \qquad \det(A - \lambda I) = -\lambda(\lambda+3)^2$$

set $\lambda = -3$: $A - \lambda I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Any real symmetric matrix is complete.

then $\underline{v}$ is eigenvector if

$v_1 + v_2 + v_3 = 0$.

(two degrees of freedom: pick any v_1, v_2, $\exists\, v_3$ s.t. $v_1 + v_2 + v_3 = 0$)

Another example:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{then} \quad \det(A - \lambda I) = (1-\lambda)^2$$

Set $\lambda = 1$: $A - \lambda I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow v_2 = 0$.

So any vector of form $\begin{pmatrix} v_1 \\ 0 \end{pmatrix}$ is eigenvector, but no others.

(1-dim'l space of e-vectors. Not 2!)

Say $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is our e-vector.

Guess: $\underline{x}(t) = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{e-vector}} t e^{\overset{\text{e-value}}{1} \cdot t}$

$\underline{x}' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (te^t + e^t)$

$A\underline{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} te^t$. Doesn't work!

No flexibility here! Add some.

Since $\begin{pmatrix}1\\0\end{pmatrix}$ was e-vector, then $\begin{pmatrix}1\\0\end{pmatrix} \cdot te^t$ cancels

in $\underline{x}'$ and $A\underline{x}$. Need to match $\begin{pmatrix}1\\0\end{pmatrix} e^t$ on LHS.

Idea: $\underline{x}(t) = \left(\underline{u} + \begin{pmatrix}1\\0\end{pmatrix} t\right) e^t$

then $\underline{x}'(t) = \left(\underline{u} + \begin{pmatrix}1\\0\end{pmatrix}\right) e^t + \begin{pmatrix}1\\0\end{pmatrix} te^t$

$A\underline{x} = A\underline{u} + \begin{pmatrix}1\\0\end{pmatrix} te^t$

So must solve: $\underline{u} + \begin{pmatrix}1\\0\end{pmatrix} = A\underline{u}$ or $\underbrace{(A - I\cdot 1)}\underline{u} = \begin{pmatrix}1\\0\end{pmatrix}$

$\begin{pmatrix}1&1\\0&1\end{pmatrix} - \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix}$

so pick $\underline{u} = \begin{pmatrix}0\\1\end{pmatrix}$

so $\underline{x}(t) = c_1 \cdot \left[\begin{pmatrix}0\\1\end{pmatrix} e^t + \begin{pmatrix}1\\0\end{pmatrix} te^t\right] + c_2 \begin{pmatrix}1\\0\end{pmatrix} e^t$

is general solution.

— DO THEORY FIRST — SEE PAGE 2A —

Fundamental Matrices for systems of D.E.s

$F(t)$ = matrix whose column vectors are independent sol'ns to $\underline{x}' = A\underline{x}$.

Example: $A = \begin{pmatrix}-3&1\\1&-3\end{pmatrix} \rightsquigarrow \underline{x} = c_1 \cdot \begin{bmatrix}1\\1\end{bmatrix} e^{-2t} + c_2 \begin{bmatrix}1\\-1\end{bmatrix} e^{-4t}$

so $F = \begin{pmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t}\end{pmatrix}$. Say all sol'ns: $F \cdot \underbrace{\begin{pmatrix}c_1\\c_2\end{pmatrix}} = \underline{x}$

arbitrary constants above.

or, if you prefer: $F' = AF$ since matrix mult. respects column vector sol'ns

$F' = (\underline{x}', \underline{y}') = (A\underline{x}, A\underline{y})$
$= A(\underline{x}, \underline{y}) = AF$, for two sol'ns $\underline{x}, \underline{y}$.

A little bit of theory:

We've been studying constant coeff. systems:

$$x_1' = ax_1 + bx_2 \qquad a,b,c,d \text{ const.} \qquad \underline{x}' = A\underline{x}$$
$$x_2' = cx_1 + dx_2$$

Can allow a,b,c,d to be functions of time. Then associated matrix looks like $A = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$. Similar for $n \times n$ systems.

Remember solution vectors $\underline{x}(t), \underline{y}(t), \underline{z}(t), \ldots$

are called dependent if there exist constants c_1, c_2, c_3 s.t. $c_1\underline{x}(t) + c_2\underline{y}(t) + c_3\underline{z}(t) + \ldots = 0$ for all t.

Big thm: If $\underline{x}(t), \underline{y}(t), \underline{z}(t), \ldots$ are n linearly indep. vectors for $n \times n$ system $\underline{x}' = A(t)\underline{x}$, then general solution is of the form $c_1\underline{x}(t) + c_2\underline{y}(t) + c_3\underline{z}(t) + \ldots$

(clearly all linear combinations are solutions, as this is just the matrix version of superposition. Other direction harder. Idea same as in single ODE case: prove $\underbrace{\text{existence / uniqueness of}}_{\text{HARD}}$ IVP, then use $\underbrace{\text{it to deduce}}_{\text{NOT SO HARD}}$ general sol'n.

NOT SO HARD - See Mattuck's L9.5 p. 22-23.

Check for linear independence of $\begin{bmatrix} | & | & | \\ \underline{x}(t) & \underline{y}(t) & \underline{z}(t) \\ | & | & | \end{bmatrix}$ using Wronskian. (det. of this matrix)

Solutions to system are dependent iff Wronskian $\equiv 0$ } very strict dichotomy here!
indep. Wronskian never 0.

Use $\underline{x} = F \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ to solve IVP. If $\underline{x}(t_0) = \underline{x}_0$

functions of t

then $\underline{x}_0 = F(t_0) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

so $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = F^{-1}(t_0) \cdot \underline{x}_0$

Note: this matrix invertible since $\det(F(t_0)) \neq 0$

Reason: $F(t)$ is made of solutions to system, which are linearly independent for all t.

So if $\underline{x} = F \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

then sol'n to IVP: $\underline{x} = F(t) \cdot F^{-1}(t_0)\, \underline{x}_0$

Do example of this with $t_0 = 0$. Then $F(0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

using $A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$ above

so $F^{-1}(0) = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\text{so} \quad \underline{x} = \underbrace{\begin{pmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}}_{\frac{1}{2} \begin{pmatrix} e^{-4t} - e^{-2t} & e^{-4t} + e^{-2t} \\ -e^{-4t} - e^{-2t} & e^{-2t} - e^{-4t} \end{pmatrix}} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\frac{1}{2} \begin{pmatrix} e^{-4t} - e^{-2t} & e^{-4t} + e^{-2t} \\ -e^{-4t} - e^{-2t} & e^{-2t} - e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Exponential Matrix : Recall Taylor series for e^x at $x=0$ given by

$$e^x = \sum_{n\geq 0}^{\infty} \frac{x^n}{n!} .$$

Make similar definition for matrices :

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

where now A^3 is matrix multiplication $A \cdot A \cdot A$

All these are $n\times n$ matrices if A is $n \times n$, so can add them.

Their $(i,j)^{th}$ entries form convergent infinite series.

Also $$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$$

(as t is just scalar multiplying A)

Thm : $e^{At} = F(t)\,F(0)^{-1}$ so

$e^{At}\underline{x}_0$ is sol'n to $\underline{x}' = A\underline{x}$ with $\underline{x}(0) = \underline{x}_0$.

pf : use power series def'n of e^{At} to show it solves

$\underline{x}' = A\underline{x}$, with initial condition. Then since

both e^{At}, $F(t)F(0)^{-1}$ solve same IVP, must be same

by uniqueness thm.

Two examples : $\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$ from equality w/ $F(t)\,F(0)^{-1}$.

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ by infinite series

18.03 Recitation 12/6

(only 3 students)

(m2 q1)
If top row is multiple of bottom row

$$\begin{pmatrix} \lambda x & \lambda y \\ x & y \end{pmatrix}$$

- det = 0
- matrix not invertable
- is a vector

$$\begin{pmatrix} \lambda x & \lambda y \\ x & y \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 0$$

is always

$$\begin{pmatrix} -y \\ x \end{pmatrix}$$

So if want to solve

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 0$$

We don't have options besides ~~[illegible]~~ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

②

Example

$$\dot{x} = A x$$

A: 2×2 matrix

① There is a 2 dimensional space of solutions

$$\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

② Its better to collect terms into Fundamental matrix

$$F = \begin{pmatrix} f_1(t) & g_1(t) \\ f_2(t) & g_2(t) \end{pmatrix}$$

What is x in terms of this?

$$x = F \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 f_1(t) + c_2 g_1(t) \\ c_1 f_2(t) + c_2 g_2(t) \end{pmatrix}$$

Once we know that we know everything

F satisfies diff eq $\dot{F} = A F$

3

So sol'n to this diff eq

If F is 1×1 (a number)

$$F = e^{At}$$

↓ take deriv

$$\frac{d}{dt} e^{At} = A e^{At}$$

Now if F is a matrix ($> 1 \times 1$)

$$e^{At} = \sum \frac{A^n t^n}{n!}$$

↓ take deriv (term by term)

$$\frac{d}{dt} e^{At} = \sum \frac{n A^n t^{n-1}}{n!}$$

$$= A \sum \frac{t^{n-1}}{(n-1)!} A^{n-1}$$

$$= A e^{At}$$

(4)

So sol to $\dot{F} = AF$

is $F = \underbrace{e^{At}}_{\text{like a notation for}} = \underbrace{\sum \frac{A^n t^n}{n!}}$

We've reduced question to solving system to doing exponenciation

└ same thing as last week, but different notation

(TA: All Fs capital - mistake if f)

IVP digression:

$$\dot{F} = AF$$

$$F = F_0 e^{At}$$

some constant matrix $\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$ ← when you plug in $t=0$ you get that

IVP: $x(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ given to you

5

Now you want to know $x(t)$

$$x(t) = A^{At}\, x(0)$$

Reflection of this in the Fs ...

$$x(t) = F\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = F(0) \cdot e^{At} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$x(0) = F(0) \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

If already have $F(t)$ and want $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Then

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = F(0)^{-1}\, x(0)$$

Apply $F(t)$ to both sides to get $x(t)$

$$F(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \cancel{F(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}} = x(t)$$

$$\boxed{F(t)\, F(0)^{-1}\, x(0)}$$

(Once you have found $F(t)$)

6

Back to how to expontiate a matix?

$$e^{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} t} =$$

$$= \begin{pmatrix} e^{At} & 0 \\ 0 & e^{Bt} \end{pmatrix}$$

this only works in this form

$$M = Z \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Z^{-1}$$

$$e^{Mt} = e^{Z \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Z^{-1} t} = \sum \frac{t^n}{n!} \left(Z \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Z^{-1} \right)^n$$

$$\left(Z \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Z^{-1} \right)^2 = Z \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \overbrace{Z^{-1} Z}^{\text{Cancles}} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Z^{-1}$$

$$= Z \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^2 Z^{-1}$$

$$\left(Z \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Z^{-1} \right)^n = Z \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n Z^{-1}$$

$$e^{Mt} = Z \left(\sum \frac{t^n}{n!} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n \right) Z^{-1}$$

$$= Z e^{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}} Z^{-1}$$

7

Suppose can write

$$M = Z \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Z^{-1}$$

$$e^{Mt} = Z \, e^{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} Z^{-1}$$

$$= Z \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n =$

Diagonitizing a matrix

$$Z \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Z^{-1}$$

$$Z = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

$$Z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} w \\ y \end{pmatrix}$$

$$Z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix}$$

8

$$Z^{-1}\begin{pmatrix} w \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$Z^{-1}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Often constant multiple that characterizes vector

$$\overbrace{Z \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Z^{-1}}^{M}\begin{pmatrix} w \\ y \end{pmatrix} =$$

$$= Z \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{?? bad step}$$

$$= Z a \quad \text{?? bad step}$$

$$= a \begin{pmatrix} w \\ y \end{pmatrix}$$

$$Z \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Z^{-1} \begin{pmatrix} x \\ z \end{pmatrix} = b \begin{pmatrix} x \\ z \end{pmatrix}$$

The columns of Z are the eigenvectors of M

The entries a, b are the eigenvalues of M

ⓐ

↓ Reduced problem of Finding z to this

How do you find those?

eigenvalues of M are the roots of the charastic polynomial ↳ a, b

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \cdot V = aV$$

Looking for eigen value ↓

$$\begin{pmatrix} M_{11} - a & M_{12} \\ M_{21} & M_{22} - a \end{pmatrix} V = 0$$

So rows are multiples of each other
So only have to check one

$$\begin{pmatrix} (M_{11} - a) v_1 + M_{12} v_2 \\ \cdots \quad \cdots \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -M_{12} \\ M_{11} - a \end{pmatrix}$$ using the very begining

So there you have it
2nd column is identical
Now can solve inital cv...
Won't solve it here

(10)

Solve $\dot{x} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} x$

using
what we did

With a 2x2 the other method may be faster

This is the conceptual way to do it

Eaiser on large matrices

Still need to find Eigen values

$(1-\lambda)(4-\lambda) - 6$

$4\lambda - 5\lambda + \lambda^2 - 6$

$-2 - 5\lambda + \lambda^2$

$$\frac{5 \pm \sqrt{25 + 4 \cdot 2}}{2}$$

$$= \frac{5 \pm \sqrt{33}}{2}$$

Call them a_1, a_2

awful 1 and awful 2 – use in rest of problem

(11)

$$\begin{bmatrix} 1-a_1 & 2 \\ 3 & 4-a_1 \end{bmatrix} \begin{bmatrix} \\ \end{bmatrix}$$

Want one to ~~be~~ be multiple of the other

$$\begin{bmatrix} 2 \\ -1+a_1 \end{bmatrix}$$ ← try to make 1st row = 0
don't need to worry 2nd row since is a multiple

for a_1

$$\begin{bmatrix} 2 \\ -1+a_2 \end{bmatrix}$$

for a_2

Try to diagonalize
↳ to make easy to exponentiate

$$A = \begin{bmatrix} 2 & 2 \\ -1+a_1 & -1+a_1 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

12

Take inverse

- - -

TA: May be conceptually eaiser
But harder to calculate

- - -

Ask in OH

12/7

1. Finish last recitation
2. Complex hw qu
3. Decoupling hw qu

---

2. Works same in real case

Jethro OH

$$4ic_1 - 4c_2 = 0$$

$$4c_1 - 4ic_2 = 0$$

So ~~iff~~ $4c_1 - 4c_2 = 0$

$$c_1 = c_2$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

now $ic_1 = c_2$

$$\begin{pmatrix} c_1 \\ \ \end{pmatrix}$$

anything

then $c_2 = ic_1$

$$\begin{pmatrix} c_1 \\ ic_1 \end{pmatrix}$$

②

So if $c_1 = 1$

$$\begin{pmatrix} 1 \\ i\,1 \end{pmatrix}$$

So that is the answer for the other one

$$x(t) = A\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{3+4i} + B\begin{pmatrix} 1 \\ i \end{pmatrix} e^{3-4i}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cancel{A\begin{pmatrix} 1 \\ -i \end{pmatrix}\cos}$$

$$e^{3t}\, e^{4it}$$

$$e^{3t}\left[\cos(4t) + i\sin(4t)\right]$$

$$\cancel{A\begin{pmatrix} 1 \\ -i \end{pmatrix}}$$

But when doing regular 2nd order eqns

$$x'' + 2x' + 2x = 0$$

$$r^2 + 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$$

(3) Could say $\boxed{x = \underbrace{A e^{(-1+i)t}} + B e^{(-1-i)t}}$

So if ~~want~~ to end up w/ real, Im sol

take Re, Im of

gives 2 fns that are ~~an~~ solution

don't need to ~~automatical~~ do B term - automatic since get same thing

$e^{-t}(\cos t + i \sin t)$

$Re() = e^{-t} \cos t$

$Im() = e^{-t} \sin t$

So general solution

$\boxed{x(t) = A e^{-t} \cos t + B e^{-t} \sin t}$ ← both valid answers

So same here - Just need to work at

$x(t) = A \underbrace{e^{3t} \cos 4t}_{Re} + B \underbrace{e^{3t} \sin 4t}_{Im}$ ⊗ Not valid since vectors

(4)

Still need to multiply by vector here

$$x(t) = A\begin{bmatrix}1\\-i\end{bmatrix} e^{3t}\cos 4t + B\begin{bmatrix}1\\i\end{bmatrix} e^{3t}\sin 4t\, i$$

Could write out as 1 vector

Review

$$\begin{pmatrix}1\\-i\end{pmatrix} e^{(3+4i)t} = \begin{pmatrix}1\\-i\end{pmatrix} e^{3t} e^{4it}$$

$$= \begin{pmatrix}1\\-i\end{pmatrix} e^{3t}(\cos 4t + i\sin 4t)$$

Vector times a number

So multiply each entry by that number

$$2\begin{pmatrix}1\\i\end{pmatrix} = \begin{pmatrix}2\\2i\end{pmatrix}$$

$$= \begin{pmatrix} e^{3t}(\cos 4t + i\sin 4t) \\ -i\, e^{3t}(\cos 4t + i\sin 4t)\end{pmatrix}$$

bottom multiply is

$$= \begin{pmatrix} \text{same} \\ e^{3t}(-i\cos 4t + \sin 4t)\end{pmatrix}$$

5

So $Re() = \begin{pmatrix} e^{3t} \cos 4t \\ e^{3t} \sin 4t \end{pmatrix}$

$Im() = \begin{pmatrix} e^{3t} \sin 4t \\ -e^{3t} \cos 4t \end{pmatrix}$

$x(t) = A \cdot Re() + B\, Im()$ ← general solution!

Decoupling

~~the~~ Prof said not covering
But still hw
Won't be tested

$$\begin{pmatrix} z \\ w \end{pmatrix}' = \begin{pmatrix} \cdots & 0 \\ 0 & \cdots \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}$$

~~same~~ some # on diagonal

(6)

System is decoupled if matrix is diagonal

↓

Values on diagonal
Zeros elsewhere

<u>decoupled</u> - if write out as 2 eqn
each equation is only 1 variable
So simple

$$\begin{pmatrix} z \\ w \end{pmatrix}' = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}$$

$$z' = 2z$$
$$w' = -w$$

Qu: So how to change variables to de couple?

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(7) The #s will be the Eigenvalues of matrix,

So step 1) Solve like normal

find eigenvalues + eigenvectors

$$\begin{pmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{pmatrix}$$

$$(4-\lambda)(-1-\lambda) - 6 = 0$$

$$\lambda^2 - 4\lambda + \lambda - 4 - 6 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$\lambda = (x+2)(x-5)$$

$$\lambda = 5, -2$$

$\underline{\lambda = -2}$

$$\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3a + 1b = 0$$

$$a = -1 \quad b = 3$$

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

⑧

$\underline{\lambda = 5}$ $\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-1c + 2d = 0$

$c = 2 \quad d = +1$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$x(t) = A\begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-2t} + B\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}$

Now $\underline{\text{Step 2}}$ Decouple

L need diagonal form

variables such that derivs are multiples of themselves

We have $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Call $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$x' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$= 4x + 3y$

(9)

x itself is not what we want

└ since matrix are not diagonal (dcs)

Need z, w that are linear combos of x + y

$z = \_\_ x + \_\_ y$

$w = \_\_ x + \_\_ y$

~~z' z A z~~ ____________________

Short way - just ans

If $z = \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-2t}$ then $z' = -2z$

↑ answer for "A" term

$w = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}$ then $w' = 5w$

(true)

$z = -x + 3y$

$w = 2x + y$

↑ got # from ans to previous

└ eigen vectors

10

$$z' = -x' + 3y'$$

$$= (-1, 3) \begin{pmatrix} x \\ y \end{pmatrix}'$$

$$= (-1, 3) \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= (5, -5) \begin{pmatrix} x \\ y \end{pmatrix}$$

... made a mistake

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Gausian ellimination

└ By steps turn matrix into diagonal form

It will tell you the change of variables you need

Another way to think about it

$$\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \left(\begin{array}{c|c} -1 & 2 \\ 3 & 1 \end{array}\right) =$$

↑ no problem

Just notation – not doing anything

$\begin{pmatrix} -1 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

when actually multiply ends up same

11

When multiply you get

$$= \left(\begin{array}{c|c} 2 & 10 \\ -6 & 5 \end{array}\right)$$

what we expect from eigen vectors

It's same as

$$= \cancel{\begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}\left(\begin{array}{c|c} -1 & 2 \\ 3 & 1 \end{array}\right)}$$

$$= \left(\begin{array}{c|c} -1 & 2 \\ 3 & 1 \end{array}\right)\begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$$

Can check they eq ✓

original · new.

= new · diagonal

So diagonal

take inverse

$$\begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} = \left(\begin{array}{c|c} -1 & 2 \\ 3 & 1 \end{array}\right)^{-1} \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}$$

That's how we diagonalize a matrix

(12)

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} = m$$

$$= M \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} \underbrace{m^{-1} \begin{pmatrix} x \\ y \end{pmatrix}}_{\text{Then need to work out}}$$

+ call $\begin{pmatrix} z \\ w \end{pmatrix}$ ← the 2 components

like $4x + 12y$

$z \neq m^{-1} x$

w

18.03 Lecture 33

12/7

Linear stuff will be ommitted completly
↑ nonlinear

Yesterday: Discussing exponential matrix

$$e^A = I + A + \frac{A^2}{2!} + \cdots$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

So
$$e^A = \underset{\boxed{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}}{I} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}{2!} + \cdots$$

↑ like scaler $\frac{1}{2} \cdot$ matrix

most things 0

$$= \begin{pmatrix} (1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots) & 0 \\ 0 & 1 \end{pmatrix}$$

↑ since identity

$$= \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$$

(2)

In general, can't compute e^A from power series expansion

- for arbitrary matrix - its a weird combo of entries
- Can't compute e^A at all!
- So why do we care?

Usually we study

$$\cancel{e^{At}}\ e^{At} = I + \frac{At}{1} + \frac{A^2t^2}{2!} + \cdots$$

eg $At = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} t = \lambda t \cdot I$

Then $e^{At} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}$

But we can't compute e^{At} unless its really easy

$A = 1 \cdot 1$ matrix

- called a number
- designate w/ lower a

$e^{At} = e^{at}$

same as if is a # - since it is a #!!!

(3)

So it solves the ODE

$\cancel{\underline{x}' = A\underline{x}}$

$\underline{x}' = a\underline{x}$
↑for 1×1 matrix

So it would be nice if it worked for any matrix

any sol is $e^{at} \cdot$ constant

Want $e^{At} \cdot \underline{c}$ to solve $\underline{x}' = A\underline{x}$
↑constant vector

Check: If $\underline{x} = e^{At} \cdot \underline{c}$

then $x' = \left(I + At + \frac{A^2t^2}{2} + \dots\right)' \underline{c}$ ← take deriv in t

$= 0 + A + A^2t + \frac{A^3t^2}{2!} + \dots$

$= A(e^{At})\underline{c}$

So e^{At} is a sol to our ODE

Works w/ any linear system w/ constant coeffs

↳inc. repeated roots

(4)

Even works for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Fantastic except slight problem

L impossible to compute e^{At}

but we can compute it

L but not from power series expansion

L but can w/ more devious ways

Rather find $e^{At} \cdot \underline{c}$ for some good choices of $\underline{c}$.

All you care about when solving ODE

Since general sol to system ($n \times n$) requires n linearly ind sols

L Then everything is a linear combo of

L every solution

So need to evaluate for n lin. ind solutions

So what good choices for c?

Idea Pick vector $\underline{c}$ so that $e^{At} \cdot \underline{c}$ has only finitely many non-zero terms in the power series.

(5)

So very smart idea;

All about the first step

$$\underbrace{e^{At}}_{(I + At + \ldots)} \cdot \underline{c} =$$

actually multipling each part

When do we get a nice answer?

When chose c to be eigenvector for nice eigenvalue.

$$= \underbrace{e^{(A-\lambda I)t}\, e^{\lambda I t}}_{e^{At}}\, \underline{c}$$

$\boxed{\text{sine } e^{A+B} = e^{A} e^{B}}$

But! ~~only~~ something sneaky

only true b/c

$$(A-\lambda I)(\lambda I) = (\lambda I)(A-\lambda I)$$

prove using power series

Can only rearrange if commute

Is true here - so can do it

$$= e^{(A-\lambda I)t}\, e^{\lambda t}\, \underline{c}$$

$$= e^{\lambda t}\, e^{(A-\lambda I)t}\, \underline{c}$$

(6)

So what do we have?

$$= e^{\lambda t}\left(I + (A-\lambda I)t + \frac{(A-\lambda I)^2 t^2}{2!} + \dots\right)\underline{c}$$

What happens if c is an eigenvector w/ eigenvalue λ?

$$(A-\lambda I)\underline{c} = 0$$

$$(A-\lambda I)^2\underline{c} = 0 \leftarrow \text{since first term killed it}$$

$$(A-\lambda I)^n\underline{c} = 0$$

~~All~~ / Most terms now 0

$$= e^{\lambda t}(I + 0 + 0 + 0 \dots)\underline{c}$$

$$= e^{\lambda t}\underline{c}$$

But works even in more sophisticated cases

If $\underline{c}$ is killed by $(A-\lambda I)^2$

but not $(A-\lambda I)$

Then sol'n

$$e^{At}\underline{c} = e^{\lambda t}(I + (A-\lambda I)t)\underline{c}$$

$\uparrow$ first 2 terms survive

7

So **Example** $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A$

looking to calc $e^{At} \cdot \underline{c}$

So write char. polynomial

$$\det(A - \lambda I) = (\lambda - 1)^2$$

One possibility. $\underline{c}$ = eigenvector w/ eigenvalue 1

$$A - 1I = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So anything of form

$\begin{pmatrix} c_1 \\ 0 \end{pmatrix}$ works.

Pick $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

But we need 2nd, ind sol to find system general sol

8

So find one killed by $(A-\lambda I)^2$

$\boxed{e^{At}\underline{c}}$

$(A-\lambda I)^2 \underline{c} = \underline{0}$

but $(A-\lambda I)\underline{c} \neq 0$

$=$ since want to find new

we found before $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

So can choose $\underline{c}$ such that

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Anything!

Just don't pick a multiple of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

So pick $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ← anything non 0 in 2nd the

So now we have the answer

Plug choice for c into eqn

Gives $e^{At}\begin{pmatrix}0\\1\end{pmatrix} = e^{1t}\left[I + \begin{pmatrix}0&1\\0&0\end{pmatrix}t\right]\begin{pmatrix}0\\1\end{pmatrix}$

So do matrix multiplication

$$\underbrace{\underbrace{\left[I + \begin{pmatrix}0&1\\0&0\end{pmatrix}t\right]}_{\begin{pmatrix}1&t\\0&1\end{pmatrix}}\underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{\begin{pmatrix}0\\1\end{pmatrix}}}_{\begin{pmatrix}t\\1\end{pmatrix}}$$

$$= e^{t}\begin{bmatrix}t\\1\end{bmatrix} \text{ is a sol}$$

Other ind sol. ~~the~~
from other version of formula

$$= e^{1t}\begin{bmatrix}1\\0\end{bmatrix}$$

So general sol

$$x(t) = A e^{t}\begin{bmatrix}t\\1\end{bmatrix} + B e^{t}\begin{bmatrix}1\\0\end{bmatrix}$$

That's why you guessed (what we guessed before)

10

~~1~~ last ~~things~~ comments

~~1~~ e^{At} was difficult to calculate

- still have not solved
- just said for certain vectors

But actually

$$e^{At} = F(t) \cdot F(0)^{-1}$$

where $F(t)$ = fundamental matrix

place sols into column vectors to make F

So same because both sols to ~~[illegible]~~ same IVP

Can use uniqueness of sols to conclude that they are the same

---

<u>Next time</u> draw pictures of sols to systems of 2x2 matricies

(won't be restricted to constant matricies or linear eqns)

methods apply generally

11

Preview: Focus on autonomous eqns

↳ Systems where RHS fn(x_1, x_2)

└ no t appears on RHS!

$$x_1' = f(x_1, x_2)$$

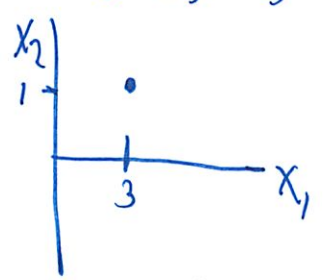
$$x_2' = g(x_1, x_2)$$

still fn of t | but does not depend on t

Plan: Draw pictures like in direction field days

└ little tick marks (will need on final)

For each (x_1, x_2) compute vector (x_1', x_2')



Plug 3,1 into the 2 Eqs

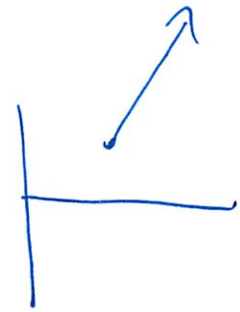
ex $x_1' = x_1 x_2 + x_2$

$x_2' = x_1^3 + 5$

then get

$$(x_1', x_2') = (6, 6)$$

(12)



Diff here + sol field

Here lenght (magnitude) of vector matters

└ tells us the speed at which traversing

And the direction (arrow heads)

└ to record direction in which time is increasing

So before next class; Go back to Linear Systems mathlet

lets you input a matrix $\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix} \to$ plot of sol curves in plane

So can see sol curves following along the trajectories

Qu on hw; Asks about stable sols

└ sols for which $x(t) \to 0$ as $t \to \infty$

(13)

Characterize them according to Eigenvalues

In mathlet, $x(t) \to$ origin

$\llcorner$ by following arrowheads

Try which one where all sample curves point to origin

Then translate $\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}$ to a fact about det + trace

trace $= d$

det $= -c$

Lecture 33 The exponential of a matrix

12/[redacted]7

Discussing exponential matrix last time:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

In general, hard to compute this for particular A. But we can do it in simple examples.

$$A = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} = t \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{then } A^2 = t^2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\cdots \quad A^n = t^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

so $$e^A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{t^2}{2!} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1+t+t^2/2!+\cdots & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}.$$

Also do example $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} t = \lambda t I$

Why do this? if A is 1×1 matrix (i.e. number a) then

$$e^{At} = e^{at} \leftarrow \text{usual exponential function.}$$

solves 1×1 system: $x' = ax \longrightarrow x(t) = c \cdot e^{at}$

So might hope that $e^{At} \underline{c}$ is a solution to $\underline{x}' = A\underline{x}$.

check it: if $\underline{x} = e^{At} \cdot \underline{c}$ then $\underline{x}' = A e^{At} \cdot \underline{c}$ ✓.

since $$\frac{d}{dt}(e^{At}) = \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \cdots\right)$$

$$= A + A^2 t + \frac{3A^3t^2}{3!} + \cdots = A e^{At}$$

<u>Amazing fact</u>: Can't evaluate e^{At}, but can evaluate by summing series

$e^{At}\underline{c}$ for good choices of $\underline{c}$. (n linearly indep. vectors) enough to determine general sol'n to $\underline{x}' = A\underline{x}$

<u>Plan</u>: Pick $\underline{c}$ so that power series

$e^{At}\underline{c}$ terminates after finitely many terms.

Indeed, write

$$e^{At}\underline{c} = e^{(A-\lambda I)t} e^{\lambda I t}\underline{c}$$

$$= e^{(A-\lambda I)t}\left(e^{\lambda t}\cdot\underline{c}\right) \leftarrow$$ $e^{\lambda t}$ now scalar mult. all components

(this step requires that $A-\lambda I$ & λI commute.)

$$= e^{\lambda t}\underbrace{e^{(A-\lambda I)t}\underline{c}}$$

in series form: $I\underline{c} + t(A-\lambda I)\underline{c} + \frac{t^2}{2!}(A-\lambda I)^2\underline{c} + \cdots$

so series will terminate if $\underline{c}$ is eigenvector for A with e-value λ, or more generally, if $(A-\lambda I)^m\cdot\underline{c} = \underline{0}$ for some m

(then all larger powers disappear as well since $(A-\lambda I)^{m+k} = (A-\lambda I)^k\cdot(A-\lambda I)^m$)

This explains how to solve all linear, const. coeff. systems:

① – distinct roots: of char. poly: $\lambda_1, \dots, \lambda_n$

pick $\underline{c}$ = eigenvector for λ_i

$\underline{\text{sol'n}}$: $e^{At} \cdot \underline{c} = e^{\lambda_i t} \underbrace{e^{(A-\lambda_i I)t} \underline{c}}_{\underline{c}}$

$\underline{c}$ since all higher terms in power series die!

② – repeated roots

e.g. $\underbrace{(r-\lambda_i)^2}(\underbrace{r-\lambda_2})$

find as many indep. e-vectors for λ_i as possible. If there's only 1, then try finding $\underline{c}$ s.t. $(A-\lambda_i I)^2 \underline{c} = 0$

handle as in step 1

then the corresponding sol'n is $e^{At} \cdot \underline{c} = e^{\lambda_i t} e^{(A-\lambda_i I)t}$

$= e^{\lambda t}(I\underline{c} + (A-\lambda I)t \cdot \underline{c})$

since all higher order terms die.

$\underline{\text{Example}}$: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

characteristic poly: $(\lambda-1)^2$

$A - \lambda I = A - 1 \cdot I$

$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

so $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

then $v_2 = 0$

Any $\begin{pmatrix} v_1 \\ 0 \end{pmatrix}$ is e-vector.

pick $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. But this e-value is incomplete.

now find $\underline{c}$ s.t.

$(A-\lambda I)^2 \cdot \underline{c} = \underline{0}$ (and $\underline{c}$ indep. from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$)

$(A-\lambda I)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

so any $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ not a multiple of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ will do.

In particular $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is simple choice.

Then <u>sol'n</u>: $\underline{x} = e^{At}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{\lambda t} e^{(A-\lambda I)t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= e^{\lambda t}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A-\lambda I}\begin{pmatrix} 0 \\ 1 \end{pmatrix} t \right)$$

$$= e^{\lambda t}\begin{pmatrix} t \\ 1 \end{pmatrix}$$

<u>Next time</u>: Graphing systems of ODEs using phase plane.

Review mathlet on this, which described behaviors of solutions according to companion matrix $\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}$. Think about that color coding in terms of e-values.

—

Focus on autonomous systems $\quad x' = f(x,y)$, $y' = g(x,y)$ ← don't depend on t.

Draw little arrows representing vectors $\begin{pmatrix} x' \\ y' \end{pmatrix}$ ← these have magnitude so a little different than direction fields.

arrowheads point in direction of increasing time.

2nd to last one

Hope we do e^{At} stuff I did not get last night

$$X' = \begin{pmatrix} 3 & & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix} X$$

$\det(A - \lambda)$

$$\det\begin{pmatrix} 3-\lambda & & & & \\ & 2-\lambda & 1 & & \\ & & 2-\lambda & & \\ & & & 5-\lambda & 1 \\ & & & & 5-\lambda \end{pmatrix}$$

$$= (3-\lambda)(2-\lambda)^2(5-\lambda)^2$$

TA: 1st step is to start squaring - multiply by itself

(I don't remember anything here)

(2)

$$e^{At} = \sum \frac{A^n t^n}{n!}$$

A^2 ta: Don't think too hard about qu

$$= \begin{pmatrix} 3 & & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix} \begin{pmatrix} 3 & & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \cdot 3 + 0 \\ 2 \cdot 2 + 0 \\ \end{pmatrix}$$

Simpler

$$\begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix}^n$$

$$\begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ & 4 \end{pmatrix}$$ squared

$$\begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix}\begin{pmatrix} 4 & 4 \\ & 4 \end{pmatrix} = \begin{pmatrix} 8 & 12 \\ & 8 \end{pmatrix}$$ cubed

③

$$\begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} \begin{pmatrix} 8 & 12 \\ & 8 \end{pmatrix} = \begin{pmatrix} 16 & 32 \\ & 16 \end{pmatrix} \quad m^4$$

$$\begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} \begin{pmatrix} 16 & 32 \\ & 16 \end{pmatrix} = \begin{pmatrix} 32 & 80 \\ & 32 \end{pmatrix} \quad m^5$$

$$\vdots$$

$$\begin{pmatrix} 2^n & 2^{n-1} \cdot n \\ & 2^n \end{pmatrix} = m^n$$

One' student's answer

$$\bar{x}' = A\bar{x}$$

$$x = e^{At}\vec{x}_0$$

$$e^{At} = \sum_n^{\infty} \frac{t^n A^n}{n!}$$

get
$$\begin{pmatrix} e^{3t} & & & & & \\ & e^{2t} & te^{2t} & & & \\ & & e^{2t} & & & \\ & & & e^{5t} & te^{5t} & \\ & & & & e^{5t} & \end{pmatrix}$$

(4)

Since

$$A^2 = A \cdot A = \begin{pmatrix} 3 & & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix} \begin{pmatrix} 3 & & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 3^2 & & & & \\ & 2^2 & 2\cdot 2 & & \\ & & 2^2 & & \\ & & & 5^2 & 5+5 \\ & & & & 5^2 \end{pmatrix}$$

Then

$$A^3 = \begin{pmatrix} 3 & & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix} \begin{pmatrix} 9 & & & & \\ & 4 & 4 & & \\ & & 4 & & \\ & & & 25 & 10 \\ & & & & 25 \end{pmatrix}$$

$$= \begin{pmatrix} 3^3 & & & & \\ & 2^3 & 2\cdot 2^2 + 2^2 = 3\cdot 2^2 & & \\ & & 2^3 & & \\ & & & 5^3 & 5^2 + 2\cdot 5^2 = 3\cdot 5^2 \\ & & & & 5^3 \end{pmatrix}$$

$$A^n = \begin{pmatrix} 3^n & & & & \\ & 2^n & n\cdot 2^{n-1} & & \\ & & 2^n & & \\ & & & 5^n & n\cdot 5^{n-1} \\ & & & & 5^n \end{pmatrix}$$

5

So plug back into

$$e^{At} = \sum \begin{pmatrix} \frac{t^n 3^n}{n!} & & & & \\ & \frac{t^n 2^n}{n!} & \frac{t^n 2^{n-1}}{(n-1)!} & & \\ & & \frac{t^n 2^n}{n!} & & \\ & & & \frac{t^n 5^n}{n!} & \frac{t^n 5^{n-1}}{(n-1)!} \\ & & & & \frac{t^n 5^n}{n!} \end{pmatrix}$$

$$= \begin{pmatrix} e^{3t} & & & & \\ & e^{2t} & te^{2t} & & \\ & & e^{2t} & & \\ & & & e^{5t} & te^{2t} \\ & & & & e^{5t} \end{pmatrix}$$

$$\vec{X} = \left(\quad \right) \vec{X}_0$$

If had IVP we could plug in values there

(6) If you don't like multiplying large matricies by themselves

Note

$$\begin{pmatrix} 3 & & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 3 & 1 \\ & & & & 3 \end{pmatrix} \begin{pmatrix} 3 & & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 3 & 1 \\ & & & & 3 \end{pmatrix} =$$

Only each square part interact!

$$= \begin{pmatrix} 3^n & & \\ & \begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix}^n & \\ & & \begin{pmatrix} 5 & 1 \\ & 5 \end{pmatrix}^n \end{pmatrix}$$

When can we do it?

When "blocks on diagonal"

- if 2 items in row, will interact w/ row below

7

#2 $X' = \begin{pmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & 1 \\ & & & 2 \end{pmatrix}$

but I forgot to ~~wright~~ write this

? so $\begin{pmatrix} e^{2t} & te^{2t} & & \\ & e^{2t} & & \\ & & e^{2t} & te^{2t} \\ & & & e^{2t} \end{pmatrix}$

Q: Is there a pattern to this?

└ Yes - will do later

But ~~on~~ here repeated eigen values

8

So an eaiser example

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$

9

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}\begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ & & & \end{pmatrix} = 0$$

So can do the problem the long way

Or write as $2 + \eta$

$$\eta = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

η

η^2

η^3

$\eta^4 = 0$

Matricies commute

└ So can use Binomial expansion

$$(2+\eta)^n = \sum_{i=0}^{3} \binom{n}{i} 2^{n-i} \eta$$

(10)

So $A^n = (2+M)^n = 2^n + n2^{n-1}M + 2^{n-2}M^2\binom{n}{2}$

$+ 2^{n-3}M^3\binom{n}{3} + 0$

$$= \begin{pmatrix} 2^n & n2^{n-1} & \binom{n}{2}2^{n-2} & \binom{n}{3}2^{n-3} \\ & 2^n & n2^{n-1} & \binom{n}{2}2^{n-2} \\ & & 2^n & n2^{n-1} \\ & & & 2^n \end{pmatrix}$$

Sum series

$$e^{2t} = \sum \frac{2^n t^n}{n!}$$

$$te^{2t} = \sum \frac{2^n t^n}{(n-1)!}$$

$$\frac{1}{2}t^2e^{2t} = \sum \frac{2^{n-2}\overbrace{\frac{n \cdot n-1}{2}}^{\binom{n}{2}} t^n}{(n-2)!}$$

~~[illegible]~~ ...

$$\frac{1}{6}t^3e^{2t} = \ldots$$

(11)

So get answer

$$\begin{pmatrix} e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} & \frac{t^3}{6}e^{2t} \\ & e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} \\ & & e^{2t} & te^{2t} \\ & & & e^{2t} \end{pmatrix}$$

Same as

$$\begin{pmatrix} a & 1 & & & & & \\ & a & 1 & & & & \\ & & a & 1 & & & \\ & & & a & 1 & & \\ & & & & a & 1 & \\ & & & & & a & 1 \\ & & & & & & \ddots \end{pmatrix}$$

$$\frac{t^n}{n!} e^{at}$$

Last part of story - break matrix into these building blocks last one

18.03 FALL 2011 – Problem Set 10

Due FRIDAY 12/09/11, high noon in 2-106

To encourage you to keep up with homework as it appears in lecture, both Part I and Part II problems are listed with the accompanying lecture in which the material will be covered.

Part I (30 points)

Lecture 31. Fri. Dec. 2: Complex and repeated eigenvalues
READ: EP 5.4, 5.6, Notes LS.3 HW: Notes 4C-1ab, 2, 6, 4D-2, 3

Lecture 32. Mon. Dec. 5: Decoupling and Solution Matrices
READ: EP 5.7, Notes LS.4-LS.6 HW: Notes 4E-1, 4F-1, 2, 4G-1, 2

Lecture 33. Wed. Dec. 7: Exponential matrices and inhomogeneous equations
READ: EP 5.8, Notes LS.6 HW: Notes 4H-1,3, 4I-1, 2

Lecture 34. Fri. Dec. 9: Introduction to nonlinear systems
READ: EP 7.2, 7.3, Notes GS

Part II (19 points)

0. (3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-102," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

1. (Friday, 6 pts)

a) Given the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -9 & -6 \end{pmatrix},$$

find two independent solutions to the linear system $\mathbf{x}' = A\mathbf{x}$.

b) Find the second order ODE whose companion matrix is equal to A. Find two linearly independent solutions $x(t)$ and $y(t)$ to this second-order equation. Then show that the vectors

$$\begin{pmatrix} x \\ x' \end{pmatrix}, \quad \begin{pmatrix} y \\ y' \end{pmatrix}$$

are expressible as linear combinations of the solutions you found in part (a).

2. (Monday, 5 pts) Recall that in the differential equation $\mathbf{x}' = A\mathbf{x}$, with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the characteristic equation of A is given by

$$\det(A - \lambda I) = \lambda^2 - (a+d)\lambda + (ad - bc) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A)\lambda$$

where tr denotes the trace of the matrix and det is the determinant. Give conditions in terms of $\operatorname{tr}(A), \det(A)$ such that every solution $\mathbf{x}(t)$ of the corresponding differential equation satisfies

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}.$$

Such differential equations are called "stable."

3. (Wednesday, 5 pts) Find e^{At} for each of the following matrices A:

a) A is the matrix in 1(a) above.

b) $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, for some constants a and b.

Michael Plasmeier

18.03 P-Set 10

12/7

-11

38/49

Part 1

Lecture 31 Complex + Repeated Eigenvalues

4C-1a Solve $x' = Ax$ for a) $\begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}$

So this is recitation from last week

$$\begin{pmatrix} -3-\lambda & 4 \\ -2 & 3-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

When can this be 0?

Charastic eq $(-3-\lambda)(3-\lambda) - 4 \cdot -2$

Charastic polynomial $\lambda^2 - (a+d) + ad - bc = 0$

$$\lambda^2 - (-3+3) + 3(-3) - (4 \cdot -2)) = 0$$

$$\lambda^2 - 0 + -9 + 8 = 0$$

$$\lambda^2 - 1 = 0$$

(2)

Try something else

$$(-3-\lambda)(3-\lambda)+8=0$$

$$\lambda^2+0-9+8=0$$

$$\lambda^2-1=0 \qquad \text{same!}$$

Factor

$$\lambda=1,-1$$

$\underline{\lambda=1}$

$$\begin{pmatrix} -4 & 4 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=0$$

$$-4c_1+4c_2=0$$

$$-2c_1+2c_2=0$$

$$-c_1+c_2=0$$

$$c_2=c_1$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\underline{\lambda=-1}$

$$\begin{pmatrix} -2 & 4 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=0$$

$$-2c_1+4c_2=0$$

$$-2c_1+4c_2=0$$

3

So possible solutions

$$c_1 = 2 \quad c_2 = 1 \qquad \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-1t} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

✓ Figured it out

b) $\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$

$$\begin{pmatrix} 4-\lambda & -3 \\ 8 & -6-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$\lambda^2 - (a+d) + ad - bc = 0$$

$$\lambda^2 - (4-6) + 4(-6) - (-3)(8) = 0$$

$$\lambda^2 + 2 - 24 + 24 = 0$$

$$\lambda^2 + 2 = 0$$

$$x = i\sqrt{2}, -i\sqrt{2}$$

does not match online

(4)

Try something else

$$(4-\lambda)(-6-\lambda)-(-3)(8)=0$$

$$\lambda^2-4\lambda+6\lambda-24+24=0$$

so I forgot these

fixed notes

$$\lambda^2+2\lambda=0$$ ✓ matches notes

$$\lambda=0,-2$$

$\underline{\lambda=0}$

$$\begin{pmatrix}4 & -3\\ 8 & -6\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$4c_1-3c_2=0$$

$$8c_1-6c_2=0$$

multiple of

what does this mean again?

det=0

not invatable

$$8c_1=6c_2$$

$$4c_1=3c_2$$

$$\frac{4}{3}c_1=c_2$$

$$4c_1-3\left(\frac{4}{3}\right)c_1=0$$

$$4c_1-4c_1=0$$

$$0=0$$

(5)

That did not work

Oh just need to do 1

That might be the multiple thing

$4c_1 - 3c_2 = 0$

$c_1 = 3 \quad c_2 = 4$

$\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ ✓ matches book

$\underline{\lambda = -2}$

$\begin{pmatrix} 6 & -3 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ 2 alternate as well

$6c_1 - 3c_2 = 0$

$c_1 = 3 \quad c_2 = 6$

reduce

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ✓ you can do that

$x(t) = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{0} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$

$= c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$ ✓ Correct

6

2. Prove that $m = 0$ is a eigen value of $n \times n$ constant matrix A iff A is a signular matrix $\det A = 0$

Charistic eq is

$$\det(A - mI) = 0$$

If $m = 0$ is a root, this says

$$\det(A) = 0$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

??

(7)

6. Farmer Smith + Jones have rabbit colonies.

Rabbits swap between farms

a) Write ODEs

$$S' = S - aS + bJ$$

Starting value — fraction that leave — fraction entering

$$J' = J - bJ + aS$$

? how do we account for the the 1 added each year?

$$S' = (1-a)S + bJ$$
$$J' = (1-b)J + aS$$

$$S' = (1-a)S + bJ$$
$$J = aS + (1-b)J$$

(8)

b) assume $a = b = \frac{1}{2}$

$S = 20$
$J = 10$
IVP (deal w/ later)

$$S' = \left(1-\frac{1}{2}\right)S + \frac{1}{2}J$$
$$J' = \frac{1}{2}S + \left(1-\frac{1}{2}\right)J$$

$$S' = \frac{1}{2}S + \frac{1}{2}J$$
$$J' = \frac{1}{2}S + \frac{1}{2}J$$

$$\begin{bmatrix} S \\ J \end{bmatrix}' = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} S \\ J \end{bmatrix}$$

So now like a normal problem

$$A' = \begin{pmatrix} 1/2 - \lambda & 1/2 \\ 1/2 & 1/2 - \lambda \end{pmatrix} A$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$
$$\lambda^2 - \left(\frac{1}{2} + \frac{1}{2}\right)\lambda + \left(\frac{1}{2} \cdot \frac{1}{2}\right) - \left(\frac{1}{2} \cdot \frac{1}{2}\right) = 0$$
$$\lambda^2 - \lambda = 0$$
$$\lambda = 0, +1$$

(9)

$\underline{\lambda = 0}$

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$\frac{1}{2} c_1 + \frac{1}{2} c_2 = 0$$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 5 \\ -5 \end{pmatrix} \text{ etc} \quad ✓$$

$\underline{\lambda = +1}$

$$\begin{pmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$ 2 = -1 so only need 1

$$-\frac{1}{2} c_1 + \frac{1}{2} c_2 = 0$$

$$c_1 = 1 \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad ✓$$
$$c_2 = 1$$

$$x(t) = A \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^0 + B \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{+t} \quad ✓$$

Now IVP

$S = 20$
$J = 10$

$$x(0) = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

$$20 = A(1)e^0 + B(1)e^{+(0)}$$
$$= A + B$$

(10)

$$10 = A - 1 e^0 + B (1) e^{+0}$$

$$= -A + B$$

2 eqn 2 unknows

$$20 = A + B$$

$$10 = -A + B$$

$$B = 10 + A$$

$$20 = A + (10 + A)$$

$$10 = 2A$$

$$A = 5$$

$$20 = 5 + B$$

$$B = 15$$

$$X(t) = 5 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^0 + 15 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{+t}$$ Correct

c) Show that S, J never ossilate

Charastic polynomial does not have complex roots
- which lead to sin/cos

So won't ossilate

(11)

4D-2 Complex + Repeated

$$x' = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} x$$

$$\lambda^2 - (3+3)\lambda + (3\cdot 3) - (-4\cdot 4) = 0$$

$$\lambda^2 - 6\lambda + 9 + 16 = 0$$

$$\lambda^2 - 6\lambda + 25$$

$$\frac{6 \pm \sqrt{36 - 4\cdot 1\cdot 25}}{2}$$

$$3 \pm \frac{\sqrt{-64}}{2}$$

$$3 \pm \frac{8}{2} i$$

$\lambda = 3 \pm 4i$ ✓ Correct

λ 3+4i

? precede as normal ?

$$\begin{pmatrix} 3-(3+4i) & -4 \\ 4 & 3-(3+4i) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$\begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

(12)

$-4i c_1 - 4 c_2 = 0$

$4c_1 - 4i c_2 = 0$

¿¿ Is that right ¿ Look at notes L5.3

✓ is in notes

c_i will be complex

¿So how to solve ¿ like normal)

$4c_1 = 4i c_2$

$c_1 = i c_2$

$-4i(i c_2) - 4c_2 = 0$

$-4c_2 - 4c_2 = 0$

hmm perhaps do multiple case thing

~~Try $c_1 = 1i$ $c_2 = -1i$~~

~~-4~~

WA says $y = -i x$

So ¿¿¿ $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

(13)

Does that work?

$$-4i(1) - 4(-i) = 0$$

$$4i - 4i = 0 \checkmark$$

$$4(1) - 4i(-i) = 0$$

$$4 - 4 = 0$$ Remember $i \cdot i = -1$

? So how would we figure that out

Sols say $(3-m)\alpha_1 - 4\alpha_2 = 0$

So $\alpha = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Using my variables

$$(3-\lambda)c_1 - 4c_2 = 0$$

$$c \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

So they rearranged $\lambda = 3+4i$?

Ask in OH ✓

(14)

Post OH :

if you have $4c_1 - 4c_2 = 0$

$c_1 = c_2$

this is just $ic_1 = c_2$

So set c_1 to anything

Then $c_2 = ic_1$

$$\begin{pmatrix} c_1 \\ ic_2 \end{pmatrix}$$

So if $c_1 = 1$

$$\begin{pmatrix} 1 \\ i \end{pmatrix}$$

← actually don't need!

$$X(t) = A\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(3+4i)t} + B\begin{pmatrix} 1 \\ i \end{pmatrix} e^{(3-4i)t}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{(3+4i)t} = e^{3t}\left[\cos(4t) + i\sin(4t)\right]$$

$$\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(3+4i)t} = \begin{bmatrix} e^{3t}(\cos(4t) + i\sin(4t)) \\ -ie^{3t}(\cos(4t) + i\sin(4t)) \end{bmatrix}$$

15

Multiply bottom out

$$= \begin{bmatrix} e^{3t}\left(\cos(4t) + i\sin(4t)\right) \\ e^{3t}\left(-i\cos(4t) + \sin(4t)\right) \end{bmatrix}$$

So $\mathrm{Re}() = \begin{pmatrix} e^{3t} \cos 4t \\ e^{3t} \sin 4t \end{pmatrix}$

$\mathrm{Im}() = \begin{pmatrix} e^{3t} \sin 4t \\ -e^{3t} \cos 4t \end{pmatrix}$

$X(t) = A \,\mathrm{Re}() + B \,\mathrm{Im}()$

$$= A \begin{pmatrix} e^{3t} \cos 4t \\ e^{3t} \sin 4t \end{pmatrix} + B \begin{pmatrix} e^{3t} \sin 4t \\ -e^{3t} \cos 4t \end{pmatrix}$$

Didn't need earlier other side since drops out due to some property...

16

3. $$x' = \begin{pmatrix} 2 & 3 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & 2 \end{pmatrix} x$$

Hmm have not done a 3-one

$$\begin{pmatrix} 2-\lambda & 3 & 3 \\ 0 & -1-\lambda & -3 \\ 0 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(2-\lambda)(-1-\lambda)(2-\lambda) = ??$$

not sure how/what to do
Oh do sections

Book

$$(2-\lambda)^2(-1-\lambda) = 0$$

$$\lambda = -1, 2$$

$\underline{\lambda = -1}$

$$\begin{pmatrix} 3 & 3 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$$

$$3c_3 = 0$$

$$c_3 = 0$$? can we do that ?

$$-2c_2 - 3c_3 = 0$$

$$? \; c_2 = 0$$

(17)

No this is all 0 - boring

But seems no other way

Book recommends $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

But I think middle vector $-1-(-1)$ they have 0

$-1+1=0$

Yeah opps wrong

So middle row $-3c_3=0$

So now top row

$3c_1 + 3c_2 + 3c_3 = 0$

$\begin{array}{ccc} ? & ? & ? \\ 1 & -1 & 0 \end{array}$

Works and is non-boring

Now other

$\underline{\lambda = 21}$ $\begin{pmatrix} 0 & 3 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$3c_2 + 3c_3 = 0$

$-3c_2 + -3c_3 = 0$

(18)

So same $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ but can't do that, instead $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

; Book has $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ - where did they get this?

And claiming not a complete eigenvalue

Lecture 31 (12/2); complete eigenvalues - have 2 lin. ind. eigenvectors

Must be $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ $x_1' = \alpha x_1$

$x_2' = \alpha x_2$

But we take two possible solutions

└ for some reason??

So

$$x(t) = A \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{-1t} + B \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + C \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t}$$

✓ I think that would be acceptable

(19)

Lecture 32 Decoupling + Solution Matricies

4E-1 A system $x' = 4x + 2y$
$y = 3x - y$

Give a new set of variables u, v that decouples the system that is linearly related to x, y

(Asked in OH)

Step 1. Solve like normal

$$\begin{pmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{pmatrix}$$

$$(4-\lambda)(-1-\lambda) - 6 = 0$$

$$\lambda^2 - 4\lambda + \lambda - 4 - 6 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$(\lambda + 2)(\lambda - 5)$$

$$\lambda = 5, -2$$

(20)

$\underline{\lambda = -2}$ $\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$3a + b = 0$$

$$a = -1 \quad b = 3$$

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$\underline{\lambda = 5}$ $\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$-1c + 2d = 0$$

$$c = 2 \quad d = 1$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x(t) = A \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-2t} + B \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}$$

Now <u>Step 2 Decouple</u>

↳ need diagonal form

variables such that derivs are multiples of themselves

(21)

z, w are linear combos of x, y

$$z = \_\_ x + \_\_ y$$
$$w = \_\_ x + \_\_ y$$

$$\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \left(\begin{array}{c|c} -1 & 2 \\ 3 & 1 \end{array}\right) =$$

original problem

our answers same as $\begin{pmatrix} -1 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 10 \\ -6 & 5 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} -1 & 2 \\ 3 & 1 \end{array}\right) \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} = \left(\begin{array}{c|c} -1 & 2 \\ 2 & 1 \end{array}\right)^{-1} \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

So $z' = -2z$

$w = 5w$

(I had diff letters in OH)

(22)

4F+1 Take the 2nd order ODE

$$x'' + p(t)x' + q(t)x = 0$$

a) Change it to first order system $x' = Ax$

ahh been awhile since we did this

$$x_1' = x_2$$

$$x_2' = -q(t)x_1 - p(t)x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

b) Show that the Wronskian of two sols is same as Wronskian in new eq

$$W(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}$$

So det of matrix

$$x_{11}(t)\,x_{22}(t) - x_{12}(t)\,x_{21}(t)$$

(23)

So

$$0 \cdot -p(t) - 1 \cdot -q(t)$$

$$= q(t)$$

i, How find Wronskian of the other?

ii, Don't get book's solution

$$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$$ is normal Wronskian of $\bar{x}_1$ and $\bar{x}_2$

Which I call

$$\begin{vmatrix} x_1 & x_2 \\ x_2 & x_3 \end{vmatrix}$$

$$x_1 x_3 - x_2^2$$

(24)

4F-2 Let $\underline{x}_1 = \begin{pmatrix} t \\ 1 \end{pmatrix}$ $\underline{x}_2 = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ be two vector functions

a) Prove $\underline{x}_1, \underline{x}_2$ are linearly ind.

L i is that using Wronskian - or want to do in b)

But can see that neither is a constant multiple of the other

b) Calc the Wronskian

$$w(\bar{x}_1, \bar{x}_2) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix}$$

$$= t(2t) - t^2 \cdot 1$$

$$= 2t^2 - t^2$$

$$= t^2$$

B Matches

c) How to reconcile w/ LS.5 Theorm 5C

Theorm 5.C a) $W = 0$ on I and x_1, x_2 lin dep

(not true

25

5.C b) $W(t)$ is never 0 on I, x_1 x_2 lin ind
This seems to be true here
(anything more than that?)

Solutions

? Don't get - these seem to reference other theorm

Oh $W=0$ at $t=0$

L So $\bar{x}_1, \bar{x}_2$ can't be solutions of $\bar{x}' = A(t)$
Where entries of $A(t)$ are cont'ous

d) Find a linear system $x' = Ax$ having x_1, x_2 as sols

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2t \\ 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} t^2 \\ 2t \end{bmatrix}$$

(26)

Solve 4 eq, 4 unknowns

$$1 = a t + b$$

$$0 = c t + d$$

$$2t = a t^2 + 2bt$$

$$2 = c t^2 + d\,2t$$

$$b = 1 - a t$$

$$2t = a t^2 + 2(1 - a t) t$$

$$2t = a t^2 + 2t - 2a t^2$$

$$2t = 2t$$

hah

Look at seperatly

$a = 0$

$b = 1$ ✓ works

$c = 0$

$d = 0$) boring

Use st method from before

$$c = \frac{-2}{t^2}$$

$$d = \frac{2}{t}$$

not contiuous at $t = 0$

(27)

Fundemental Matrix

46-1 Two ind. sols to $x' = Ax$ are

$$x_1 = \begin{pmatrix}1\\1\end{pmatrix} e^{3t} \qquad x_2 = \begin{pmatrix}1\\2\end{pmatrix} e^{2t}$$

a) Find a sol satisfying

$$x(0) = \begin{pmatrix}0\\1\end{pmatrix} \quad \text{and} \quad x(0) = \begin{pmatrix}1\\0\end{pmatrix}$$

$$F = \begin{pmatrix} 1e^{3t} & 1e^{2t} \\ 1e^{3t} & 2e^{2t} \end{pmatrix}$$

All sols $\underline{x} = F \cdot \begin{pmatrix}c_1\\c_2\end{pmatrix}$

$$F' = AF$$

Can use to solve IVPs like here

if $x(t_0) = x_0$

then $x_0 = F(t_0)\begin{pmatrix}c_1\\c_2\end{pmatrix}$

ie $\begin{pmatrix}c_1\\c_2\end{pmatrix} = F(t_0)^{-1}$

(28)

So $F(t_0) = F(0) = \left(\begin{array}{c|c} 1 & 1 \\ 1 & 2 \end{array}\right)$

Take inverse

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{2-1}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

So sol to IVP

$$X(t) = F(t)\, F(0)^{-1}\, X_0$$

$$= \begin{pmatrix} e^{3t} & e^{2t} \\ e^{3t} & 2e^{2t} \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftarrow X_0$$

$$= \begin{pmatrix} e^{3t} \cdot 2 + e^{2t} \cdot -1 \\ e^{3t} \cdot -1 + 2e^{2t} \cdot 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= 2e^{2t} - e^{3t}$$

i did matrix multiplication wrong.
i book does something very diff.

(29)

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

is

$$= 2e^{3t} - e^{2t}$$

Unless did matrix multiply wrong
↳ p-set is too long?

b) Using a, find a way to solve $\begin{pmatrix} a \\ b \end{pmatrix}$

$$= \begin{pmatrix} 2ae^{3t} - ae^{2t} \\ -be^{3t} + 2be^{2t} \end{pmatrix}$$

30

2] For the system $X' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} X$

a) Find a fundamental matrix

So basically solve like normal? Yeah

$$\begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix}$$

$$(5-\lambda)(1-\lambda)+3=0$$

$$\lambda^2-6\lambda+5+3=0$$

$$\lambda^2-6\lambda+8$$

$$\lambda=2,4$$

$\underline{\lambda=2}$ $$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=0$$

$$3c_1-1c_2=0$$

$$c_1=1 \quad c_2=3 \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$\underline{\lambda=4}$ $$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=0$$ } multiple

$$c_1-c_2=0$$

$$c_1=c_2 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(31)

$$F = \begin{pmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{pmatrix}$$

Now use to solve IVP $x(0) = 2$
$y(0) = -1$

So $F(0) = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$

$$F(0)^{-1} = \frac{1}{1-3}\begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{pmatrix} \cdot -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{pmatrix}\begin{pmatrix} -3/2 \\ 7/2 \end{pmatrix}$$ ← oh can multiple end first!

$$= -\frac{3}{2}\begin{pmatrix} e^{2t} \\ 3e^{2t} \end{pmatrix} + \frac{7}{2}\begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}$$

(32)

b) Now normalize at $t=0$

L ? what does this mean?

Solutions just show a diff step of multiplying

L this looks silly

$$\begin{bmatrix} -\frac{1}{2}e^{2t}+\frac{3}{2}e^{4t} & \frac{1}{2}e^{2t}-\frac{1}{2}e^{4t} \\ -\frac{3}{2}e^{2t}+\frac{3}{2}e^{4t} & \frac{3}{2}e^{2t}-\frac{1}{2}e^{4t} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

< Same ..

(33)

Lecture 33 Exponential matricies + inhomogeneous eqn's

<u>4H-1</u> a) Calculate e^{At} if $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

b) Verify that $X = e^{At} X_0$ is a sol to $X' = AX$, $X(0) = X_0$

So saw

$$e^{At} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} \text{ in class}$$

for $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

So $e^{At} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}$ ✓

Much longer proof in sols

b) Verify.

$$\begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{at} \\ c_2 e^{bt} \end{pmatrix}$$

$x = c_1 e^{at}$, $y = c_2 e^{bt}$ is a sol of $\begin{cases} x' = ax \\ y' = by \end{cases}$ ← simplest problem ✓

for $x(0) = c_1$ $y(0) = c_2$

(34)

3) Calculate e^{At} directly from ∞ series !

if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then ans same as 4H-1

¿¡ So what does it actually want to do ¿

So e^{At} can only be computed if very easy

but can be computed from $\underline{c}$

$$e^{At}\,\underline{c} =$$

$$e^{(A-\lambda I)t}\,e^{\lambda I t}\,\underline{c}$$

Found $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in class as example

¿¡ Don't get what this qu is asking

No solutions ...

(35)

Inhomogeneous Systems

← ¿when did we cover? ¿is it in the notes?

4I-1

Solve $x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x + \begin{pmatrix} 2 \\ -8 \end{pmatrix} - \begin{pmatrix} 5 \\ 8 \end{pmatrix} t$

by variation of parameters.

Step 1 Solve the reduced eqn $\bar{x} = A\bar{x}$

Char eqn $\lambda^2 + \lambda - 6 = 0$

$(\lambda + 3)(\lambda - 2) = 0$

$\lambda = 2, -3$

$\lambda = 2$

$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 4 & 0 \end{pmatrix}$$

$-c_1 + c_2 = 0$

$c_2 = c_1$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(36)

$\underline{\lambda = -3}$

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}$$

$$4c_1 + c_2 = 0$$

$$\begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

So Fund matrix

$$F = \begin{bmatrix} e^{-3t} & e^{2t} \\ 4e^{-3t} & e^{2t} \end{bmatrix}$$

$$F^{-1} = \frac{1}{5e^{t}} \begin{bmatrix} e^{2t} & e^{-2t} \\ 4e^{-3t} & e^{-3t} \end{bmatrix}$$

$$V' = F^{-1} \begin{bmatrix} -5t + 2 \\ -3t - 8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{3t}}{5}(-5t+2) - \frac{e^{3t}}{5}(-8t+8) \\ \frac{9}{5}e^{-4t}(-5t+2) + e^{-5t}(-8t-8) \end{bmatrix}$$

(37)

$$= \begin{bmatrix} \frac{3e^{3t}t}{5} + 2e^{3t} \\ -\frac{3}{5}e^{-3t}t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{te^{3t}}{5} + \frac{3}{5}e^{3t} \\ \frac{14}{5}e^{-2t}t + \frac{1}{5}e^{-3t} \end{bmatrix}$$

We never learned this stuff!

(38)

2]

Solve $x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$

$$v' = \frac{1}{5}\begin{bmatrix} e^{3t} & -e^{3t} \\ 4e^{-2t} & e^{-2t} \end{bmatrix}\begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} e^{t} + 2e^{7t} \\ 4e^{-4t} - 2e^{-t} \end{bmatrix}$$

$$v = \frac{1}{5}\begin{bmatrix} e^{t} + \frac{e^{4t}}{2} \\ e^{-4t} + 2e^{-t} \end{bmatrix}$$

$$\bar{X} = F v$$

Thus

$$\bar{X}_r = \frac{1}{5}\begin{bmatrix} 5/2\, e^{t} \\ -5e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{t}/2 \\ -e^{-2t} \end{bmatrix}$$

(38)

Add to $\bar{x}_p$ the $\bar{x}_h$

$$= c_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

$\bar{x}_p = ce^{-2t} + de^{t}$ Substitute in to the eqn

$$-2\bar{c}\, e^{-2t} + \bar{d}e^{t} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \bar{c}\, e^{-2t} +$$

$$+ \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} de^{t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} e^{t}$$

$$\therefore -2\vec{c} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{c} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$d = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix} \vec{d} + \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Then LHS

$$-2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{c}$$

Subtract $-2\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{c}$ from both sides

$$\begin{bmatrix} -3 & -1 \\ -4 & -0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$C_1 = 0$

$C_2 = -1$

40

Then the other system

$$\begin{bmatrix} 0 & -1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$-d_2 = 0$$
$$-4d_1 + 3d_2 = -2$$

$$d_1 = \frac{1}{2} \qquad d_2 = 0$$

Thus

$$\vec{X}_p = \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{t}$$

$$= \begin{bmatrix} e^{t/2} \\ -2^{-2t} \end{bmatrix}$$

41

Part 2

8. No one yet
   Met w/ Jethro OH for part 1 - very helpful

1. Given the matrix $A = \begin{pmatrix} 0 & 1 \\ -9 & -6 \end{pmatrix}$
   find two ind. sols to linear system $\underline{x}' = A\underline{x}$

So $\begin{pmatrix} 0-\lambda & 1 \\ -9 & -6-\lambda \end{pmatrix}$

$$(0-\lambda)(-6-\lambda) - -9 = 0$$

$$\lambda^2 + 6\lambda + 0 + 9 = 0$$

$$(\lambda + 3)^2$$

$\underline{\lambda = -3}$

$$\begin{pmatrix} 3 & 1 \\ -9 & -3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \quad \text{multiple}$$

$$3c_1 + 1c_2 = 0$$

$$c_1 = -1$$
$$c_2 = 3$$

42

So need another sol - not a multiple

; Not possible

Oh duh $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$$x(t) = A \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-3t} + B \cancel{t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} t e^{-3t}$$

?

-1 .

(43)

b) Find the 2nd order ODE whose companion matrix $= A$. Find two linearly ind. sols $x(t)$ $y(t)$ to this 2nd order ODE.

?: Companion matrix

$$\boxed{\begin{array}{l} \text{for } p(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1} + t^n \\ C(p) = \begin{vmatrix} 1 & 0 & \cdots & -c_0 \\ 0 & 1 & & -c_1 \\ \vdots & 0 & 1 \ \ddots & -c_2 \\ \vdots & \vdots & & \vdots \\ \vdots & & & \end{vmatrix} \end{array}}$$

?: is this the replacement (decoupling) thing?

(44)

Then show that the vectors

$$\begin{pmatrix} x \\ x' \end{pmatrix}, \begin{pmatrix} y \\ y' \end{pmatrix}$$

are expressible as a linear combo of solutions found in a)

$x' = \ldots x$

$y' = \ldots y$

Problem 2?

?

−5

45

2. Recall that in the diff eq $x' = Ax$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The charastic eq'n of A is

$$\det(A - \lambda I) = \lambda^2 - (a+d)\lambda + (ad - bc)$$
$$= \lambda^2 - \text{trace}(A)\lambda + \det(A)\lambda$$

Give conditions in terms of $\text{tr}(A)$ $\det(A)$ such that every sol $x(t)$ of corresponding diff eq satisfies

$$\lim_{t \to \infty} x(t) = 0$$

Such are called <u>stable</u>

Well we have eq'n

$$x' = a\,x + b\,y$$
$$y' = c\,x + d\,y$$

if those are 0s - then no change

46

But we want no change <u>over time</u>

? something like $a = c$
$b = d$

Then det is

$$ab - ab = 0$$

Trace

$$a + b = \text{anything?}$$

47

3. Find e^{At} for each of the following matricies

a) $\begin{pmatrix} 0 & 1 \\ -9 & -6 \end{pmatrix}$

So no easy way

Need to do w/ $\underline{C}$

But I don't really get that method

-2

2-1

b) $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for some consants a, b

-3

A —
4 from B   from participation

85 → 4
86 → 4.5
66 → 3

Pset 10 Part II Solutions.

(1)

a) $A = \begin{pmatrix} 0 & 1 \\ -9 & -6 \end{pmatrix}$, $x' = Ax$.

Guess
$X(t) = ve^{\lambda t}$.
$\Rightarrow Av = \lambda v$.

$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -9 & -6-\lambda \end{pmatrix}$

$= \lambda^2 + 6\lambda + 9 = (\lambda+3)^2$. $\quad \det(\ldots) = 0 \Rightarrow \lambda = -3, -3$.

$\underline{\lambda = -3}$:

$\rightsquigarrow \begin{pmatrix} 3 & 1 \\ -9 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \rightsquigarrow v = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$$\boxed{\vec{x}_1(t) = \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t}}$$

Now guess $x(t) = ve^{\lambda t} + wte^{\lambda t}$.

$(x' = Ax) \Rightarrow \quad Av = \lambda v + w$
$\quad Aw = \lambda w$.

$\therefore$ Take $w = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

Get v : solve $\begin{pmatrix} 3 & 1 \\ -9 & -3 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

$3c + d = 1$, $\quad c = 0, d = 1$ will do. $\therefore v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\boxed{\vec{x}_2(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ -3 \end{pmatrix} te^{-3t}}$$

General soln : $\alpha x_1(t) + \beta x_2(t) \quad (\alpha, \beta \in \mathbb{R})$.

b) Recall $x'' + c_1 x' + c_2 x = 0$ becomes

$$\begin{pmatrix} x \\ x' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -c_2 & -c_1 \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}$$

So in this case $x'' + 6x' + 9x = 0$.

Char eq^n: $r^2 + 6r + 9 = (r+3)^2 = 0$

So $r = -3, -3$.

General solution: $x(t) = Ae^{-3t} + Bte^{-3t}$

~~e^{-3t} & te^{-3t} lin. indep. solns~~.

So $x = e^{-3t}$ & $y = te^{-3t}$ lin. indep. solns.

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} e^{-3t} \\ -3e^{-3t} \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t} = \vec{x}_1.$$

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} te^{-3t} \\ e^{-3t} - 3te^{-3t} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ -3 \end{pmatrix} te^{-3t} = \vec{x}_2.$$

~~(2) Stability basically depends on the signs of the eigenvalues, which are the solutions to~~

~~$\lambda^2 - Tr(A)\lambda + det(A) = 0$.~~

~~If λ_1, λ_2 are the eigenvalues (possibly repeated) then $Tr(A) = \lambda_1 + \lambda_2$ & $det(A) = \lambda_1 \lambda_2$.~~

~~If det < 0 then λ_1, λ_2 complex conjugate~~

②

Let λ_1, λ_2 be the eigenvalues of A.

So $$\lambda_1 + \lambda_2 = \mathrm{Tr}\,A$$

& $$\lambda_i \lambda_2 = \det A.$$

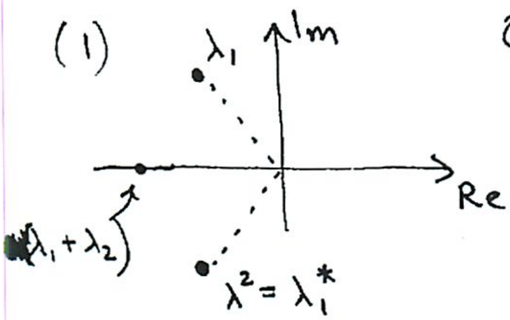
If $\mathrm{Tr}\,A \geq 0$ then λ_1 or λ_2 (or both) has real part ≥ 0 & so the corresponding $e^{\lambda t}$ is <u>not</u> stable.

So we <u>need</u> $\mathrm{Tr}\,A < 0$.

Assume $\mathrm{Tr}\,A < 0$ then.

There are several possible arrangements of λ_1, λ_2:

(1)

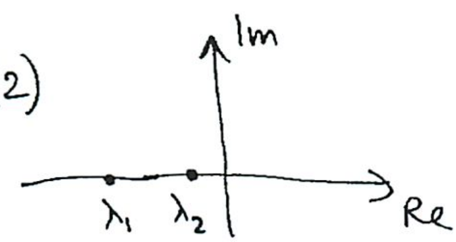


$0 \neq \lambda_1 = \lambda_2^*$, $\mathrm{Tr}\,A = \lambda_1 + \lambda_2 < 0$

$\Rightarrow \mathrm{Re}(\lambda_{1,2}) < 0$

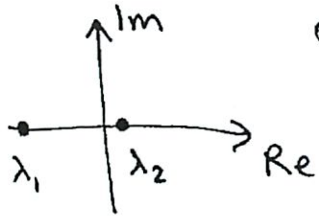
$\Rightarrow$ STABILITY.

(2)



λ_1, λ_2 both < 0. Obviously also STABLE.

(3)



One of λ_1, λ_2 is ≥ 0, but other has larger abs. value & is negative, So $\mathrm{Tr}\,A = \lambda_1 + \lambda_2 < 0$.

This case is <u>unstable</u>

The determinant $\det A$ is:

Case 1: $\det A = |\lambda_1|^2 > 0$.

Case 2: $\det A = \lambda_1 \lambda_2 > 0$.

Case 3: $\det A = \lambda_1 \lambda_2 \leq 0$.

So STABILITY $\iff$ (Case 1) or (Case 2) $\iff$ $\det A > 0$.

Finally: The system is stable exactly when

$\underline{\mathrm{Tr}\, A < 0}$ & $\underline{\det A > 0}$.

3 (a)

Recall the theorem in notes (lec. 32) that if $F(t)$ is a fundamental matrix for $x' = Ax$, then $e^{At} = F(t) \cdot F(0)^{-1}$.

Well, $\begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ -3 \end{pmatrix} t e^{-3t}$ are fundamenta solutions, so take

$$F(t) = \begin{pmatrix} 1 & t \\ -3 & 1-3t \end{pmatrix} e^{-3t}.$$

$$F(0) = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}.$$

$$F(0)^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}.$$

So $$e^{At} = e^{-3t} \begin{pmatrix} 1 & t \\ -3 & 1-3t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+3t & t \\ -9t & 1-3t \end{pmatrix} e^{-3t}.$$

3b $\quad A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

Eigenvalue eqn: $\quad (a-\lambda)^2 + b^2 = 0.$

$$\lambda^2 - 2a\lambda + (a^2+b^2) = 0.$$

$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2}$$

$$= a \pm ib.$$

Eigenvectors:

$\lambda = a+ib$: $\quad \begin{pmatrix} -ib & -b \\ b & -ib \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$

$\lambda = a-ib$: $\quad \begin{pmatrix} ib & -b \\ b & ib \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}.$

So let $\quad F(t) = \begin{pmatrix} e^{(a+ib)t} & e^{(a-ib)t} \\ -ie^{(a+ib)t} & ie^{(a-ib)t} \end{pmatrix}.$

$$F(0) = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}. \qquad F(0)^{-1} = \frac{1}{2i}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}$$

$$e^{At} = F(t)F(0)^{-1} = \frac{1}{2i}\begin{pmatrix} ie^{(a-ib)t} + ie^{(a+ib)t} & e^{(a-ib)t} - e^{(a+ib)t} \\ e^{(a+ib)t} - e^{(a-ib)t} & ie^{(a+ib)t} + ie^{(a-ib)t} \end{pmatrix}$$

$$= \frac{e^{at}}{2i}\begin{pmatrix} i(e^{-ibt} + e^{ibt}) & (e^{-ibt} - e^{ibt}) \\ (e^{ibt} - e^{-ibt}) & i(e^{ibt} + e^{-ibt}) \end{pmatrix} = e^{at}\begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}.$$

A tricky way:

$A = a\begin{pmatrix}1&0\\0&1\end{pmatrix} + b\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, these matrices commute & it is known that $e^{X+Y} = e^X e^Y$ if X, Y commute.

So $e^{At} = e^{a\begin{pmatrix}1&0\\0&1\end{pmatrix}t}\, e^{b\begin{pmatrix}0&-1\\1&0\end{pmatrix}t}$.

$= e^{at}\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

What does this = ?

$$e^{bt\begin{pmatrix}0&-1\\1&0\end{pmatrix}} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + bt\begin{pmatrix}0&-1\\1&0\end{pmatrix} + \frac{(bt)^2}{2}\begin{pmatrix}-1&0\\0&-1\end{pmatrix} + \frac{(bt)^3}{6}\begin{pmatrix}0&1\\-1&0\end{pmatrix} + \frac{(bt)^4}{24}\begin{pmatrix}1&0\\0&1\end{pmatrix} + \cdots$$

the matrix cycles through $\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&-1\\1&0\end{pmatrix}, \begin{pmatrix}-1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}$ & repeats.

$$\text{So } e^{bt\begin{pmatrix}0&-1\\1&0\end{pmatrix}} = \begin{pmatrix} 1-\frac{(bt)^2}{2}+\frac{(bt)^4}{24}-\frac{(bt)^6}{720}+-\cdots & -(bt)+\frac{(bt)^3}{6}-\frac{(bt)^5}{120}+-\cdots \\ (bt)-\frac{(bt)^3}{6}+\frac{(bt)^5}{120}-+\cdots & 1-\frac{(bt)^2}{2}+\frac{(bt)^4}{24}-\frac{(bt)^6}{720}+-\cdots \end{pmatrix}$$

$$= \begin{pmatrix}\cos bt & -\sin bt\\ \sin bt & \cos bt\end{pmatrix}.$$

OK... maybe that was harder than the first way.

18.03 Lecture 34

Drawing sol curves

(Prof Brubaker knew my name as I was coming in)

(last day of new material)

Drawing pictures of systems

Mostly 2x2 of the form

$$\frac{dx_1}{dt} = F(x_1, x_2)$$

$$\frac{dx_2}{dt} = G(x_1, x_2)$$

← autonomous (since no t appears)
(but both fns of t)

includes $\underline{x}' = A\,\underline{x}$
└ const coeff matrix
(kinda boring, but included)

critical pts
└ one of the most important things

→ a point (a,b) where $(x_1', x_2') = (0,0)$

ie (a,b) s.t. $F(a,b) = 0$
$G(a,b) = 0$

② If a solution $(x_1(t), x_2(t))$ ever arive at this point (a,b) it will never leave!

L since no change in position

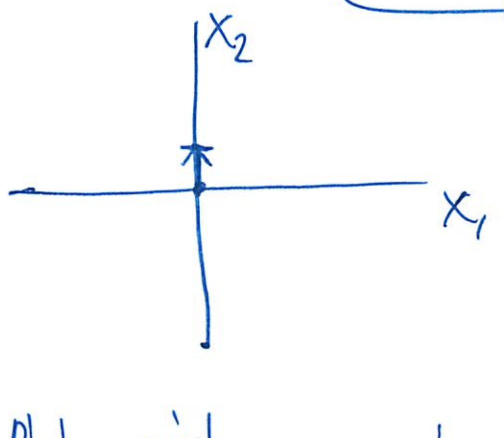
Q: (could be a sink or some other things)

Ex $x_1' = x_1 - \frac{1}{2} x_2$

$x_2' = 1 - x_1^2$ ← not linear

so no method to solve systematically

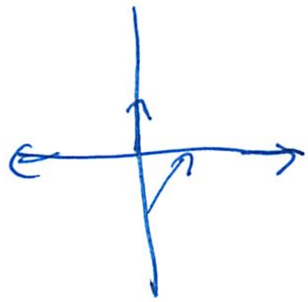
But can plot phase plane



1. Pick point ie $(0,0)$
2. Plug value in ie $(x_1', x_2') = (0,1)$
3. Draw arrow (unit vector) $(0,1)$ $(0,0)$

(3)

Do some more



Now find critical points

$x_2' = 0$

$1 - x_1^2 = 0$

$\boxed{x_1 = \pm 1}$ true at any critical pts

$x_1^2 = 0$

$\boxed{2x_1 = x_2}$

Combine facts 2

$(1,2), (-1,-2)$

So should be interesting objects around here

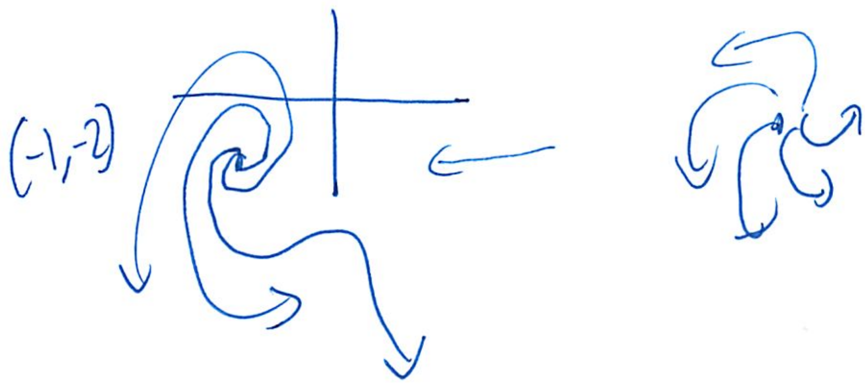
These are isolated critical points

we will focus on today

(4)

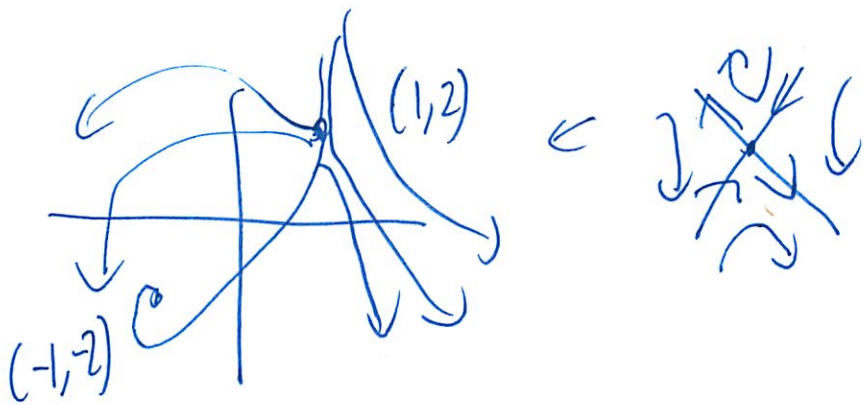
PPlane applet from Rice

Can ~~pl~~ click on points to complete
sol around



"unstable spiral"

arrow of time going out = source



"saddle"

various different names

Goal for lecture: Can take nonlinear sol
Approximate it to ~~sol~~ linear
Solve linear
Can know approx what happens at critical pts

(5)

Plan: Study behavior of critical pt in linear, Constant coefficents case
TTI

Use it to study linear examples

$$x' = Ax$$

then if $\det(A) \neq 0$ then only critical pt is the origin $(0,0)$

i.e. solving
$$ax_1 + bx_2 = 0$$
$$cx_1 + dx_2 = 0$$

So all linear examples give you same behavior at the origins

Sol's to $\underline{x}' = A\underline{x}$ just depend on eigenvalues + eigenvectors

---

example

$$\lambda_1 = 1 \to \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1 \to \underline{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

(e)

General soln $c_1\begin{bmatrix}1\\1\end{bmatrix}e^{t} + c_2\begin{bmatrix}3\\-1\end{bmatrix}e^{-t}$

$= c_1\begin{bmatrix}e^{t}\\e^{t}\end{bmatrix} + c_2\begin{bmatrix}3e^{-t}\\-e^{-t}\end{bmatrix}$

Now choose eiter $c_1, c_2 = 0$

and other one non zero

Then sol's

if $c_1 = 0$ $\quad x(t) = c_2\begin{bmatrix}3e^{-t}\\-e^{-t}\end{bmatrix}$

so point is ratio of $\frac{x_2}{x_1} = -\frac{1}{3}$

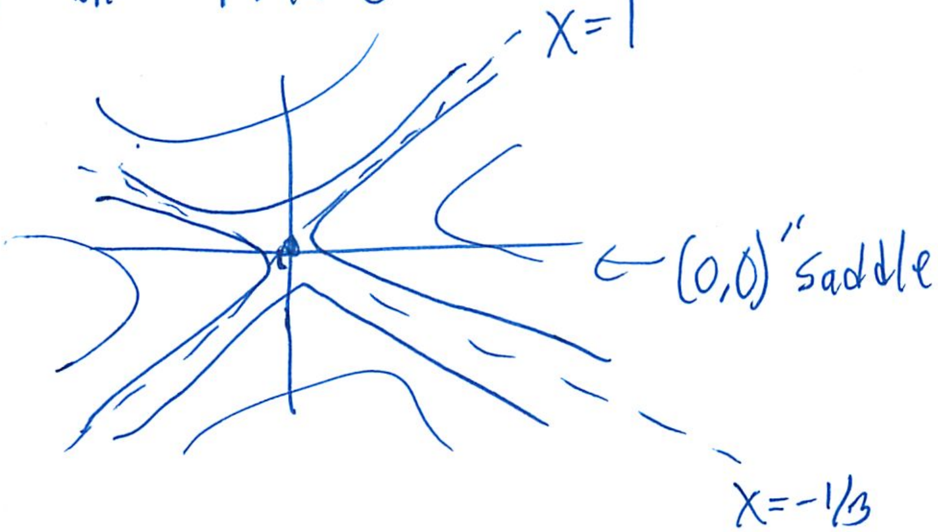
$c_2 = 0$

ratio $\frac{x_2}{x_1} = 1$

So this tells you for every pair (x_1, x_2) it follows the line slope $-\frac{1}{3}$

7

So plot on PPlane



Which dominates as $t \to \infty$

So e^t gets big (e^{-t} gets smaller)

So sol vector approaching (1, 1) as $t \to \infty$

So resulting behavior has crit pt at origin

Had to do this in part 2 of HW 10

Just compute eigenvalues

└ saddle b/c 1 sol shrinks $t \to \infty$ ⊖

1 sol grows $t \to \infty$ ⊕

So since $\lambda_1 > 0 > \lambda_2$

or $\lambda_2 > 0 > \lambda_1$

8

Can build a dictionary of point behaviors

Can prob guess we get a spiral when sols look like

$$\left(e^{at}\cos bt,\ e^{at}\sin bt\right)$$

winding around circle

but growing so outward spiral if $a > 0$

(if $a < 0$, spiral in)

Sol's occur when eigenvalues of A are complex of form $a + bi$

Lots of possible combos

- real
- complex
- repeated
- etc

in 6.5 section of notes

But don't need to memorize

Just can you do the simple graphs

9

Now do for nonlinear systems

Use multi-variable taylor approx

at $(x_1, x_2) = (a, b)$

↳ critical pts

$$F(x_1, x_2) \approx F(a,b) + \underset{\frac{dF}{dx}\big|_{(a,b)}}{F_x}(a,b)(x_1 - a) + F_{x_2}(a,b)(x_2 - b)$$

↳ take all possible derivitives

add more higher order terms

↳ but we will just do linear approx.

Good when (x_1, x_2) near (a,b)

provided F is contineous partial differentiable fn

· So do this w/ our example

$$x_1' = x_1 - \frac{1}{2} x_2$$

$$x_2' = 1 - x_1^2$$

(10)

Need to linearize

top - is at $(0,0)$ critical pt

We have C. P. $(1,2), (-1,-2)$

do linear approx at

$$F(x_1, x_2) = x_1 - \tfrac{1}{2} x_2 \quad \text{near } (1,2)$$

(what is the value at here)

$$\approx \underbrace{0}_{\text{always since C.P.}} + 1(x_1-1) + \left(-\tfrac{1}{2}\right)(x_2-2)$$

do they almost = ?

Can check

$$x_1 - 1 - \tfrac{1}{2} x_2 + 1 \quad ✓$$

$$G(x_1, x_2) = 1 - x_1^2$$

$$\approx 0 + (-2x_1)\Big|_{(1,2)} (x_1 - 1) + 0(x_2 - 2)$$

11

After linearizing at $(1,2)$:

$$x_1' = 1(x_1 - 1) + \left(-\tfrac{1}{2}\right)(x_2 - 2)$$

$$x_2' = -2(x_1 - 1) + 0(x_2 - 2)$$

Could change variables so move $(0,0)$ to $(1,2)$

↳ but could also not

- if did - linear sol homogeneous, w/ those coeffs

$$A = \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & 0 \end{bmatrix}$$

In general, formula for this matrix

$$A = \begin{bmatrix} F_{x_1}(a,b) & F_{x_2}(a,b) \\ G_{x_1}(a,b) & G_{x_2}(a,b) \end{bmatrix}$$ Jacobian Matrix

Can use behavior of cp to solve non linear cp idea

~~Jacobian matrix~~

Eigenvalues $\begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & 0 \end{pmatrix}$ char polynomial: $\det(A - \lambda I)$

$$\lambda^2 - \lambda - 1$$

12

Roots $\lambda = \frac{1 \pm \sqrt{5}}{2}$

real #s

one >0

one <0

So get 2 eigenvalues that are real

One pos
one neg) → "saddle"

Now can do this for examples

S = squirl

H = hawk

$$\frac{dS}{dt} = \underset{\text{multiply}}{aS} - p\underset{\text{when hawks}}{S \cdot H}$$

$$\frac{dH}{dt} = -\underset{\text{die}}{bH} + qS \cdot H$$

Can solve for crit pts $(0,0)$ - duh - no items doesn't change "saddle"

$\left(\frac{b}{q}, \frac{a}{p}\right)$ "stable circular"

(13)

Can solve any non linear eq by linearizing

Will see 1 non linear eq to linearize on exam

Next Mon, Wed: Review Mon - exams 1-2

Wed - rest

Lecture 34 Drawing Sol Curves

Draw pictures of solution curves $(x_1(t), x_2(t))$ in plane

to $\underline{x}' = A\underline{x}$. or more generally in (possibly non-linear)

system $\dfrac{dx_1}{dt} = F(x_1, x_2)$

$\dfrac{dx_2}{dt} = G(x_1, x_2)$

A critical point (a,b) is one for which $F(a,b) = G(a,b) = 0$.

This says that at $(x_1, x_2) = (a,b)$ both $x_1', x_2' = 0$

(i.e. ~~point~~ solution doesn't move away from point)

Example: $x_1' = x_1 - \frac{1}{2}x_2$

$x_2' = 1 - x_1^2$

critical points: $1 - x_1^2 = 0 \rightarrow x_1 = \pm 1$

(which could be hard to solve for in general, since non-linear equations)

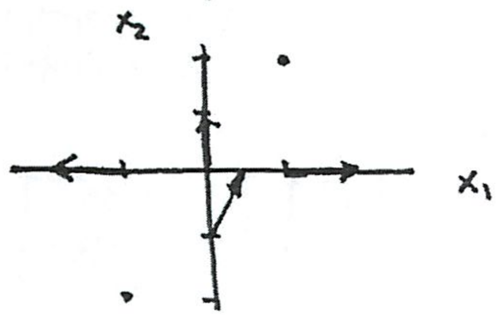
$x_1 - \frac{1}{2}x_2 = 0$

$2x_1 = x_2 \Rightarrow (1,2)$ and $(-1,-2)$ are critical points.

in pictures, plot $1 - x_1^2 = 0$ and $2x_1 = x_2$, observe intersections.

(We'll deal mostly with isolated critical pts.)

Now draw phase plane.

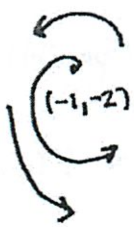


... using computer, we got much nicer picture.

From picture, (1,2) critical point looks like:

"saddle point"

for $(-1,-2)$, trajectories spiraling away from critical point " unstable spiral"



<u>stability</u>: Solutions close enough to critical point remain close for all time.

roughly – in stable solutions, arrows of vectors in phase plane point toward a critical point: e.g.

or

and in unstable solutions, they point <u>away</u> from critical point.

—

In order to understand possible behaviors at (or near) critical points

classify them in linear case:

$$x_1' = ax_1 + bx_2$$
$$x_2' = cx_1 + dx_2$$

then shape of solutions depends on eigenvalues of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and their eigenvectors.

if $\det(A) \neq 0$ only critical pt. is $(0,0)$

Note: stability very important in practical applications. No real world systems stay exactly fixed at a point. What happens with slight fluctuations?

<u>Examples</u>: $\lambda_1 = 1, \ \lambda_2 = -1$

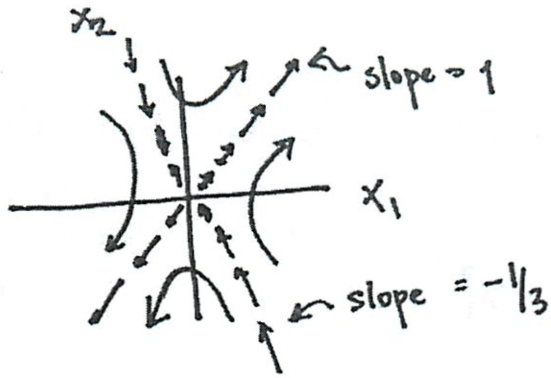
$\underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \underline{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ e.g. $A = \begin{pmatrix} -1/2 & 3/2 \\ 1/2 & 1/2 \end{pmatrix}$

then <u>sol'n</u>: $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-t} = c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} 3e^{-t} \\ -1e^{-t} \end{bmatrix}$

if $c_1 = 0$ then ratio of $\frac{x_1}{x_2} = -3$ $(x_1, x_2 \to 0$ as $t \to \infty)$

if $c_2 = 0$ then ratio of $\frac{x_1}{x_2} = 1$ $(x_1, x_2 \to \infty$ as $t \to \infty)$

picture



since want $\frac{x_2}{x_1} = -\frac{1}{3}$, e.g.,

to compute slopes.

In general, if we assume that $F(x_1, x_2)$ and $G(x_1, x_2)$ are "nice" (continuous partial derivatives) then trajectories do not cross according to uniqueness thm. for solutions to IVPs of systems.

See Mattuck's "Graphing ODE Systems" for lots of pictures of behavior near isolated critical point (always $(0,0)$) of homog. linear, const. coeff. systems.

Can Guess: spiraling: $(e^{at} \sin bt, \; e^{at} \cos bt)$ if $a > 0$, spiral out
$a < 0$, spiral in

which occurs when e-values are complex $a + bi$.

etc. ...

Given non-linear system, can linearize! using Taylor approximation.

$$F(x,y) \approx F(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \cdot (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \cdot (y - y_0)$$

$$+ \underbrace{\text{higher order terms involving various mixes of partial derivs.}}_{\text{ignore if } (x,y) \text{ close to } (x_0,y_0)}$$

As in one variable case, this is a good approximation near (x_0, y_0).

Back to original example :

$$x_1' = x_1 - \tfrac{1}{2} x_2$$

$$x_2' = 1 - x_1^2$$

already linear, but still need to do linear approx. near critical pt.

$1 - x_1^2$

Pick $(a, b) = (1, 2)$

$$F(x_1, x_2) = x_1 - \tfrac{1}{2} x_2 \approx F(1,2) + F_{x_1}(1,2) \cdot (x_1 - 1) + F_{x_2}(1,2)(x_2 - 2)$$

$$\approx 0 + (x_1 - 1) - \tfrac{1}{2}(x_2 - 2)$$

$$G(x_1, x_2) = 1 - x_1^2 \approx G(1,2) + G_{x_1}(1,2)(x_1 - 1) + G_{x_2}(1,2)(x_2 - 2)$$

$$= 0 + (-2)(x_1 - 1) + 0 (x_2 - 2)$$

so linearizing:

$$x_1' = (x_1 - 1) - \tfrac{1}{2}(x_2 - 2)$$

$$x_2' = -2(x_1 - 1) + 0 (x_2 - 2)$$

could change coords so that critical point is at $(0,0)$ not $(1,2)$

But either way, see this linear system with $A = \begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & 0 \end{pmatrix}$

$$\det(A - \lambda I) = \lambda^2 - \lambda - 1 =$$

so $\lambda = \dfrac{1 \pm \sqrt{5}}{2}$ are e-values.

one positive evalue, one neg. evalue.

Makes saddle!

(just as we saw in earlier picture)

Notice A given by

$$A = \begin{pmatrix} F_{x_1}(a,b) & F_{x_2}(a,b) \\ G_{x_1}(a,b) & G_{x_2}(a,b) \end{pmatrix}$$

"Jacobian matrix"

Classic example of non-linear system: Predator-Prey Model.

without predators, the prey population grows.

S: squirrels H: hawks. so $\frac{dS}{dt} = a \cdot S$ w/o hawks.

without prey (food), the predator population declines

$$\frac{dH}{dt} = -b \cdot H \quad \text{w/o squirrels.}$$

But when populations mix:

$$\frac{dS}{dt} = aS - p \cdot S \cdot H$$

$$\frac{dH}{dt} = -bH + q \cdot S \cdot H$$

non-linear system but can linearize, near critical points:

$(0,0)$, $(b/q, a/p)$

$$J(0,0) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix} \sim \text{saddle}$$

$$J(b/q, a/p) = \begin{pmatrix} 0 & -pb/q \\ aq/p & 0 \end{pmatrix} \sim \text{"stable center"}$$

corresponding to pure imag. e-values.

LS. LINEAR SYSTEMS

LS.1 Review of Linear Algebra

In these notes, we will investigate a way of handling a linear system of ODE's directly, instead of using elimination to reduce it to a single higher-order equation. This gives important new insights into such systems, and it is usually a more convenient and faster way of solving them.

The method makes use of some elementary ideas about linear algebra and matrices, which we will assume you know from your work in multivariable calculus. Your textbook contains a section (5.3) reviewing most of these facts, with numerical examples. Another source is the 18.02 Supplementary Notes, which contains a beginning section on linear algebra covering approximately the right material.

For your convenience, what you need to know is summarized briefly in this section. Consult the above references for more details and for numerical examples.

1. Vectors. A **vector** (or *n-vector*) is an n-tuple of numbers; they are usually real numbers, but we will sometimes allow them to be complex numbers, and all the rules and operations below apply just as well to n-tuples of complex numbers. (In the context of vectors, a single real or complex number, i.e., a constant, is called a **scalar**.)

The n-tuple can be written horizontally as a **row vector** or vertically as a **column vector**. In these notes it will almost always be a column. To save space, we will sometimes write the column vector as shown below; the small T stands for **transpose**, and means: change the row to a column.

$$\mathbf{a} = (a_1, \ldots, a_n) \quad \textit{row vector} \qquad \mathbf{a} = (a_1, \ldots, a_n)^T \quad \textit{column vector}$$

These notes use boldface for vectors; in handwriting, place an arrow $\vec{a}$ over the letter.

Vector operations. The three standard operations on n-vectors are:

addition: $(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$.

multiplication by a scalar: $c\,(a_1, \ldots, a_n) = (ca_1, \ldots, ca_n)$

scalar product: $(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = a_1b_1 + \ldots + a_nb_n$.

2. Matrices and Determinants. An $m \times n$ **matrix** A is a rectangular array of numbers (real or complex) having m rows and n columns. The element in the i-th row and j-th column is called the *ij-th entry* and written a_{ij}. The matrix itself is sometimes written (a_{ij}), i.e., by giving its generic entry, inside the matrix parentheses.

A $1 \times n$ matrix is a row vector; an $n \times 1$ matrix is a column vector.

Matrix operations. These are

addition: if A and B are both $m \times n$ matrices, they are added by adding the corresponding entries; i.e., if $A = (a_{ij})$ and $B = (b_{ij})$, then $A + B = (a_{ij} + b_{ij})$.

multiplication by a scalar: to get cA, multiply every entry of A by the scalar c; i.e., if $A = (a_{ij})$, then $cA = (ca_{ij})$.

matrix multiplication: if A is an $m \times n$ matrix and B is an $n \times k$ matrix, their product AB is an $m \times k$ matrix, defined by using the scalar product operation:

$$ij\text{-th entry of } AB = (i\text{-th row of } A) \cdot (j\text{-th column of } B)^T \ .$$

The definition makes sense since both vectors on the right are n-vectors. In what follows, the most important cases of matrix multiplication will be

(i) *A and B are square matrices of the same size*, i.e., both A and B are $n \times n$ matrices. In this case, multiplication is always possible, and the product AB is again an $n \times n$ matrix.

(ii) *A is an $n \times n$ matrix and $B = \mathbf{b}$, a column n-vector.* In this case, the matrix product $A\mathbf{b}$ is again a column n-vector.

Laws satisfied by the matrix operations.

For any matrices for which the products and sums below are defined, we have

$$\begin{aligned} (AB)C &= A(BC) && \textit{(associative law)} \\ A(B+C) &= AB+AC, \quad (A+B)C = AB+AC && \textit{(distributive laws)} \\ AB &\neq BA && \textit{(commutative law fails in general)} \end{aligned}$$

Identity matrix. We denote by I_n the $n \times n$ matrix with 1's on the main diagonal (upper left to bottom right), and 0's elsewhere. If A is an arbitrary $n \times n$ matrix, it is easy to check from the definition of matrix multiplication that

$$A I_n = A \quad \text{and} \quad I_n A = A \ .$$

I_n is called the **identity matrix** of order n; the subscript n is often omitted.

Determinants. Associated with every *square* matrix A is a number, written $|A|$ or $\det A$, and called the **determinant** of A. For these notes, it will be enough if you can calculate the determinant of 2×2 and 3×3 matrices, by any method you like.

Theoretically, the determinant should not be confused with the matrix itself; the determinant is a *number*, the matrix is the *square array*. But everyone puts vertical lines on either side of the matrix to indicate its determinant, and then uses phrases like "the first row of the determinant", meaning the first row of the corresponding matrix.

An important formula which everyone uses and no one can prove is

$$(1) \qquad \det(AB) = \det A \cdot \det B \ .$$

Inverse matrix. A square matrix A is called **nonsingular** or **invertible** if $\det A \neq 0$.

If A is nonsingular, there is a unique square matrix B of the same size, called the **inverse** to A, having the property that

$$BA = I, \quad \text{and} \quad AB = I \ .$$

This matrix B is denoted by A^{-1}. To confirm that a given matrix B is the inverse to A, you only have to check one of the above equations; the other is then automatically true.

Different ways of calculating A^{-1} are given in the references. However, if A is a 2×2 matrix, as it usually will be in the notes, it is easiest simply to use the formula for it:

$$(2) \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Remember this as a procedure, rather than as a formula: switch the entries on the main diagonal, change the sign of the other two entries, and divide every entry by the determinant. (Often it is better for subsequent calculations to leave the determinant factor outside, rather than to divide all the terms in the matrix by det$A$.) As an example of (2),

$$\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

To calculate the inverse of a nonsingular 3×3 matrix, see for example the 18.02 notes.

3. Square systems of linear equations. Matrices and determinants were originally invented to handle in an efficient way the solution of a system of simultaneous linear equations. This is still one of their most important uses. We give a brief account of what you need to know. This is not in your textbook, but can be found in the 18.02 Notes. We will restrict ourselves to *square* systems — those having as many equations as they have variables (or "unknowns", as they are frequently called). Our notation will be:

$$\begin{aligned} A &= (a_{ij}), \quad \text{a square } n \times n \text{ matrix of constants,} \\ \mathbf{x} &= (x_1, \dots, x_n)^T, \quad \text{a column vector of unknowns,} \\ \mathbf{b} &= (b_1, \dots, b_n)^T, \quad \text{a column vector of constants;} \end{aligned}$$

then the square system

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ldots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

can be abbreviated by the matrix equation

$$(3) \qquad A\mathbf{x} = \mathbf{b}.$$

If $\mathbf{b} = \mathbf{0} = (0, \dots, 0)^T$, the system (3) is called **homogeneous**; if this is not assumed, it is called **inhomogeneous**. The distinction between the two kinds of system is significant. There are two important theorems about solving square systems: an easy one about inhomogeneous systems, and a more subtle one about homogeneous systems.

Theorem about square inhomogeneous systems.

If A is nonsingular, the system (3) has a unique solution, given by

$$(4) \qquad \mathbf{x} = A^{-1}\mathbf{b}.$$

Proof. Suppose $\mathbf{x}$ represents a solution to (3). We have

$$\begin{aligned} A\mathbf{x} = \mathbf{b} \Rightarrow A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{b}, \\ \Rightarrow (A^{-1}A)\mathbf{x} &= A^{-1}\mathbf{b}, && \text{by associativity;} \\ \Rightarrow \qquad I\mathbf{x} &= A^{-1}\mathbf{b}, && \text{definition of inverse;} \\ \Rightarrow \qquad \mathbf{x} &= A^{-1}\mathbf{b}, && \text{definition of } I. \end{aligned}$$

This gives a formula for the solution, and therefore shows it is unique if it exists. It does exist, since it is easy to check that $A^{-1}\mathbf{b}$ is a solution to (3). □

The situation with respect to a homogeneous square system $A\mathbf{x} = \mathbf{0}$ is different. This always has the solution $\mathbf{x} = \mathbf{0}$, which we call the *trivial* solution; the question is: when does it have a nontrivial solution?

Theorem about square homogeneous systems. *Let A be a square matrix.*

$$(5) \qquad A\mathbf{x} = \mathbf{0} \quad \textit{has a nontrivial solution} \quad \Leftrightarrow \quad \det A = 0 \quad \textit{(i.e., A is singular).}$$

Proof. The direction $\Rightarrow$ follows from (4), since if A is nonsingular, (4) tells us that $A\mathbf{x} = \mathbf{0}$ can have only the trivial solution $\mathbf{x} = \mathbf{0}$.

The direction $\Leftarrow$ follows from the criterion for linear independence below, which we are not going to prove. But in 18.03, you will always be able to show by calculation that the system has a nontrivial solution if A is singular.

4. Linear independence of vectors.

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, x_k$ be a set of n-vectors. We say they are **linearly dependent** (or simply, *dependent*) if there is a non-zero relation connecting them:

$$(6) \qquad c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k = \mathbf{0}, \qquad (c_i \text{ constants, not all } 0).$$

If there is no such relation, they are called **linearly independent** (or simply, *independent*). This is usually phrased in a positive way: the vectors are *linearly independent* if the only relation among them is the zero relation, i.e.,

$$(7) \qquad c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k = \mathbf{0} \quad \Rightarrow \quad c_i = 0 \quad \text{for all } i.$$

We will use this definition mostly for just two or three vectors, so it is useful to see what it says in these low-dimensional cases. For $k = 2$, it says

$$(8) \qquad \mathbf{x}_1 \textit{ and } \mathbf{x}_2 \textit{ are dependent} \quad \Leftrightarrow \quad \textit{one is a constant multiple of the other.}$$

For if say $\mathbf{x}_2 = c\mathbf{x}_1$, then $c\mathbf{x}_1 - x_2 = \mathbf{0}$ is a non-zero relation; while conversely, if we have non-zero relation $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{0}$, with say $c_2 \neq 0$, then $\mathbf{x}_2 = -(c_1/c_2)\,\mathbf{x}_1$.

By similar reasoning, one can show that

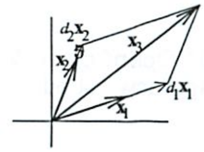
$$(9) \qquad \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \textit{ are dependent} \Leftrightarrow \textit{one of them is a linear combination of the other two.}$$

Here by a **linear combination** of vectors we mean a sum of scalar multiples of them, i.e., an expression like that on the left side of (6). If we think of the three vectors as origin vectors in three space, the geometric interpretation of (9) is

$$(10) \qquad \textit{three origin vectors in 3-space are dependent} \quad \Leftrightarrow \quad \textit{they lie in the same plane.}$$

For if they are dependent, say $\mathbf{x}_3 = d_1\mathbf{x}_1 + d_2\mathbf{x}_2$, then (thinking of them as origin vectors) the parallelogram law for vector addition shows that $\mathbf{x}_3$ lies in the plane of $\mathbf{x}_1$ and $\mathbf{x}_2$ — see the figure.

Conversely, the same figure shows that if the vectors lie in the same plane and say $\mathbf{x}_1$ and $\mathbf{x}_2$ span the plane (i.e., don't lie on a line), then by completing the



parallelogram, $\mathbf{x}_3$ can be expressed as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$. (If they all lie on a line, they are scalar multiples of each other and therefore dependent.)

Linear independence and determinants. We can use (10) to see that

(11) *the rows of a* 3×3 *matrix* A *are dependent* $\Leftrightarrow$ $\det A = \mathbf{0}$.

Proof. If we denote the rows by $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$, then from 18.02,

$$\begin{array}{c}\text{volume of the parallelepiped} \\ \text{spanned by } \mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3\end{array} = \mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3) = \det A,$$

so that

$$\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3 \quad \text{lie in a plane} \quad \Leftrightarrow \quad \det A = 0.$$

The above statement (11) generalizes to an $n \times n$ matrix A ; we rephrase it in the statement below by changing both sides to their negatives. (We will not prove it, however.)

Determinantal criterion for linear independence

Let $\mathbf{a}_1, \dots, \mathbf{a}_n$ *be* n*-vectors, and* A *the square matrix having these vectors for its rows (or columns). Then*

(12) $\mathbf{a}_1, \dots, \mathbf{a}_n$ *are linearly independent* $\Leftrightarrow$ $\det A \neq 0$.

Remark. The theorem on square homogeneous systems (5) follows from the criterion (12), for if we let $\mathbf{x}$ be the column vector of n variables, and A the matrix whose columns are $\mathbf{a}_1, \dots, \mathbf{a}_n$, then

$$(13) \qquad A\mathbf{x} = (\mathbf{a}_1 \dots \mathbf{a}_n) \begin{pmatrix} x_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} = \mathbf{a}_1 x_1 + \dots + \mathbf{a}_n x_n$$

and therefore

$$\begin{aligned} A\mathbf{x} = \mathbf{0} \quad & \text{has only the solution } \mathbf{x} = \mathbf{0} \\ \Leftrightarrow \quad & \mathbf{a}_1 x_1 + \dots + \mathbf{a}_n x_n = \mathbf{0} \quad \text{has only the solution} \quad \mathbf{x} = \mathbf{0}, \quad \text{by (13);} \\ \Leftrightarrow \quad & \mathbf{a}_1, \dots, \mathbf{a}_n \quad \text{are linearly independent, by (7);} \\ \Leftrightarrow \quad & \det A \neq 0, \quad \text{by the criterion (12).} \end{aligned}$$

Exercises: Section 4A

LS.2 Homogeneous Linear Systems with Constant Coefficients

1. Using matrices to solve linear systems.

The naive way to solve a linear system of ODE's with constant coefficients is by eliminating variables, so as to change it into a single higher-order equation. For instance, if

$$(1) \qquad \begin{aligned} x' &= x + 3y \\ y' &= x - y \end{aligned}$$

we can eliminate x by solving the second equation for x, getting $x = y + y'$, then replacing x everywhere by $y + y'$ in the first equation. This gives

$$y'' - 4y = 0 ;$$

the characteristic equation is $(r-2)(r+2) = 0$, so the general solution for y is

$$y = c_1 e^{2t} + c_2 e^{-2t} .$$

From this we get x from the equation $x = y + y'$ originally used to eliminate x; the whole solution to the system is then

$$(2) \qquad \begin{aligned} x &= 3c_1 e^{2t} - c_2 e^{-2t} \\ y &= c_1 e^{2t} + c_2 e^{-2t} . \end{aligned}$$

We now want to introduce linear algebra and matrices into the study of systems like the one above. Our first task is to see how the above equations look when written using matrices and matrix multiplication.

When we do this, the system (1) and its general solution (2) take the forms

$$(4) \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} ,$$

$$(5) \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3c_1 e^{2t} - c_2 e^{-2t} \\ c_1 e^{2t} + c_2 e^{-2t} \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} .$$

Study the above until it is clear to you how the matrices and column vectors are being used to write the system (1) and its solution (2). Note that when we multiply the column vectors by scalars or scalar functions, it does not matter whether we write them behind or in front of the column vector; the way it is written above on the right of (5) is the one usually used, since it is easiest to read and interpret.

We are now going to show a new method of solving the system (1), which makes use of the matrix form (4) for writing it. We begin by noting from (5) that two particular solutions to the system (4) are

$$(6) \qquad \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} .$$

Based on this, our new method is to look for solutions to (4) of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t}, \tag{7}$$

where a_1, a_2 and λ are unknown constants. We substitute (7) into the system (4) to determine what these unknown constants should be. This gives

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \lambda e^{\lambda t} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t}. \tag{8}$$

We can cancel the factor $e^{\lambda t}$ from both sides, getting

$$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{9}$$

We have to solve the matrix equation (9) for the three constants. It is not very clear how to do this. When faced with equations in unfamiliar notation, a reasonable strategy is to rewrite them in more familiar notation. If we try this, (9) becomes the pair of equations

$$\begin{aligned} \lambda a_1 &= a_1 + 3a_2 \\ \lambda a_2 &= a_1 - a_2 . \end{aligned} \tag{10}$$

Technically speaking, these are a pair of non-linear equations in three variables. The trick in solving them is to look at them as a pair of linear equations in the unknowns a_i, with λ viewed as a parameter. If we think of them this way, it immediately suggests writing them in standard form

$$\begin{aligned} (1-\lambda)a_1 + 3a_2 &= 0 \\ a_1 + (-1-\lambda)a_2 &= 0 . \end{aligned} \tag{11}$$

In this form, we recognize them as forming a square system of homogeneous linear equations. According to the theorem on square systems (LS.1, (5)), they have a non-zero solution for the a's if and only if the determinant of coefficients is zero:

$$\begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = 0, \tag{12}$$

which after calculation of the determinant becomes the equation

$$\lambda^2 - 4 = 0 . \tag{13}$$

The roots of this equation are 2 and -2; what the argument shows is that the equations (10) or (11) (and therefore also (8)) have non-trivial solutions for the a's exactly when $\lambda = 2$ or $\lambda = -2$.

To complete the work, we see that for these values of the parameter λ, the system (11) becomes respectively

$$\begin{aligned} -a_1 + 3a_2 &= 0 \\ a_1 - 3a_2 &= 0 \\ &(\text{for } \lambda = 2) \end{aligned} \qquad\qquad \begin{aligned} 3a_1 + 3a_2 &= 0 \\ a_1 + a_2 &= 0 \\ &(\text{for } \lambda = -2) \end{aligned} \tag{14}$$

It is of course no accident that in each case the two equations of the system become dependent, i.e., one is a constant multiple of the other. If this were not so, the two equations would have only the trivial solution $(0, 0)$. All of our effort has been to locate the two values of λ for which this will *not* be so. The dependency of the two equations is thus a check on the correctness of the value of λ.

To conclude, we solve the two systems in (14). This is best done by assigning the value 1 to one of the unknowns, and solving for the other. We get

$$(15) \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{for } \lambda = 2\,; \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{for } \lambda = -2\,,$$

which thus gives us, in view of (7), essentially the two solutions (6) we had found previously by the method of elimination. Note that the solutions (6) could be multiplied by an arbitrary non-zero constant without changing the validity of the general solution (5); this corresponds in the new method to selecting an arbitrary value of one of the a's, and then solving for the other value.

One final point before we discuss this method in general. Is there some way of passing from (9) (the point at which we were temporarily stuck) to (11) or (12) by using matrices, without writing out the equations separately? The temptation in (9) is to try to combine the two column vectors $\mathbf{a}$ by subtraction, but this is impossible as the matrix equation stands. If we rewrite it however as

$$(9') \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

it now makes sense to subtract the left side from the right; using the distributive law for matrix multiplication, the matrix equation (9′) then becomes

$$(11') \qquad \begin{pmatrix} 1-\lambda & 3 \\ 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is just the matrix form for (11). Now if we apply the theorem on square homogeneous systems, we see that (11′) has a nontrivial solution for the $\mathbf{a}$ if and only if its coefficient determinant is zero, and this is precisely (12). The trick therefore was in (9) to replace the scalar λ by the diagonal matrix λI .

2. Eigenvalues and eigenvectors.

With the experience of the preceding example behind us, we are now ready to consider the general case of a homogeneous linear 2×2 system of ODE's with constant coefficients:

$$(16) \qquad \begin{aligned} x' &= ax + by \\ y' &= cx - dy\,, \end{aligned}$$

where the a, b, c, d are constants. We write this system in matrix form as

$$(17) \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We look for solutions to (17) having the form

$$(18) \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} a_1 e^{\lambda t} \\ a_2 e^{\lambda t} \end{pmatrix},$$

where a_1, a_2, and λ are unkinown constants. We substitute (18) into the system (17) to determine these unknown constants. Since $D(ae^{\lambda t}) = \lambda a e^{\lambda t}$, we arrive at

$$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} . \tag{19}$$

We can cancel the factor $e^{\lambda t}$ from both sides, getting

$$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} . \tag{20}$$

As the equation (20) stands, we cannot combine the two sides by subtraction, since the scalar λ cannot be subtracted from the square matrix on the right. As in the previously worked example however (9′), the trick is to replace the scalar λ by the diagonal matrix λI; then (20) becomes

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} . \tag{21}$$

If we now proceed as we did in the example, subtracting the left side of (6) from the right side and using the distributive law for matrix addition and multiplication, we get a 2×2 homogeneous linear system of equations:

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{22}$$

Written out without using matrices, the equations are

$$\begin{aligned} (a-\lambda)a_1 + ba_2 &= 0 \\ ca_1 + (d-\lambda)a_2 &= 0 . \end{aligned} \tag{23}$$

According to the theorem on square homogeneous systems, this system has a non-zero solution for the a's if and only if the determinant of the coefficients is zero:

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 . \tag{24}$$

The equation (24) is a quadratic equation in λ, evaluating the determinant, we see that it can be written

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0 . \tag{25}$$

Definition. *The equation (24) or (25) is called the* **characteristic equation** *of the matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tag{26}$$

Its roots λ_1 and λ_2 are called the **eigenvalues** *or characteristic values of the matrix A.*

There are now various cases to consider, according to whether the eigenvalues of the matrix A are two distinct real numbers, a single repeated real number, or a pair of conjugate complex numbers. We begin with the first case: *we assume for the rest of this chapter that the eigenvalues are two distinct real numbers λ_1 and λ_2.*

To complete our work, we have to find the solutions to the system (23) corresponding to the eigenvalues λ_1 and λ_2. Formally, the systems become

$$(27) \qquad \begin{aligned} (a-\lambda_1)a_1 + ba_2 &= 0 \\ ca_1 + (d-\lambda_1)a_2 &= 0 \end{aligned} \qquad\qquad \begin{aligned} (a-\lambda_2)a_1 + ba_2 &= 0 \\ ca_1 + (d-\lambda_2)a_2 &= 0 \end{aligned}$$

The solutions to these two systems are column vectors, for which we will use Greek letters rather than boldface.

Definition. *The respective solutions* $\mathbf{a} = \vec{\alpha}_1$ *and* $\mathbf{a} = \vec{\alpha}_2$ *to the systems (27) are called the* **eigenvectors** *(or characteristic vectors) corresponding to the eigenvalues* λ_1 *and* λ_2.

If the work has been done correctly, in each of the two systems in (27), the two equations will be dependent, i.e., one will be a constant multiple of the other. Namely, the two values of λ have been selected so that in each case the coefficient determinant of the system will be zero, which means the equations will be dependent. The solution $\vec{\alpha}$ is determined only up to an arbitrary non-zero constant factor. A convenient way of finding the eigenvector $\vec{\alpha}$ is to assign the value 1 to one of the a_i, then use the equation to solve for the corresponding value of the other a_i.

Once the eigenvalues and their corresponding eigenvectors have been found, we have two independent solutions to the system (16); According to (19), they are

$$(28) \qquad \mathbf{x}_1 = \vec{\alpha}_1 e^{\lambda_1 t}, \qquad \mathbf{x}_2 = \vec{\alpha}_2 e^{\lambda_2 t}, \qquad \text{where } \mathbf{x}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}.$$

Then the general solution to the system (16) is

$$(29) \qquad \mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\vec{\alpha}_1 e^{\lambda_1 t} + c_2\vec{\alpha}_2 e^{\lambda_2 t}.$$

At this point, you should stop and work another example, like the one we did earlier. Try **5.4** Example 1 in your book; work it out yourself, using the book's solution to check your work. Note that the book uses $\mathbf{v}$ instead of $\vec{\alpha}$ for an eigenvector, and v_i or a, b instead of a_i for its components.

We are still not done with the general case; without changing any of the preceding work, you still need to see how it appears when written out using an even more abridged notation. Once you get used to it (and it is important to do so), the compact notation makes the essential ideas stand out very clearly.

As before, we let A denote the matrix of constants, as in (26). Below, on the left side of each line, we will give the compact matrix notation, and on the right, the expanded version. The equation numbers are the same as the ones above.

We start with the system (16), written in matrix form, with A as in (26):

$$(17') \qquad \mathbf{x}' = A\mathbf{x} \qquad\qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

We use as the trial solution

$$(18') \qquad \mathbf{x} = \mathbf{a}e^{\lambda t} \qquad\qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t}.$$

We substitute this expression for $\mathbf{x}$ into the system (17′), using $\mathbf{x}' = \lambda \mathbf{a} e^{\lambda t}$:

$$(19') \qquad \lambda \mathbf{a} e^{\lambda t} = A \mathbf{a} e^{\lambda t} \qquad \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} .$$

Cancel the exponential factor from both sides, and replace λ by λI, where I is the identity matrix:

$$(21') \qquad \lambda I \mathbf{a} = A \mathbf{a} \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} .$$

Subtract the left side from the right and combine terms, getting

$$(22') \qquad (A - \lambda I)\mathbf{a} = 0 \qquad \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

This square homogeneous system has a non-trivial solution if and only if the coefficient determinant is zero:

$$(24') \qquad |A - \lambda I| = 0 \qquad \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 .$$

Definition. *Let A be a square matrix of constants, Then by definition*

(i) $|A - \lambda I| = 0$ is the **characteristic equation** *of A ;*

(ii) its roots λ_i are the **eigenvalues** *(or characteristic values) of A ;*

(iii) for each eigenvalue λ_i, the corresonding solution $\vec{\alpha}_i$ to (22′) is the **eigenvector** *(or characteristic vector) associated with λ_i .*

If the eigenvalues are distinct and real, as we are assuming in this chapter, we obtain in this way two independent solutions to the system (17′):

$$(28) \qquad \mathbf{x}_1 = \vec{\alpha}_1 e^{\lambda_1 t}, \qquad \mathbf{x}_2 = \vec{\alpha}_2 e^{\lambda_2 t}, \qquad \text{where } \mathbf{x}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} .$$

Then the general solution to the system (16) is

$$(29) \qquad \mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 = c_1 \vec{\alpha}_1 e^{\lambda_1 t} + c_2 \vec{\alpha}_2 e^{\lambda_2 t} .$$

The matrix notation on the left above in (17′) to (24′) is compact to write, makes the derivation look simpler. Moreover, when written in matrix notation, *the derivation applies to square systems of any size*: $n \times n$ just as well as 2×2. This goes for the subsequent definition as well: it defines *characteristic equation, eigenvalue* and *associated eigenvector* for a square matrix of any size.

The chief disadvantage of the matrix notation on the left is that for beginners it is very abridged. Practice writing the sequence of matrix equations so you get some skill in using the notation. Until you acquire some confidence, keep referring to the written-out form on the right above, so you are sure you understand what the abridged form is actually saying.

Since in the compact notation, the definitions and derivations are valid for square systems of any size, you now know for example how to solve a 3×3 system, if its eigenvalues turn out to be real and distinct; 5.4 Example 2 in your book is such a system. First however read the following remarks which are meant to be helpful in doing calculations: remember and *use* them.

Remark 1. Calculating the characteristic equation.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its characteristic equation is given by (cf. (24) and (25)):

$$(30) \qquad \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0, \qquad \text{or} \qquad \lambda^2 - (a+d)\lambda + (ad-bc) = 0\ .$$

Since you will be calculating the characteristic equation frequently, learn to do it using the second form given in (30). The two coefficients have analogs for any square matrix:

$$ad - bc = \det A \qquad\qquad a + d = \operatorname{tr} A \quad (\textit{trace } A)$$

where the **trace** of a square matrix A is the sum of the elements on the main diagonal. Using this, the characteristic equation (30) for a 2×2 matrix A can be written

$$(31) \qquad \lambda^2 - (\operatorname{tr} A)\,\lambda + \det A = 0\ .$$

In this form, the characteristic equation of A can be written down by inspection; you don't need the intermediate step of writing down $|A - \lambda I| = 0$. For an $n \times n$ matrix, the characteristic equation reads in part (watch the signs!)

$$(32) \qquad |A - \lambda I| = (-\lambda)^n + \operatorname{tr} A(-\lambda)^{n-1} + \ldots + \det A = 0\ .$$

In one of the exercises you are asked to derive the two coefficients specified.

Equation (32) shows that the characteristic polynomial $|A - \lambda I|$ of an $n \times n$ matrix A is a polynomial of degree n, so that such a matrix has at most n real eigenvalues. The trace and determinant of A give two of the coefficients of the polynomial. Even for $n = 3$ however this is not enough, and you will have to calculate the characteristic equation by expanding out $|A - \lambda I|$. Nonetheless, (32) is still very valuable, as it enables you to get an independent check on your work. Use it whenever $n > 2$.

Remark 2. Calculating the eigenvectors.

This is a matter of solving a homogeneous system of linear equations (22′).

For $n = 2$, there will be just one equation (the other will be a multiple of it); give one of the a_i's the value 1 (or any other convenient non-zero value), and solve for the other a_i.

For $n = 3$, two of the equations will usually be independent (i.e., neither a multiple of the other). Using just these two equations, give one of the a's a convenient value (say 1), and solve for the other two a's. (The case where the three equations are all multiples of a single one occurs less often and will be dealt with later.)

Remark 3. Normal modes.

When the eigenvalues of A are all real and distinct, the corresponding solutions (28)

$$\mathbf{x}_i = \vec{\alpha}_i e^{\lambda_i t}, \qquad i = 1, \ldots, n,$$

are usually called the *normal modes* in science and engineering applications. They often have physical interpretations, which sometimes makes it possible to find them just by inspection of the physical problem. The exercises will illustrate this.

Exercises: Section 4C

LS.3 Complex and Repeated Eigenvalues

1. Complex eigenvalues.

In the previous chapter, we obtained the solutions to a homogeneous linear system with constant coefficients

$$A\mathbf{x} = \mathbf{0}$$

under the assumption that the roots of its characteristic equation $|A - \lambda I| = 0$ — i.e., the eigenvalues of A — were *real* and *distinct.*

In this section we consider what to do if there are complex eigenvalues. Since the chanracteristic equation has real coefficients, its complex roots must occur in conjugate pairs:

$$\lambda = a + bi, \qquad \bar{\lambda} = a - bi \ .$$

Let's start with the eigenvalue $a + bi$. According to the solution method described in Chapter LS.2, the next step would be to find the corresponding eigenvector $\vec{\alpha}$, by solving the equations (LS.2, (27))

$$\begin{aligned} (a - \lambda)a_1 + ba_2 &= 0 \\ ca_1 + (d - \lambda)a_2 &= 0 \end{aligned}$$

for its components a_1 and a_2. Since λ is complex, the a_i will also be complex, and therefore the eigenvector $\vec{\alpha}$ corresponding to λ will have complex components.

Putting together the eigenvalue and eigenvector gives us formally the complex solution

$$\mathbf{x} = \vec{\alpha}\, e^{(a+bi)t} \ . \tag{1}$$

Naturally, we want real solutions to the system, since it was real to start with. To get them, the following theorem tells us to just take the real and imaginary parts of (1).

Theorem 3.1 *Given a system* $\mathbf{x}' = A\mathbf{x}$, *where* A *is a real matrix. If* $\mathbf{x} = \mathbf{x}_1 + i\,\mathbf{x}_2$ *is a complex solution, then its real and imaginary parts* $\mathbf{x}_1, \mathbf{x}_2$ *are also solutions to the system.*

Proof. Since $\mathbf{x}_1 + i\,\mathbf{x}_2$ is a solution, we have

$$(\mathbf{x} + i\,\mathbf{x}_2)' = A\,(\mathbf{x} + i\,\mathbf{x}_2) \ ;$$

equating real and imaginary parts of this equation,

$$\mathbf{x}_1' = A\mathbf{x}_1 \ , \qquad \mathbf{x}_2' = A\mathbf{x}_2 \ ,$$

which shows that the real vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $\mathbf{x}' = A\mathbf{x}$. □

Example 1. Find the corresponding two real solutions to $\mathbf{x}' = A\mathbf{x}$ if a complex eigenvalue and corresponding eigenvector are

$$\lambda = -1 + 2i \ , \qquad \vec{\alpha} = \begin{pmatrix} i \\ 2 - 2i \end{pmatrix} \ .$$

Solution. First write $\vec{\alpha}$ in terms of its real and imaginary parts:

$$\vec{\alpha} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ -2 \end{pmatrix} \ .$$

The corresponding complex solution $\mathbf{x} = \vec{\alpha}\, e^{\lambda t}$ to the system can then be written

$$\mathbf{x} = \left(\begin{pmatrix} 0 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right) e^{-t}(\cos 2t + i \sin 2t)\ ,$$

so that we get respectively for the real and imaginary parts of $\mathbf{x}$

$$x_1 = e^{-t}\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos 2t - i \begin{pmatrix} 1 \\ -2 \end{pmatrix} \sin 2t\right) = e^{-t} \begin{pmatrix} -\sin 2t \\ 2\cos 2t + 2\sin 2t \end{pmatrix},$$

$$x_2 = e^{-t}\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix} \cos 2t - i \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin 2t\right) = e^{-t} \begin{pmatrix} -\cos 2t \\ -2\cos 2t + 2\sin 2t \end{pmatrix};$$

these are the two real solutions to the system. □

In general, if the complex eigenvalue is $a + bi$, to get the real solutions to the system, we write the corresponding complex eigenvector $\vec{\alpha}$ in terms of its real and imaginary part:

$$\vec{\alpha} = \vec{\alpha}_1 + i\,\vec{\alpha}_2, \qquad \vec{\alpha}_i \ \text{ real vectors;}$$

(study carefully in the above example how this is done in practice). Then we substitute into (1) and calculate as in the example:

$$\mathbf{x} = (\vec{\alpha}_1 + i\vec{\alpha}_2)\, e^a t(\cos bt + i \sin bt),$$

so that the real and imaginary parts of $\mathbf{x}$ give respectively the two real solutions

$$\text{(3)} \qquad \begin{aligned} \mathbf{x}_1 &= e^{at}(\vec{\alpha}_1 \cos bt - \vec{\alpha}_2 \sin bt)\ , \\ \mathbf{x}_2 &= e^{at}(\vec{\alpha}_1 \sin bt + \vec{\alpha}_2 \cos bt)\ . \end{aligned}$$

These solutions are linearly independent if $n = 2$. If $n > 2$, that portion of the general solution corresonding to the eigenvalues $a \pm bi$ will be

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2\ .$$

Note that, as for second-order ODE's, the complex conjugate eigenvalue $a - bi$ gives up to sign the same two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$.

The expression (3) was not written down for you to memorize, learn, or even use; the point was just for you to get some practice in seeing how a calculation like that in Example 1 looks when written out in general. To actually solve ODE systems having complex eigenvalues, imitate the procedure in Example 1.

Stop at this point, and practice on an example (try Example 3, p. 377).

2. Repeated eigenvalues. Again we start with the real $n \times n$ system

$$\text{(4)} \qquad \mathbf{x}' = A\mathbf{x}\ .$$

We say an eigenvalue λ_1 of A is **repeated** if it is a multiple root of the characteristic equation of A—in other words, the characteristic polynomial $|A - \lambda I|$ has $(\lambda - \lambda_1)^2$ as a factor. Let's suppose that λ_1 is a double root; then we need to find *two* linearly independent solutions to the system (4) corresponding to λ_1.

One solution we can get: we find in the usual way an eigenvector $\vec{\alpha}_1$ corresponding to λ_1 by solving the system

$$\text{(5)} \qquad (A - \lambda_1 I)\,\mathbf{a} = \mathbf{0}\ .$$

This gives the solution $\mathbf{x}_1 = \vec{\alpha}_1 e^{\lambda_1 t}$ to the system (4). Our problem is to find a second solution. To do this we have to distinguish two cases. The first one is easy.

A. The complete case.

Still assuming λ_1 is a real double root of the characteristic equation of A, we say λ_1 is a **complete** eigenvalue if there are two linearly independent eigenvectors $\vec{\alpha}_1$ and $\vec{\alpha}_2$ corresponding to λ_1; i.e., if these two vectors are two linearly independent solutions to the system (5). Using them we get two independent solutions to (4), namely

$$\mathbf{x}_1 = \vec{\alpha}_1 e^{\lambda_1 t}, \qquad \text{and} \qquad \mathbf{x}_2 = \vec{\alpha}_2 e^{\lambda_1 t}\,. \tag{6}$$

Naturally we would like to see an example of this for $n = 2$, but the following theorem explains why there aren't any good examples: if the matrix A has a repeated eigenvalue, it is so simple that no one would solve the system (4) by fussing with eigenvectors!

Theorem 3.2 *Let A be a 2×2 matrix and λ_1 an eigenvalue. Then*

$$\lambda_1 \text{ is repeated and complete} \quad \Leftrightarrow \quad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

Proof. Let $\vec{\alpha}_1$ and $\vec{\alpha}_2$ be two independent solutions to (5). Then any 2-vector $\vec{\alpha}$ is a solution to (5), for by using the parallelogram law of addition, $\vec{\alpha}$ can be written in the form

$$\vec{\alpha} = c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2\,,$$

and this shows it is a solution to (5), since $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are:

$$(A - \lambda_1 I)\vec{\alpha} = c_1(A - \lambda_1 I)\vec{\alpha}_1 + c_2(A - \lambda_1 I)\vec{\alpha}_2 = 0 + 0\,.$$

In particular, the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ satisfy (5); letting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} a = \lambda_1 \\ c = 0 \end{cases};$$

$$\begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} b = 0 \\ d = \lambda_1 \end{cases}.$$

This proves the theorem in the direction $\Rightarrow$; in the other direction, one sees immediately that the characteristic polynomial is $(\lambda - \lambda_1)^2$, so that λ_1 is a repeated eigenvalue; it is complete since the matrix $A - \lambda_1 I$ has 0 for all its entries, and therefore every 2-vector $\vec{\alpha}$ is a solution to (5).

For $n = 3$ the situation is more interesting. Still assuming λ_1 is a double root of the characteristic equation, it will be a complete eigenvalue when the system (5) has two independent solutions; this will happen when the system (5) has essentially only one equation: the other two equations are constant multiples of it (or identically 0). You can then find two independent solutions to the system just by inspection.

Example 2. If the system (5) turns out to be three equations, each of which is a constant multiple of say

$$2a_1 - a_2 + a_3 = 0\,,$$

we can give a_1 and a_2 arbitrary values, and then a_3 will be determined by the above equation. Hence two independent solutions (eigenvectors) would be the column 3-vectors

$$(1, 0, 2)^T \qquad \text{and} \qquad (0, 1, 1)^T\,.$$

In general, if an eigenvalue λ_1 of A is k-tuply repeated, meaning the polynomial $A - \lambda I$ has the power $(\lambda - \lambda_1)^k$ as a factor, but no higher power, the eigenvalue is called **complete** if it has k independent associated eigenvectors, i.e., if the system (5) has k linearly independent solutions. These then produce k solutions to the ODE system (4).

There is an important theorem in linear algebra (it usually comes at the very end of a linear algebra course) which guarantees that all the eigenvalues of A will be complete, regardless of what their multiplicity is:

Theorem 3.3 *If the real square matrix A is symmetric, meaning $A^T = A$, then all its eigenvalues are real and complete.*

B. The defective case.

If the eigenvalue λ is a double root of the characteristic equation, but the system (5) has only one non-zero solution $\vec{\alpha}_1$ (up to constant multiples), then the eigenvalue is said to be **incomplete** or **defective**, and no second eigenvector exists. In this case, the second solution to the system (4) has a different form. It is

$$(7) \qquad \mathbf{x} = (\vec{\beta} + \vec{\alpha}_1 t)e^{\lambda_1 t} ,$$

where $\vec{\beta}$ is an unknown vector which must be found. This may be done by substituting (7) into the system (4), and using the fact that $\vec{\alpha}_1$ is an eigenvector, i.e., a solution to (5) when $\lambda = \lambda_1$. When this is done, we find that $\vec{\beta}$ must be a solution to the system

$$(8) \qquad (A - \lambda_1 I)\,\vec{\beta} = \vec{\alpha}_1 .$$

This is an inhomogeneous system of equations for determining β. It is guaranteed to have a solution, provided that the eigenvalue λ_1 really is defective.

> Notice that (8) doesn't look very solvable, because the matrix of coefficients has determinant zero! So you won't solve it by finding the inverse matrix or by using Cramer's rule. It has to be solved by elimination.

Some people do not bother with (7) or (8); when they encounter the defective case (at least when $n = 2$), they give up on eigenvalues, and simply solve the original system (4) by elimination.

Example. Try Example 2 (section 5.6) in your book; do it by using (7) and (8) above to find β, then check your answer by instead using elimination to solve the ODE system.

Proof of (7) for $n = 2$. Let A be a 2×2 matrix.

Since λ_1 is to be a double root of the characteristic equation, that equation must be

$$(\lambda - \lambda_1)^2 = 0, \qquad i.e., \qquad \lambda^2 - 2\lambda_1\lambda + \lambda_1^2 = 0.$$

From the known form of the characteristic equation (LS.2, (25)), we see that

$$\text{trace}\, A = 2\lambda_1, \qquad \det A = \lambda_1^2.$$

A convenient way to write two numbers whose sum is $2\lambda_1$ is: $\lambda_1 \pm a$; doing this, we see that our matrix A takes the form

$$(9) \qquad A = \begin{pmatrix} \lambda_1 + a & b \\ c & \lambda_1 - a \end{pmatrix}, \qquad \text{where } bc = -a^2, \ \ (\text{since } \det A = \lambda_1^2).$$

Now calculate the eigenvectors of such a matrix A. Note that b and c are not both zero, for if they were, $a = 0$ by (9), and the eigenvalue would be complete. If say $b \neq 0$, we may choose as the eigenvector

$$\vec{\alpha}_1 = \begin{pmatrix} b \\ -a \end{pmatrix},$$

and then by (8), we get

$$\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Exercises: Section 4D

LS.4 Decoupling Systems

1. Changing variables.

A common way of handling mathematical models of scientific or engineering problems is to look for a change of coordinates or a change of variables which simplifies the problem. We handled some types of first-order ODE's — the Bernouilli equation and the homogeneous equation, for instance — by making a change of dependent variable which converted them into equations we already knew how to solve. Another example would be the use of polar or spherical coordinates when a problem has a center of symmetry.

An example from physics is the description of the acceleration of a particle moving in the plane: to get insight into the acceleration vector, a new coordinate system is introduced whose basis vectors are $\mathbf{t}$ and $\mathbf{n}$ (the unit tangent and normal to the motion), with the result that $\mathbf{F} = m\mathbf{a}$ becomes simpler to handle.

We are going to do something like that here. Starting with a homogeneous linear system with constant coefficients, we want to make a linear change of coordinates which simplifies the system. We will work with $n = 2$, though what we say will be true for $n > 2$ also.

How would a simple system look? The simplest system is one with a diagonal matrix: written first in matrix form and then in equation form, it is

$$(1) \qquad \begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \text{or} \qquad \begin{aligned} u' &= \lambda_1 u \\ v' &= \lambda_2 v \end{aligned} .$$

As you can see, if the coefficient matrix has only diagonal entries, the resulting "system" really consists of a set of first-order ODE's, side-by-side as it were, each involving only its own variable. Such a system is said to be **decoupled** since the variables do not interact with each other; each variable can be solved for independently, without knowing anything about the others. Thus, solving the system on the right of (1) gives

$$(2) \qquad \begin{aligned} u &= c_1 e^{\lambda_1 t} \\ v &= c_2 e^{\lambda_2 t} \end{aligned}, \qquad \text{or} \qquad \mathbf{u} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda_2 t}.$$

So we start with a 2×2 homogeneous system with constant coefficients,

$$(3) \qquad \mathbf{x}' = A\mathbf{x},$$

and we want to introduce new dependent variables u and v, related to x and y by a linear change of coordinates, i.e., one of the form (we write it three ways):

$$(4) \qquad \mathbf{u} = D\mathbf{x}, \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{aligned} u &= ax + by \\ v &= cx + dy \end{aligned} .$$

We call D the **decoupling matrix**. After the change of variables, we want the system to be decoupled, i.e., to look like the system (1). What should we choose as D?

The matrix D will define the new variables u and v in terms of the old ones x and y. But in order to substitute into the system (3), it is really the inverse to D that we need; we shall denote it by E :

$$\text{(5)} \qquad \mathbf{u} = D\mathbf{x}, \qquad \mathbf{x} = E\mathbf{u}\,, \qquad E = D^{-1}\,.$$

In the decoupling, we first produce E ; then D is calculated as its inverse. We need both matrices: D to define the new variables, E to do the substitutions.

We are now going to assume that the ODE system $\mathbf{x}' = A\mathbf{x}$ has *two real and distinct eigenvalues*; with their associated eigenvectors, they are denoted as usual in these notes by

$$\text{(6)} \qquad \lambda_1, \quad \vec{\alpha}_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}; \qquad \lambda_2, \quad \vec{\alpha}_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}.$$

The idea now is the following. Since these eigenvectors are somehow "special" to the system, let us choose the new coordinates so that the eigenvectors become the unit vectors $\mathbf{i}$ and $\mathbf{j}$ in the uv-system. To do this, we make the eigenvectors the two columns of the matrix E ; that is, we make the change of coordinates

$$\text{(7)} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \qquad E = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}.$$

With this choice for the matrix E,

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in the uv-system correspond in the xy-system respectively to the first and second columns of E, as you can see from (7).

We now have to show that this change to the uv-system decouples the ODE system $\mathbf{x}' = A\mathbf{x}$. This rests on the following very important equation connecting a matrix A, one of its eigenvalues λ, and a corresponding eigenvector $\vec{\alpha}$:

$$\text{(8)} \qquad A\vec{\alpha} = \lambda\vec{\alpha}\,,$$

which follows immediately from the equation used to calculate the eigenvector:

$$(A - \lambda I)\vec{\alpha} = 0 \quad \Rightarrow \quad A\vec{\alpha} = (\lambda I)\vec{\alpha} = \lambda(I\vec{\alpha}) = \lambda\vec{\alpha}\,.$$

> The equation (8) is often used as the definition of eigenvector and eigenvalue: an *eigenvector of* A is a vector which changes by some scalar factor λ when multiplied by A; the factor λ is the *eigenvalue* associated with the vector.

As it stands, (8) deals with only one eigenvector at a time. We recast it into the standard form in which it deals with both eigenvectors simultaneously. Namely, (8) says that

$$A\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \lambda_1\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \qquad A\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \lambda_2\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}.$$

These two equations can be combined into the single matrix equation

$$\text{(9)} \quad A\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \text{or} \qquad AE = E\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

as is easily checked. Note that the diagonal matrix of λ's must be placed on the right in order to multiply the *columns* by the λ's; if we had placed it on the left, it would have multiplied the *rows* by the λ's, which is not what we wanted.

From this point on, the rest is easy. We want to show that the change of variables $\mathbf{x} = E\mathbf{u}$ decouples the system $\mathbf{x}' = A\mathbf{x}$, where E is defined by (7). We have, substituting $\mathbf{x} = E\mathbf{u}$ into the system, the successive equations

$$\begin{aligned}
\mathbf{x}' &= A\mathbf{x} \\
E\mathbf{u}' &= A\,E\mathbf{u} \\
E\mathbf{u}' &= E\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\mathbf{u}, \qquad \text{by (9);}
\end{aligned}$$

multiplying both sides on the left by $D = E^{-1}$ then shows the system is decoupled:

$$\mathbf{u}' = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\mathbf{u} \;.$$

Definition. *For a matrix A with two real and distinct eigenvalues, the matrix E in (7) whose columns are the eigenvectors of A is called an* **eigenvector matrix** *for A, and the matrix $D = E^{-1}$ is called the* **decoupling matrix** *for the system $\mathbf{x}' = A\mathbf{x}$; the new variables u, v in (7) are called the* **canonical variables.**

One can alter the matrices by switching the columns, or multiplying a column by a non-zero scalar, with a corresponding alteration in the new variables; apart from that, they are unique.

Example 1. For the system

$$\begin{aligned}
x' &= x - y \\
y' &= 2x + 4y
\end{aligned}$$

make a linear change of coordinates which decouples the system; verify by direct substitution that the system becomes decoupled.

Solution. In matrix form the system is $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix}1 & -1\\ 2 & 4\end{pmatrix}$.

We calculate first E, as defined by (7); for this we need the eigenvectors. The characteristic polynomial of A is

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)\;;$$

the eigenvalues and corresponding eigenvectors are, by the usual calculation,

$$\lambda_1 = 2, \quad \vec{\alpha}_1 = \begin{pmatrix}1\\ -1\end{pmatrix}; \qquad \lambda_2 = 3, \quad \vec{\alpha}_2 = \begin{pmatrix}1\\ -2\end{pmatrix}.$$

The matrix E has the eigenvectors as its columns; then $D = E^{-1}$. We get (cf. LS.1, (2) to calculate the inverse matrix to E)

$$E = \begin{pmatrix}1 & 1\\ -1 & -2\end{pmatrix}, \qquad D = \begin{pmatrix}2 & 1\\ -1 & -1\end{pmatrix}.$$

By (4), the new variables are defined by

$$\mathbf{u} = D\mathbf{x}\,, \qquad \begin{aligned} u &= 2x + y \\ v &= -x - y\,. \end{aligned}$$

To substitute these into the system and check they they decouple we use

$$\mathbf{x} = E\mathbf{u}\,, \qquad \begin{aligned} x &= u+v \\ y &= -u-2v \end{aligned}\,.$$

Substituting these into the original system (on the left below) gives us the pair of equations on the right:

$$\begin{aligned} x' &= x-y \\ y' &= 2x+4y \end{aligned} \qquad \begin{aligned} u'+v' &= 2u+3v \\ -u'-2v' &= -2u-6v \end{aligned}\;;$$

adding the equations eliminates u; multiplying the top equation by 2 and adding eliminates v, giving the system

$$\begin{aligned} u' &= 2y \\ v' &= 3v \end{aligned}$$

which shows that in the new coordinates the system is decoupled.

The work up to this point assumes that $n = 2$ and the eigenvalues are real and distinct. What if this is not so?

If the eigenvalues are complex, the corresponding eigenvectors will also be complex, i.e., have complex components. All of the above remains formally true, provided we allow all the matrices to have complex entries. This means the new variables u and v will be expressed in terms of x and y using complex coefficients, and the decoupled system will have complex coefficients.

In some branches of science and engineering, this is all perfectly acceptable, and one gets in this way a complex decoupling. If one insists on using real variables only, a decoupling is not possible.

If there is only one (repeated) eigenvalue, there are two cases, as discussed in LS.3 . In the complete case, there are two independent eigenvalues, but as pointed out there (Theorem 3.2), the system will be be automatically decoupled, i.e. A will be a diagonal matrix. In the incomplete case, there is only one eigenvector, and decoupling is impossible (since in the decoupled system, both $\mathbf{i}$ and $\mathbf{j}$ would be eigenvectors).

For $n \geq 3$, real decoupling requires us to find n linearly independent real eigenvectors, to form the columns of the nonsingular matrix E. This is possible if

a) all the eigenvalues are real and distinct, or

b) all the eigenvalues are real, and each repeated eigenvalue is complete.

Repeating the end of LS.3, we note again the important theorem in linear algebra which guarantees decoupling is possible:

Theorem. *If the matrix A is real and symmetric, i.e., $A^T = A$, all its eigenvalues will be real and complete, so that the system $\mathbf{x}' = A\mathbf{x}$ can always be decoupled.*

Exercises: Section 4E

LS.5 Theory of Linear Systems

1. General linear ODE systems and independent solutions.

We have studied the homogeneous system of ODE's with constant coefficients,

$$(1) \qquad \mathbf{x}' = A\mathbf{x}\,,$$

where A is an $n \times n$ matrix of constants ($n = 2, 3$). We described how to calculate the eigenvalues and corresponding eigenvectors for the matrix A, and how to use them to find n independent solutions to the system (1).

With this concrete experience solving low-order systems with constant coefficients, what can be said in general when the coefficients are not constant, but functions of the independent variable t? We can still write the linear system in the matrix form (1), but now the matrix entries will be functions of t:

$$(2) \qquad \begin{aligned} x' &= a(t)x + b(t)y \\ y' &= c(t)x + d(t)y \end{aligned}\,, \qquad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\,,$$

or in more abridged notation, valid for $n \times n$ linear homogeneous systems,

$$(3) \qquad \mathbf{x}' = A(t)\mathbf{x}\,.$$

> Note how the matrix becomes a function of t — we call it a "matrix-valued function" of t, since to each value of t the function rule assigns a matrix:
>
> $$t_0 \to A(t_0) = \begin{pmatrix} a(t_0) & b(t_0) \\ c(t_0) & d(t_0) \end{pmatrix}$$
>
> In the rest of this chapter we will often not write the variable t explicitly, but it is always understood that the matrix entries are functions of t.
>
> We will sometimes use $n = 2$ or 3 in the statements and examples in order to simplify the exposition, but the definitions, results, and the arguments which prove them are essentially the same for higher values of n.

Definition 5.1 Solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ to (3) are called **linearly dependent** if there are constants c_i, not all of which are 0, such that

$$(4) \qquad c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) = 0, \qquad \text{for all } t.$$

If there is no such relation, i.e., if

$$(5) \qquad c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) = 0 \quad \text{for all } t \quad \Rightarrow \quad \text{all } c_i = 0,$$

the solutions are called **linearly independent**, or simply *independent*.

> The phrase "for all t" is often in practice omitted, as being understood. This can lead to ambiguity; to avoid it, we will use the symbol $\equiv 0$ for **identically 0**, meaning: "zero for all t"; the symbol $\not\equiv 0$ means "not identically 0", i.e., there is some t-value for which it is not zero. For example, (4) would be written

$$c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) \;\equiv\; 0 \;.$$

Theorem 5.1 *If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is a linearly independent set of solutions to the $n \times n$ system $\mathbf{x}' = A(t)\mathbf{x}$, then the general solution to the system is*

$$\mathbf{x} \;=\; c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n \tag{6}$$

Such a linearly independent set is called a **fundamental** *set of solutions.*

This theorem is the reason for expending so much effort in LS.2 and LS.3 on finding two independent solutions, when $n = 2$ and A is a constant matrix. In this chapter, the matrix A is not constant; nevertheless, (6) is still true.

Proof. There are two things to prove:

(a) All vector functions of the form (6) really are solutions to $\mathbf{x}' \;=\; A\mathbf{x}$.

This is the *superposition principle* for solutions of the system; it's true because the system is *linear.* The matrix notation makes it really easy to prove. We have

$$\begin{aligned}(c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n)' &= c_1\mathbf{x}_1' + \ldots + c_n\mathbf{x}_n' \\ &= c_1A\mathbf{x}_1 + \ldots + c_nA\mathbf{x}_n, &&\text{since } \mathbf{x}_i' = A\mathbf{x}_i\,; \\ &= A\,(c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n), &&\text{by the distributive law (see LS.1).}\end{aligned}$$

(b) All solutions to the system are of the form (6).

This is harder to prove, and will be the main result of the next section.

2. The existence and uniqueness theorem for linear systems.

For simplicity, we stick with $n = 2$, but the results here are true for all n. There are two questions that need answering about the general linear system

$$\begin{aligned} x' &= a(t)x + b(t)y \\ y' &= c(t)x + d(t)y \end{aligned}\;; \qquad \text{in matrix form,} \quad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{2}$$

The first is from the previous section: to show that all solutions are of the form

$$\mathbf{x} = c_1\mathbf{x}_1 + x_2\mathbf{x}_2,$$

where the $\mathbf{x}_i$ form a fundamental set (i.e., neither is a constant multiple of the other). (The fact that we can write down *all* solutions to a linear system in this way is one of the main reasons why such systems are so important.)

An even more basic question for the system (2) is, how do we know that *has* two linearly independent solutions? For systems with a constant coefficient matrix A, we showed in the previous chapters how to solve them explicitly to get two independent solutions. But the general non-constant linear system (2) does not have solutions given by explicit formulas or procedures.

The answers to these questions are based on following theorem.

Theorem 5.2 Existence and uniqueness theorem for linear systems.

If the entries of the square matrix $A(t)$ are continuous on an open interval I containing t_0, then the initial value problem

$$\mathbf{x}' = A(t)\,\mathbf{x}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{7}$$

has one and only one solution $\mathbf{x}(t)$ on the interval I.

The proof is difficult, and we shall not attempt it. More important is to see how it is used. The three theorems following answer the questions posed, for the 2×2 system (2). They are true for $n > 2$ as well, and the proofs are analogous.

In the theorems, *we assume the entries of $A(t)$ are continuous on an open interval I*; then the conclusions are valid on the interval I. (For example, I could be the whole t-axis.)

Theorem 5.2A Linear independence theorem.

Let $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ be two solutions to (2) on the interval I, such that at some point t_0 in I, the vectors $\mathbf{x}_1(t_0)$ and $\mathbf{x}_2(t_0)$ are linearly independent. Then

a) the solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent on I, and

b) the vectors $\mathbf{x}_1(t_1)$ and $\mathbf{x}_2(t_1)$ are linearly independent at every point t_1 of I.

Proof. a) By contradiction. If they were dependent on I, one would be a constant multiple of the other, say $\mathbf{x}_2(t) = c_1\mathbf{x}_1(t)$; then $\mathbf{x}_2(t_0) = c_1\mathbf{x}_1(t_0)$, showing them dependent at t_0. □

b) By contradiction. If there were a point t_1 on I where they were dependent, say $\mathbf{x}_2(t_1) = c_1\mathbf{x}_1(t_1)$, then $\mathbf{x}_2(t)$ and $c_1\mathbf{x}_1(t)$ would be solutions to (2) which agreed at t_1, hence by the uniqueness statement in Theorem 5.2, $\mathbf{x}_2(t) = c_1\mathbf{x}_1(t)$ on all of I, showing them linearly dependent on I. □

Theorem 5.2B General solution theorem.

a) The system (2) has two linearly independent solutions.

b) If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are any two linearly independent solutions, then every solution $\mathbf{x}$ can be written in the form (8), for some choice of c_1 and c_2:

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2; \tag{8}$$

Proof. Choose a point $t = t_0$ in the interval I.

a) According to Theorem 5.2, there are two solutions $\mathbf{x}_1$, $\mathbf{x}_2$ to (3), satisfying respectively the initial conditions

$$\mathbf{x}_1(t_0) = \mathbf{i}, \qquad \mathbf{x}_2(t_0) = \mathbf{j}\ , \tag{9}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the usual unit vectors in the xy-plane. Since the two solutions are linearly independent when $t = t_0$, they are linearly independent on I, by Theorem 5.2A.

b) Let $\mathbf{u}(t)$ be a solution to (2) on I. Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are independent at t_0 by Theorem 5.2, using the parallelogram law of addition we can find constants c_1' and c_2' such that

$$\mathbf{u}(t_0) = c_1'\mathbf{x}_1(t_0) + c_2'\mathbf{x}_2(t_0). \tag{10}$$

The vector equation (10) shows that the solutions $\mathbf{u}(t)$ and $c_1'\mathbf{x}_1(t) + c_2'\mathbf{x}_2(t)$ agree at t_0; therefore by the uniqueness statement in Theorem 5.2, they are equal on all of I, that is,

$$\mathbf{u}(t) = c_1'\mathbf{x}_1(t) + c_2'\mathbf{x}_2(t) \qquad \text{on } I.$$

3. The Wronskian

We saw in chapter LS.1 that a standard way of testing whether a set of n n-vectors are linearly independent is to see if the $n \times n$ determinant having them as its rows or columns is non-zero. This is also an important method when the n-vectors are solutions to a system; the determinant is given a special name. (Again, we will assume $n = 2$, but the definitions and results generalize to any n.)

Definition 5.3 *Let* $\mathbf{x}_1(t)$ *and* $\mathbf{x}_2(t)$ *be two 2-vector functions. We define their* **Wronskian** *to be the determinant*

$$(11) \qquad W(\mathbf{x}_1, \mathbf{x}_2)(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix}$$

whose columns are the two vector functions.

The independence of the two vector functions should be connected with their Wronskian not being zero. At least for points, the relationship is clear; using the result mentioned above, we can say

$$(12) \quad W(\mathbf{x}_1, \mathbf{x}_2)(t_0) = \begin{vmatrix} x_1(t_0) & x_2(t_0) \\ y_1(t_0) & y_2(t_0) \end{vmatrix} = 0 \quad \Leftrightarrow \quad \mathbf{x}_1(t_0) \text{ and } \mathbf{x}_2(t_0) \textit{ are dependent.}$$

However for vector functions, the relationship is clear-cut *only when* $\mathbf{x}_1$ *and* x_2 *are solutions to a well-behaved ODE sysem (2).* The theorem is:

Theorem 5.3 Wronskian vanishing theorem.

On an interval I *where the entries of* $A(t)$ *are continuous, let* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *be two solutions to (2), and* $W(t)$ *their Wronskian (11). Then either*

a) $W(t) \equiv 0$ *on* I*, and* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *are linearly dependent on* I*, or*

b) $W(t)$ *is never 0 on* I*, and* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *are linearly independent on* I*.*

Proof. Using (12), there are just two possibilities.

a) $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent on I; say $\mathbf{x}_2 = c_1\mathbf{x}_1$. In this case they are dependent at each point of I, and $W(t) \equiv 0$ on I, by (12);

b) $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent on I, in which case by Theorem 5.2A they are linearly independent at each point of I, and so $W(t)$ is never zero on I, by (12). □

Exercises: Section 4E

LS.6 Solution Matrices

In the literature, solutions to linear systems often are expressed using square matrices rather than vectors. You need to get used to the terminology. As before, we state the definitions and results for a 2×2 system, but they generalize immediately to $n \times n$ systems.

1. Fundamental matrices. We return to the system

$$\mathbf{x}' = A(t)\,\mathbf{x}\,, \tag{1}$$

with the general solution

$$\mathbf{x} = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)\,, \tag{2}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are two independent solutions to (1), and c_1 and c_2 are arbitrary constants.

We form the matrix whose columns are the solutions $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$X(t) = (\,\mathbf{x}_1 \quad \mathbf{x}_2\,) = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}. \tag{3}$$

Since the solutions are linearly independent, we called them in LS.5 a *fundamental* set of solutions, and therefore we call the matrix in (3) a **fundamental matrix** for the system (1).

Writing the general solution using X(t). As a first application of $X(t)$, we can use it to write the general solution (2) efficiently. For according to (2), it is

$$\mathbf{x} = c_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c_2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

which becomes using the fundamental matrix

$$\mathbf{x} = X(t)\,\mathbf{c} \qquad \text{where } \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad \textit{(general solution to (1))}. \tag{4}$$

Note that the vector $\mathbf{c}$ must be written on the right, even though the c's are usually written on the left when they are the coefficients of the solutions $\mathbf{x}_i$.

Solving the IVP using X(t). We can now write down the solution to the IVP

$$\mathbf{x}' = A(t)\,\mathbf{x}\,, \qquad \mathbf{x}(t_0) = \mathbf{x}_0. \tag{5}$$

Starting from the general solution (4), we have to choose the $\mathbf{c}$ so that the initial condition in (6) is satisfied. Substituting t_0 into (5) gives us the matrix equation for $\mathbf{c}$:

$$X(t_0)\,\mathbf{c} = \mathbf{x}_0\,.$$

Since the determinant $|X(t_0)|$ is the value at t_0 of the Wronskian of $\mathbf{x}_1$ amd $\mathbf{x}_2$, it is non-zero since the two solutions are linearly independent (Theorem 5.2C). Therefore the inverse matrix exists (by LS.1), and the matrix equation above can be solved for $\mathbf{c}$:

$$\mathbf{c} = X(t_0)^{-1}\mathbf{x}_0\,;$$

using the above value of $\mathbf{c}$ in (4), the solution to the IVP (1) can now be written

$$\mathbf{x} = X(t)X(t_0)^{-1}\mathbf{x}_0\,. \tag{6}$$

Note that when the solution is written in this form, it's "obvious" that $\mathbf{x}(t_0) = \mathbf{x}_0$, i.e., that the initial condition in (5) is satisfied.

An equation for fundamental matrices We have been saying "a" rather than "the" fundamental matrix since the system (1) doesn't have a unique fundamental matrix: thare are many different ways to pick two independent solutions of $\mathbf{x}' = A\mathbf{x}$ to form the columns of X. It is therefore useful to have a way of recognizing a fundamental matrix when you see one. The following theorem is good for this; we'll need it shortly.

Theorem 6.1 *$X(t)$ is a fundamental matrix for the system (1) if its determinant $|X(t)|$ is non-zero and it satisfies the matrix equation*

$$X' = AX\,, \tag{7}$$

where X' means that each entry of X has been differentiated.

Proof. Since $|X| \not\equiv 0$, its columns $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, by section LS.5. And writing $X = (\mathbf{x}_1 \quad \mathbf{x}_2)$, (7) becomes, according to the rules for matrix multiplication,

$$(\mathbf{x}_1' \quad \mathbf{x}_2') = A\,(\mathbf{x}_1 \quad \mathbf{x}_2) = (A\mathbf{x}_1 \quad A\mathbf{x}_2)\,,$$

which shows that

$$\mathbf{x}_1' = Ax_1 \qquad \text{and} \qquad \mathbf{x}_2' = A\mathbf{x}_2\,;$$

this last line says that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to the system (1). □

2. The normalized fundamental matrix.

Is there a "best" choice for fundamental matrix?

There are two common choices, each with its advantages. If the ODE system has constant coefficients, and its eigenvalues are real and distinct, then a natural choice for the fundamental matrix would be the one whose columns are the normal modes — the solutions of the form

$$\mathbf{x}_i = \vec{\alpha}_i e^{\lambda_i t}\,, \qquad i = 1, 2.$$

There is another choice however which is suggested by (6) and which is particularly useful in showing how the solution depends on the initial conditions. Suppose we pick $X(t)$ so that

$$X(t_0) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Referring to the definition (3), this means the solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ are picked so

$$\mathbf{x}_1(t_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_2(t_0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{8$'$}$$

Since the $\mathbf{x}_i(t)$ are uniquely determined by these initial conditions, the fundamental matrix $X(t)$ satisfying (8) is also unique; we give it a name.

Definition 6.2 The unique matrix $\widetilde{X}_{t_0}(t)$ satisfying

$$\widetilde{X}_{t_0}' = A\widetilde{X}_{t_0}\,, \qquad \widetilde{X}_{t_0}(t_0) = I \tag{9}$$

is called the **normalized fundamental matrix** at t_0 for A.

For convenience in use, the definition uses Theorem 6.1 to guarantee $\widetilde{X}_{t_0}$ will actually be a fundamental matrix; the condition $|\widetilde{X}_{t_0}(t)| \neq 0$ in Theorem 6.1 is satisfied, since the definition implies $|\widetilde{X}_{t_0}(t_0)| = 1$.

To keep the notation simple, we will assume in the rest of this section that $t_0 = 0$, as it almost always is; then $\widetilde{X}_0$ is the normalized fundamental matrix. Since $\widetilde{X}_0(0) = I$, we get from (6) the matrix form for the solution to an IVP:

(10) *The solution to the IVP* $\quad \mathbf{x}' = A(t)\,\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \qquad$ *is* $\qquad \mathbf{x}(t) = \widetilde{X}_0(t)\mathbf{x}_0.$

Calculating $\widetilde{X}_0$. One way is to find the two solutions in (8′), and use them as the columns of $\widetilde{X}_0$. This is fine if the two solutions can be determined by inspection.

If not, a simpler method is this: find any fundamental matrix $X(t)$; then

$$\widetilde{X}_0(t) \;=\; X(t)\,X(0)^{-1}. \tag{11}$$

To verify this, we have to see that the matrix on the right of (11) satisfies the two conditions in Definition 6.2. The second is trivial; the first is easy using the rule for matrix differentiation:

If $M = M(t)$ and B, C are constant matrices, then $\;(BM)' = BM', \quad (MC)' = M'C,$

from which we see that since X is a fundamental matrix,

$$(X(t)X(0)^{-1})' = X(t)'X(0)^{-1} = AX(t)X(0)^{-1} = A(X(t)X(0)^{-1}),$$

showing that $X(t)X(0)^{-1}$ also satisfies the first condition in Definition 6.2. □

Example 6.2A Find the solution to the IVP: $\mathbf{x}' \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}\,, \quad \mathbf{x}(0) = \mathbf{x}_0\,.$

Solution Since the system is $x' = y,\;\; y' = -x$, we can find by inspection the fundamental set of solutions satisfying (8′) :

$$\begin{array}{l} x = \;\;\cos t \\ y = -\sin t \end{array} \quad \text{and} \quad \begin{array}{l} x = \sin t \\ y = \cos t \end{array}\,.$$

Thus by (10) the normalized fundamental matrix at 0 and solution to the IVP is

$$\mathbf{x} \;=\; \widetilde{X}\,\mathbf{x}_0 \;=\; \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \;=\; x_0\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + y_0\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\,.$$

Example 6.2B Give the normalized fundamental matrix at 0 for $\mathbf{x}' \;=\; \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \mathbf{x}\,.$

Solution. This time the solutions (8′) cannot be obtained by inspection, so we use the second method. We calculated the normal modes for this sytem at the beginning of LS.2; using them as the columns of a fundamental matrix gives us

$$X(t) \;=\; \begin{pmatrix} 3e^{2t} & -e^{-2t} \\ e^{2t} & e^{-2t} \end{pmatrix}\,.$$

Using (11) and the formula for calculating the inverse matrix given in LS.1, we get

$$X(0) = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}, \qquad X(0)^{-1} = \tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix},$$

so that

$$\widetilde{X}(t) = \tfrac{1}{4}\begin{pmatrix} 3e^{2t} & -e^{-2t} \\ e^{2t} & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} = \tfrac{1}{4}\begin{pmatrix} 3e^{2t}+e^{2t} & 3e^{2t}-3e^{-2t} \\ e^{2t}-e^{-2t} & e^{2t}+3e^{-2t} \end{pmatrix}.$$

6.3 The Exponential matrix.

The work in the preceding section with fundamental matrices was valid for any linear homogeneous square system of ODE's,

$$\mathbf{x}' = A(t)\,\mathbf{x}\,.$$

However, if the system has *constant coefficients*, i.e., the matrix A is a constant matrix, the results are usually expressed by using the exponential matrix, which we now define.

Recall that if x is any real number, then

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\,. \tag{12}$$

Definition 6.3 Given an $n \times n$ constant matrix A, the **exponential matrix** e^A is the $n \times n$ matrix defined by

$$e^A = I + A + \frac{A^2}{2!} + \ldots + \frac{A^n}{n!} + \ldots\,. \tag{13}$$

Each term on the right side of (13) is an $n \times n$ matrix; adding up the ij-th entry of each of these matrices gives you an infinite series whose sum is the ij-th entry of e^A. (The series always converges.)

In the applications, an independent variable t is usually included:

$$e^{At} = I + At + A^2\,\frac{t^2}{2!} + \ldots + A^n\,\frac{t^n}{n!} + \ldots\,. \tag{14}$$

This is not a new definition, it's just (13) above applied to the matrix At in which every element of A has been multiplied by t, since for example

$$(At)^2 = At \cdot At = A \cdot A \cdot t^2 = A^2 t^2.$$

Try out (13) and (14) on these two examples; the first is worked out in your book (Example 2, p. 417); the second is easy, since it is not an infinite series.

Example 6.3A Let $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, show: $e^A = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$; $e^{At} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}$

Example 6.3B Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, show: $e^A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; $e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

What's the point of the exponential matrix? The answer is given by the theorem below, which says that the exponential matrix provides a royal road to the solution of a square system with constant coefficients: no eigenvectors, no eigenvalues, you just write down the answer!

Theorem 6.3 *Let A be a square constant matrix. Then*

(15) *(a)* $e^{At} = \widetilde{X}_0(t)$, *the normalized fundamental matrix at 0;*

(16) *(b) the unique solution to the IVP* $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ *is* $\mathbf{x} = e^{At}\mathbf{x}_0$.

Proof. Statement (16) follows immediately from (15), in view of (10).

We prove (15) is true, by using the description of a normalized fundamental matrix given in Definition 6.2: letting $X = e^{At}$, we must show $X' = AX$ and $X(0) = I$.

The second of these follows from substituting $t = 0$ into the infinite series definition (14) for e^{At}.

To show $X' = AX$, we assume that we can differentiate the series (14) term-by-term; then we have for the individual terms

$$\frac{d}{dt} A^n \frac{t^n}{n!} = A^n \cdot \frac{t^{n-1}}{(n-1)!},$$

since A^n is a constant matrix. Differentiating (14) term-by-term then gives

$$(18) \qquad \begin{aligned} \frac{dX}{dt} = \frac{d}{dt}e^{At} &= A + A^2 t + \ldots + A^n \frac{t^{n-1}}{(n-1)!} + \ldots \\ &= A e^{At} = AX\,. \end{aligned}$$

Calculation of e^{At}.

The main use of the exponential matrix is in (16) — writing down explicitly the solution to an IVP. If e^{At} has to be actually calculated for a specific system, several techniques are available.

a) In simple cases, it can be calculated directly as an infinite series of matrices.

b) It can always be calculated, according to Theorem 6.3, as the normalized fundamental matrix $\widetilde{X}_0(t)$, using (11): $\widetilde{X}_0(t) = X(t)X(0)^{-1}$.

c) A third technique uses the exponential law

$$(19) \qquad e^{(B+C)t} = e^{Bt}e^{Ct}, \qquad \textit{valid if } BC = CB.$$

To use it, one looks for constant matrices B and C such that

$$(20) \qquad A = B + C, \qquad BC = CB, \qquad e^{Bt} \text{ and } e^{Ct} \text{ are computable;}$$

then

$$(21) \qquad e^{At} = e^{Bt}e^{Ct}.$$

Example 6.3C Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Solve $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, using e^{At}.

Solution. We set $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; then (20) is satisfied, and

$$e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

by (21) and Examples 6.3A and 6.3B. Therefore, by (16), we get

$$\mathbf{x} = e^{At}\mathbf{x}_0 = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = e^{2t} \begin{pmatrix} 1+2t \\ 2 \end{pmatrix}.$$

Exercises: Sections 4G,H

18.03 Lecture 35

12/12

Review

Exam next Tue!
Dec 20
9 - noon
Johnson

Today: Exam Review: Unit 1 + 2

Strategies: Exam counts for a lot
Can change entire trajectory of class
Trying to emphasize your best score
(: individual weighting of scores???)
It's not easy to get an A on the final
Hardest part: which technique to use
- ~~exam~~ ~~...~~ sometimes told which
- but usually you pick
Study: Go back over old exams
He posted more pratice tests
But our old exams better

2

Today: Unit 1 mostly
Unit 2 - everyone did well
~~and methods subsumed by LaPlace~~

Reminder: Not all ~~equatio~~ ODEs involve const. coeff linear eqns

Example

$$\frac{dy}{dx} = -\overset{\text{non linear}}{y^3} + \overset{\text{oth deriv}}{3y} + \overset{x}{x}$$

Can't use Laplace or Systems
Since $y^3 \to$ non Linear ODE

Could ask:
- long term behavior of solutions w/ certain IV
  ie $y(0) = \frac{3}{2}$

use (direction fields / fences + funnels)

- Approx soln's to IVP using Eulers
  - no improvements to it on exam
- But not an exact answer!!
  - Since non linear ODE

③

- But could use power series if you had to
- If periodic → Fourier Series

So try long term / fences + funnels for

$$\frac{dy}{dx} = -y^3 + 3y + x \qquad y(0) = \frac{3}{2}$$

<u>Answer</u> curve $C(x)$

$C'(x) < f(x, C(x))$ lower fence

$\quad$ " $>$ " $\quad$ upper fence

To remember

$C'(x) = 0$

If $C'(x) < f(x, C(x))$

lines have ⊕ slope

lower

pushes lines back up

(4)

upper

Asked to make LHS or RHS easy

Choose a curve to make LHS or RHS easy

$$f(x,y) = -y^3 + 3y + x$$

Usually choose 0-isocline

$\llcorner$ set function = to 0

$$0 = -y^3 + 3y + x$$

Seperate variables

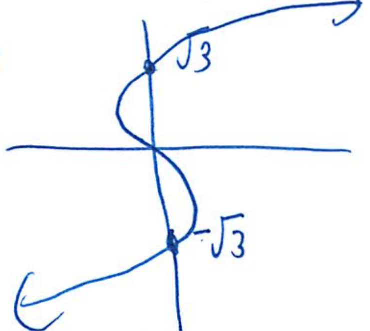
$$x = y^3 - 3y$$

Now must actually draw

Try to factor

$$x = (y)(y^2 - 3)$$

Draw

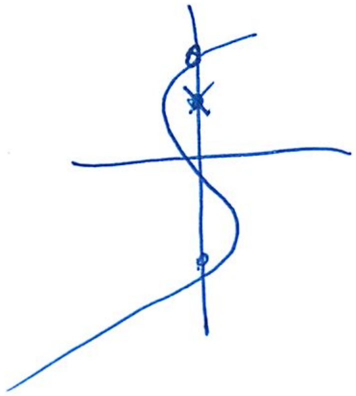


Since cubic

(5)

We are hoping this is a fence

Our point is $(0, \frac{3}{2})$



We are hopping is an upper fense

L Then just need to find a lower fence

Look at both sides of inequality. See what is bigger. RHS $= 0$

LHS $=$ (+) since function ↑ when > 0

Need to prove upper fence. Check

$$c'(x) > 0$$

Hard to solve ~~ov~~ y as fn (x)

So need implicit differentiation

L take deriv of both sides w/r. to x

(6)

$$1 = 3y^2 \frac{dy}{dx} - 3\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{1}{y^2-1}\right)$$

Now argue if $(x > 0)$ ie (true for curve to right of $(0, \sqrt{3})$)

then $\frac{1}{3}\left(\frac{1}{y^2-1}\right) > 0$

So $y > 1$

✓ upper fence

Now need to find long term behavior

Will be hard to find a asymptotic curve to our weird $\frac{1}{3}$ curve

?

→ So need to pick another isocline for a <u>lower fence</u>

want to take our S-curve - shift to right

So point is between curves

So have same behavior

(7)

So 1-isocline
└ shift to up/right by 1

Check $C'(x)$ vs $f(x, C(x)) = 1$

Can check
Same explicit differenitation as before
The 1 constant dies

$\frac{1}{3}\left(\frac{1}{\text{something}} < 1\right)$

So ✓ lower fence

At $x = \infty$ the $+1$ does not matter
the cubed term dominates

Rule of $\lim\left(\frac{\text{one}}{\text{over}}\right)$ → highest power dominates

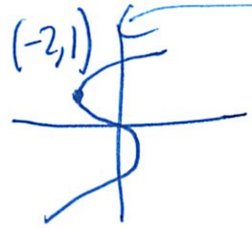
So N-isocline for any possible $n > 0$ are asymptotic

So far 8-9/10 points

8

Then if time, try to worry if inner section is $< \frac{3}{2}$
L finding which isocline ??

Could use calculus w/ head sideways to know



So use 2-isocline
Would be guaranteed to include $(0, \frac{3}{2})$
bounds initial condition
1-isocline may work - he didn't check
Prof Brubaker

Antifunnels There is 1 solution
Only do it if said prove that only
1 sol or just prove solution

There will be a fenses + funnels qu like Exam 1 #4
w/ Eulers method next

Won't cover Eulers method today

9

Modeling

hard since it depends what you model

4 possible first order problems.

1. Exponential ~~dk~~ growth + decay
   - easy, can seperate

2. Population / Logistical (Harvesting) models
   - optional addition

~~apP-P~~

$aP - bP^2$

a natural growth away from limiting pop

$\frac{a}{b}$ = limiting population

bifurcation ~~graph~~ diagrams — slope fields on line

- not dependent on time

10

3. Mixing problems (w/ salt)

Trick $\frac{d\,\text{Salt}}{dt}$ = rate of salt in − rate of Salt out

example

Fish tank w/ salt-water fish

Salinity has dropped to 30 g/L

(natural 35 g/L − wikipedia)

So need to fix this problem quickly

Can only pump in saline mixture

Plan pump 40 g/L salt water at rate of 20 L/min

Rapidly mix

Empty 20 L/min

10,000 L tank

Find time t at which back to 35 g/L

(11)

$$\frac{d\,Salt}{dt} = \underbrace{40\,g/L \cdot 20\,L/min}_{in} - \underbrace{\frac{S(t)}{10{,}000} \cdot 20\,L/min}_{out}$$

concentration of salt at time t

← why we do diff eq

divide for concentration

units g/min

* check units to confirm

Want 350 g of salt

Move all to denom

$$\frac{1}{Const - S}$$

Solve by seperation of variables

$$\sim \log \frac{1}{cont - S}$$

$$\int \frac{dS}{800 - \frac{S}{500}} = \int dt$$

do integral

4. Newton's Law of Cooling

$$\frac{dT}{dt} = k(\underbrace{T_e}_{\text{exterior tem}} - \underbrace{T}_{\text{temp of object}})$$

lead to linear const. coeff ODEs

$$\frac{dT}{dt} + kT = k\underbrace{T_e}_{\text{sinusoidal usually}}$$

So can complexify + solve ODE

this we can solve 7-8 ways
- LaPlace
- Guessing
- Complexify
- Integrating Factor
- etc

Ex Integrating Factor w/ non-constant linear coefficents

- last technique from 1st section

$$x^3\frac{dy}{dx} + x^2 y = \cos x$$

Is this linear? Yes

Constant? No - so can't seperate

(13)

Int. factor works well for

$$1\frac{dy}{dx} + P(x)\, y = Q(x)$$

So can rewrite

$$\frac{d}{dx}\left(y \cdot e^{\int P(x)dx}\right)$$

$$= Q(x)\, e^{\int P(x)dx}$$

Divide by x^3

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\cos x}{x^3}$$

So IF

$P(x) = \frac{1}{x}$ $\quad \cdot e^{\int P(x)\,dx}$

$= e^{\ln|x|}$ abs val

but only $x \geq 0$

$= e^{\ln x}$

$= x$

(14)

So y

$$y = \frac{1}{x}\int \frac{\cos x}{x^3}\, x \, dx$$

$$= \frac{1}{x}\int \frac{\cos x}{x^2}\, dx$$

leave it as like that

L for non constant coefficients

More stuff in notes

Lecture 35 Exam Review 12/12

So much with linear ODEs w/ const. coeffs. Want to remind you that there are other examples, especially w/ first-order ODEs, with their own methods.

Example 1: $\dfrac{dy}{dx} = -y^3 + 3y + x$

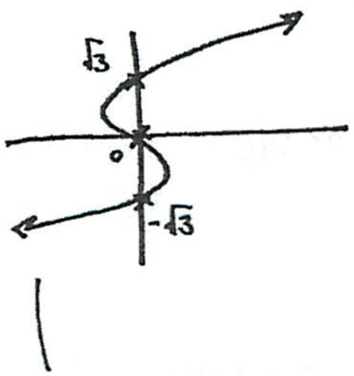
- Ask for solutions' long term behavior. ← Use direction field / isoclines
- Ask for numeric approximation using Euler's method.
- Why no exact answer? Non-linear, so few methods. Power series.

Q: Are there any solutions with initial condition $(0, a)$ for some $a > 0$ s.t. long-term behavior stays between $y = 0$ and $y = x$? Why or why not?

or what is long term behavior of solution through $(0, 3/2)$? Prove your answer is correct.

0-isocline: $x = y^3 - 3y$
$= y(y^2 - 3)$

Recall, a curve $C(x)$ is
lower fence if $C'(x) < f(x, C(x))$
upper fence if $C'(x) > f(x, C(x))$



0-isocline

on 0-isocline, $f(x, C(x)) = 0$.

To find $C'(x)$: $1 = 3y^2 \dfrac{dy}{dx} - 3\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{1}{3}\left(\dfrac{1}{y^2 - 1}\right) > 0$ if $y > 1$

(or just note increasing) so have upper fence.

1-iso-cline : $x = y^3 - 3y + 1$ then $f(x, C(x)) = 1$ but

$C'(x)$ is same. Gives lower fence. Do they bound $(0, 3/2)$ on both

sides? Better to consider 2-isocline for this. Mention antifunnels.

Euler's method : step size 1

$$f(0, 3/2) = -(3/2)^3 + 3\cdot(3/2) + 0$$
$$= -27/8 + \underbrace{9/2}_{36/8} = 9/8$$

$$(0, 3/2) \longrightarrow (1, 3/2 + 1\cdot f(0, 3/2))$$
$$(1, 3/2 + 9/8) = (1, 21/8)$$

To go further: $(1, 21/8) \longmapsto (2, 21/8 + 1\cdot f(1, 21/8)) \ldots$ overestimation since first deriv decreases.

will get messy.

Expect to see question just like this on final exam. (Similar to Exam 1, Q.4)

Modeling — (Exponential growth/decay (separable), Population Models (logistical model with harvesting),

---

bifurcation diagrams

$$aP - bP^2 = aP(1 - \tfrac{b}{a}P)$$

then a : natural growth rate

$\frac{a}{b}$: limit.

Mixing problems: Concentration In v. Out

Ex. Fish tank with salt-water fish. Pump in:

(35 g/L is salinity of salt water). 20 L/min, 40 g/L to get back to 35 g/L.

in 10000 L tank. w/ 30 g/L salinity. Leaving at 20 L/min.

Salinity in 10,000 L saltwater fish tank has fallen to 30 g/L. To fix, pump in 40 g/L saltwater at 20 L/min, with 20 L/min of well-mixed water leaving tank.

How long does it take to restore salinity to 35 g/L.

rate of salt entering: $40\ g/L \cdot 20\ L/min = 800\ g/min$

rate of salt leaving: $20\ L/min \cdot S(t)\ /\ 10{,}000\ g/L$

$S'(t)$: rate of salt increase at time t $= 800\ g/min - S(t)/500\ g/min$

Solve using separation of vars. $\underbrace{\int \frac{S'}{800 - \frac{S}{500}}\, dS}_{} = t + c$

$-500 \ln\left(800 - \frac{S}{500}\right)$

$800 - \frac{S}{500} = e^{-t/500 + c}$

$S = (800 - e^{-t/500 + c})500$

...

Newton's Law of Cooling:

$\frac{dT}{dt} = k(T - Te)$ ... linear ODE w/ const. coeff.

Integrating factor Q. $\frac{x^2}{y}\frac{dy}{dx} + 2x = 0$

clean this up by mult. by y: $x^2 \frac{dy}{dx} + 2xy = 0$

or IVP: $y - (x+y)\frac{dy}{dx} = 0 \quad y(1) = 1.$

General Form: $\frac{dy}{dx} + P(x)\,y = Q(x)$ int. factor: $e^{\int P(x)\,dx}$

For IVP:

Integrating factor example:

$$x^3 \frac{dy}{dx} + x^2 y = \cos x \qquad y(1) = 4$$

Ans. divide by x^3 to put in the form: $\frac{dy}{dx} + P(x) y = Q(x)$

$$\frac{dy}{dx} + \frac{1}{x} \cdot y = \frac{\cos x}{x^3}$$

Then $p(x) = \frac{1}{x}$ so int. factor: $e^{\int P(x)\,dx} = x$

(initial condition at $x = 1$ where anti-deriv of $\frac{1}{x}$ can be taken to be $\ln x$ not $\ln |x|$.)

Then

Soln: $y = \frac{1}{x}\left(4 + \int_1^x \frac{\cos t}{t^2}\, dt\right)$ Problem: $\frac{1}{x}$ not defined at 0.

nicer way to do integrating factors in IVP.

Other methods: Substitution (changing a variable to simplify ODE), exactness (checking partial derivs to see if ODE solvable) by prod. formula

won't be covered on exam

Theory: Give an example of an initial value problem with two solutions through the initial point.

Ex.: $y' = 2\sqrt{y}$ has solutions $y = x^2$, $y = 0$, through $(0,0)$

18.03 Recitation

12/13

Last Recitation

No office hours here on for him

Will give P-Sets back in his office afterwards

Taking QnA

A question for us

X, Y - Which is which?

$$A^{-1}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}A$$

$$B^{-1}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}B$$

Could compute Jordan form but long & hard

Or could square the maticies

(2)

$$A^{-1}\begin{pmatrix} 0 & 1 & & & & & & \\ & 0 & 1 & & & & & \\ & & 0 & & & & & \\ & & & 0 & 1 & & & \\ & & & & 0 & & & \\ & & & & & 0 & 1 & \\ & & & & & & 0 & 1 \\ & & & & & & & 0 \end{pmatrix} A = X$$

$X, \; X^2, \; X^3, \; X^4, \; \ldots$

anilating 3 vectors

anilates 6 vectors

(# of blocks +
# of blocks of size at least 2)

- everyone gets annilated

3 blocks

6 blocks + blocks size ≥ 2

8 " " " ≥ 2

+ " " ≥ 3

(3)

Can solve for # of blocks at least 2

General story

Fact: (Jordan Canonical Form)

$$X = A^{-1}\begin{pmatrix} \lambda & 1 & & & & & \\ & \lambda & & & & & \\ & & \mu & 1 & & & \\ & & & \mu & 1 & & \\ & & & & \mu & & \\ & & & & & \nu & 1 \\ & & & & & & \nu \end{pmatrix} A$$

everything else is 0

diagonal entries are eigenvalues of X

$X - \mu$

No block sitting anywhere else except μ

Gives you info about the block corresponding to μ

Broken down to

$$A^{-1}\begin{pmatrix} \mu & 1 & & & & & & \\ & \mu & 1 & & & & & \\ & & \mu & & & & & \\ & & & \mu & 1 & & & \\ & & & & \mu & & & \\ & & & & & \mu & 1 & \\ & & & & & & \mu & 1 \\ & & & & & & & \mu \end{pmatrix} A^{-1}$$

4

$\boxed{\text{(I have no idea what is going on here)}}$

But how to find the matrix A?

Need some linear algebra to know

$$A^{-1}\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} A = X - \mu$$

What does this do. It does

$$\begin{tikzcd}
0 \arrow[d] & \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \arrow[l] \arrow[d, shift right, "A^{-1}"'] & \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} \arrow[l] \arrow[d, shift right, "A^{-1}"'] & \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} \arrow[l] \arrow[d, shift right, "A^{-1}"'] & \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} \arrow[l] \arrow[d, shift right, "A^{-1}"'] & \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \arrow[l] \arrow[d, shift right, "A^{-1}"'] \\
0 & (X-\mu)^4 v \arrow[l] \arrow[u, shift right, "A"'] & (X-\mu)^3 v \arrow[l] \arrow[u, shift right, "A"'] & (X-\mu)^2 v \arrow[l] \arrow[u, shift right, "A"'] & (X-\mu) v \arrow[l] \arrow[u, shift right, "A"'] & v \arrow[l] \arrow[u, shift right, "A"']
\end{tikzcd}$$

So need any vector so $(X-\mu)^4$ does not kill v

If guess any v at all, chances are it will work

If not, try again

5

So what do you do when you have that method?

$$A^{-1} = \left((X-\lambda)^4 v, (X-\lambda)^3 v, (X-\lambda)^2 v, (X-\lambda) v, v \right)$$

What does this do / say?

It says this is what A^{-1} is
└ the prescription

Invert it to get A

Prescription for how to recover A

Why did we do all this?

Since we are exponentiating matrices to solve diff eq

$$e^{Xt} = A^{-1} e^{\Delta t} A$$

Can exponentiate block by block
└ Did this last time

6

$$e^{\underbrace{\left(\lambda I + \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}\right)}t} =$$

$$= e^{\lambda t} \cdot e^{\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} t}$$

$$= \begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} \\ & 1 & t & \frac{t^2}{2} \\ & & 1 & t \\ & & & 1 \end{pmatrix}$$

If you write things out like we did here you can get the complete solution

* All this was a way you could solve really big diff eq.
This is what the computer is doing in the bg when you ask it to solve.

7

Q: Is there a more systematic way to solve fences + funnels?

No.

Its helpful to think about isoclines
But if that is not there you need to guess functions.

(recitation over 20 min early)

18.03 Lecture 36

Review 2

12/14

(last lecture)

Last time: was doing integrating factors (IV problem)

that class ~lecture 7 and last lecture

tip: look at notes for how to handle IV problems w/ integrating factors

$\left(e^{\int P(x)\,dx}\right)$ instead $\int_1^x P(r)\,dr$

? dummy variable

Today: Examples of Higher Order Equations

When to use what:

Friday Babaler OH	12-2
Monday "	2-4

Hope comfortable solving Homogeneous Eqn

Will take 1 eqn to build on

$$y'' + y = 0$$

$$(D^2+1)y = 0$$

2

<u>charastic eq</u> $r^2+1=0$

<u>roots</u> $r=\pm i$

So $y=\pm e^{it}$ complex sol

$y=\cos t+i\sin t$ real sol

for e^{it}

Both Re() and Im() part are solutions

By superposition/ linearity, <u>general ans</u>

$y=c_1\cos t+c_2\sin t$

(This will appear 4-5 times on the exam)

<u>Inital Value Problem</u>

<u>eg</u> $y(0)=2$

could solve, but not unique

Need 2nd data pt

$y'(0)=5$

(3)

Option 1 Solve for c_1, c_2 in general eqn

Option 2 Could solve w/ LaPlace Transform
Lesp of 0
(formula on exam
(all of the same formulas
are still there)

$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$$
$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - s\,y(0) - y'(0)$$

Both use the charistic eqn
LaPlace would be slower

Inhomogeneous Equations

Input → before it was nothin
now e^{5t}

So $y'' + y = e^{5t}$

General sol $y =$ homogeneous sol + particular soln

(4)

exponential input formula

Method 1 Could use ~~exponential response formula~~ $\exists$ also

└ he thinks its the fastest way

$$y = \frac{1}{p(a)} e^{at} \quad \text{if input} = e^{at}$$

Correct ~~unless~~ ~~use~~ as long as $p(a) \neq 0$

~~Where~~ $P(r)$ = charistic polynomial

a = value of input

For input e^{5t}

response: $y = \frac{1}{a^2+1} e^{5t}$

$$= \frac{1}{5^2+1} e^{5t}$$

$$= \frac{1}{26} e^{5t}$$

Method 2 Could also do w/ guessing

└ guess a ~~potential~~ partial soltion which is made of all possible derivs including fn itself on RHS

5

So <u>guess</u> $c \cdot e^{st}$

└ works well ~~as~~ as long as RHS does not coincide w/ roots of charistic eq'n

Stick into diff eq'n and solve for c

└ $\boxed{\text{ODE for } y}$

In general <u>erf</u> (exponential response formula)

$$y = \frac{t^m e^{at}}{p^{(m)}(a)}$$

m = multiplicity of the root in p

It also works for complex

(exponential response method)

$e^{at} \cos bt$

$e^{at} \sin bt$

↑ a could be 0

Great way to handle sinusodial response formulas

└ Make sure remember if taking Re() or Im()

(6)

Different Input $t^2 \sin t$

Can't use exponential here!

Only guessing or LaPlace

Guessing

include homogeneous part we found earlier

All the derividives

$t^2 \sin t \qquad t^2 \cos t$

$t \sin t \qquad t \cos t$

$\sin t \qquad \cos t$

6 fns to guess

$c_1 t^2 \sin t + c_2 t^2 \cos t$

$+ c_1 t \sin t + c_2 t \cos t$

$+ c_1 \sin t + c_2 \cos t$

(would be simpler on test)

7

Can we conclude something about result

Product is odd

So remove all but odd fuctions

$$C_1 t^2 \sin t + C_4 t\cos t + C_5 \sin t$$

<u>So</u> Sometimes can use symmetry

<u>But</u> $\sin t$ is part of homogeneous sol!

↳ So won't give you anything new to add it to the eqn

<u>So</u> add another one

New guess:

$t^3 \sin t$, $t^3 \cos t$

$\vdots$ $\vdots$

$t \sin t$ $t \cos t$

bumped every sol up by $t^{(1)}$ since $m=1$ of root i appearing in char. polynomial

Still want odd fns

$$c_2 t^3 \cos t + c_3 t^2 \sin t + c_6 t \cos t$$

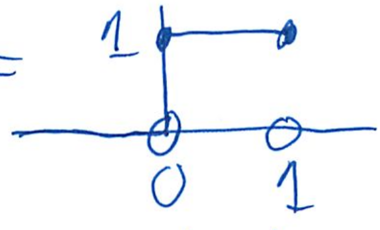
Now solve
3 eq
3 unknowns

<u>Laplace</u> Could do it
But not recommended

$$\mathcal{L}(t^2 \sin t) = \text{ugly}$$

$$\frac{d^2}{ds^2} \frac{1}{s^2+1} = \text{quotient rule} :($$

<u>Input #3</u> $b(t) =$



Now forced to use Laplace
(since discontinuous

Assume rest conditions $y(0) = y'(0) = 0$

$$y'' + y = b(t)$$

(9)

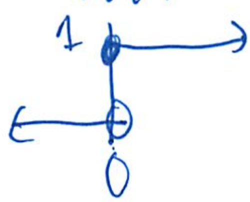
Step 1 Take LaPlace transform of both sides

LHS $(s^2+1)\mathcal{L}(y)$

no lower order terms
since rest conditions

RHS $\mathcal{L}(b(t))$

Remember $U_0(t)$ unit step function



$b(t) = u_0(t) - u_1(t)$

$= u(t) - u(t-1)$ diff notation

Table $\mathcal{L}(u_a(t)) = \dfrac{e^{-as}}{s}$

$$\mathcal{L}(b(t)) = \frac{1}{s} - \frac{e^{-as}}{s}$$

(10)

$$\mathcal{L}(y) = \frac{1}{s(s^2+1)} - \frac{e^{-as}}{s(s^2+1)} \quad a=1$$

Could combine numerators

└ but don't !

e^{-as} causes shift in t when taking $\mathcal{L}^{-1}$

$$= \frac{1}{s(s^2+1)} - e^{-s}\left(\frac{1}{s(s^2+1)}\right)$$

Finish w/ partial fractions

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

irreducable quadatic

Use cover up method

Plug in complex roots to get ans

$A = 1$

$B = -1$

$C = 0$

11

$$= \frac{1}{s} - \frac{s}{s^2+1}$$

normally $\frac{B(s-a)+C}{(s-a)^2+z^2}$

a was 0

(confused - study)

Take inverse LaPlace transforms

$$\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) =$$

$$= 1 - \cos t$$

Final solution

$$Y = 1 - \cos t - u(t-1)(1-\cos(t-1))$$

every variable t shifted by 1!

(12)

Input 4 Triangle wave function w/ period 2π

└ Fourier since periodic

(see notes for it done)

Input 5 $y'' + y = x$ $y'(0) = y'(\pi) = 0$

↑ Could have 1 qu exam w/ 15 parts

$x \in [0, \pi]$

↑ satisfies certain end point conditions

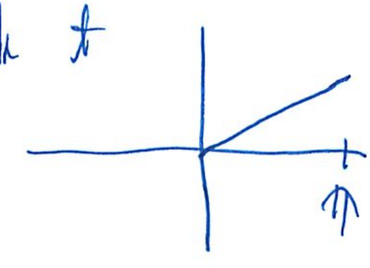
Use Fourier w/ cosin

So sol has cos-like-properties

aka Fourier cosine series

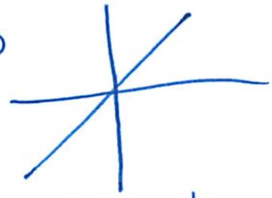
So use Fourier techniques to write fn as Fourier w/ proper end point conditions

Think x



13

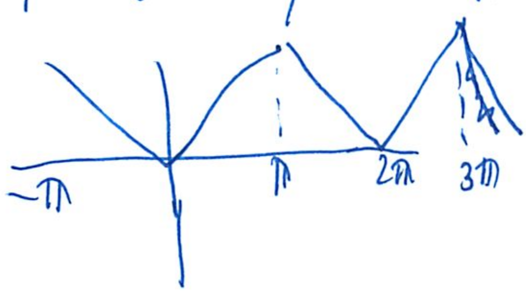
You want to



← but sine series

(common mistake on Exam 3)

But you actually want



A few more things to think about

1. Heat eqn or endpoint eq
   - same as previous qu
   - find constants

2. RLC circuits or spring mass dashpot
   - resonance
   - pratical resonance
   - amplitudes

Lecture 36 12/4

Examples / Methods of Higher Order Equations

Homogeneous equations: $y'' + y = 0 \quad ((D^2+1)y = 0)$

if general sol'n: characteristic equ.: $r^2+1 = 0 \qquad (r-i)(r+i) = 0$

cx soln $e^{\pm it}$. Taking Re + Cx parts gives solutions to original homog. equ.

$\cos t,\ \sin t$

General sol'n (using superposition): $y(t) = c_1 \cdot \cos t + c_2 \sin t$.

if initial value problem: $y(0) = 2,\ y'(0) = 5$ then two options:

- Solve for c_1, c_2 (✓, very simple here)
- Laplace transform $\mathcal{L}(y'') = s^2 \mathcal{L}(y) - s\,y(0) - y'(0)$

$\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$

(will be provided same table as on Exam 3)

With external inputs (inhomogeneous): $y''+y = e^{5t}$

if general + input is exponential → exp. response formula

→ guessing (doing diff. op that kills RHS)

(guess function and all its derivatives)

↓ Laplace trans.

E.R.F.: $y_p = \dfrac{1}{p(a)} e^{at}$ + add in homogeneous solution.

if RHS $= e^{at}$, and $p(a) \neq 0$.

more generally: $y_p = \dfrac{t^m e^{at}}{p^{(m)}(a)}$ m: mult. of root a in p(r): char. poly

In our example

$a = 5$,

$p(r) = r^2+1$

so $p(a) = 26$

By guessing: Guess $k \cdot e^{5t}$ some const k. Input equation into to solve for k.

<u>Input #2</u>: $t^2 \sin t$. (ERF out, since this is not exponential. $\sin t = \mathrm{Im}(e^{it})$ but no way to get $t^2 \sin t$)

<u>Guessing</u>: All derivatives of

$t^2 \sin t \rightsquigarrow$ $t^2 \sin t$, $t^2 \cos t$
$t \sin t$, $t \cos t$
$\sin t$, $\cos t$

already in homog. sol'n.

use <u>instead</u>: $t^3 \sin t$, $t^3 \cos t$
$\vdots$
$t \sin t$, $t \cos t$

Solve lots of equations.

<u>Laplace transform</u> (?)

$$\mathcal{L}(t^2 \sin t) = -(\underbrace{\mathcal{L}(t \sin t)}_{-\mathcal{L}(\sin t)'})' \qquad ': \frac{d}{ds}$$

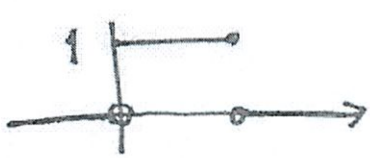
$$\mathcal{L}(\sin t) = (s^2+1)^{-1}$$

$$\frac{d}{ds}(s^2+1)^{-1} = -(s^2+1)^{-2} \cdot 2s$$

$$\frac{d}{ds}\left(\frac{2s}{(s^2+1)^2}\right) = \cdots$$

(starting to get rather messy. Will have ugly partial fractions to handle at end)

<u>Input #3</u>: $b(t)$:



<u>ODE</u>: $y'' + y = b(t)$

$b(t) = 1 - u_1(t) = 1 - u(t-1)$ where $u(t)$: unit step function

<u>Must use Laplace</u>: $\mathcal{L}(b(t)) = \mathcal{L}(1 - u_1(t))$

$$= \frac{1}{s} - \frac{e^{-s}}{s}$$

If rest conditions ($y(0) = y'(0) = 0$) then have:

$$\mathcal{L}(y'' + y) = (s^2+1)\,\mathcal{L}(y).$$

so after taking Laplace of both sides:

$$(s^2+1)\,\mathcal{L}(y) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\mathcal{L}(y) = \frac{1}{s\cdot(s^2+1)} - \frac{e^{-s}}{s(s^2+1)}$$

Take inverse transforms separately.

$e^{-s}\cdot\frac{1}{s\cdot(s^2+1)}$ will have e^{-s} acting as shift in time.

Take care of it at end.

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \underbrace{\frac{Bs+C}{s^2+1}}$$

Better in general: $B(s - \mathrm{Re}(\alpha))$

But here $\mathrm{Re}(\alpha) = 0$.

where α: cx. root of quadratic factor

Use cx. cover up method

$(s-0)$: $\frac{1}{s^2+1} = A + \ldots$ → 0 as $s = 0$

so $A = \frac{1}{0^2+1} = 1.$

B, C : $\frac{1}{s} = \ldots + Bs + C$ → 0 as $s = i$

$$\underbrace{\frac{1}{i}}_{-i} = Bi + C$$

$B = -1,\ C = 0.$

Get $\frac{1}{s\cdot(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1, \quad \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos t$$

$$y = (1 - \cos t) - u(t-1)\cdot(1 - \cos(t-1))$$

Input #4 : $Sq(t)$: square wave function with period 1.

$Tr(t)$: triangle wave function with period 2π

Example #5:

Boundary value problem.

$y'' + y = t$

$y'(0) = y'(\pi) = 0.$

18.03 Differential Equations

Grade Report

Grade Report for Michael E. Plasmeier

Assignment/Exam Name	Graph	Due Date	Points	Max Pts
Homework 1		09.16.2011	34.00	49.00
Homework 2		09.23.2011	33.00	41.00
Exam 1		09.28.2011	19.00	41.00
Homework 3		10.07.2011	53.00	66.00
Homework 4		10.14.2011	8.00	30.00
Homework 5		10.21.2011	37.50	55.00
Exam 2		10.26.2011	20.00	36.00
Homework 6		11.07.2011	40.50	59.00
Homework 7		11.13.2011	20.50	23.00
Homework 8		11.18.2011	44.00	58.00
Exam 3		11.28.2011	20.00	41.00
Homework 9		12.02.2011	25.00	33.00
Homework 10		12.09.2011		49.00
		CUMULATIVE SCORE	354.50	

Instructor's Comments

Missing PS10

Final