18.03 FINAL EXAM

Tuesday, December 20, 2011

Name: Michael Plasmeier

Recitation Instructor (Circle One): L. Chumakova / N. Olver / R. Rosales / V. Shende / J. van Ekeren / Z. Yun

Recitation Hour: #10

Instructions: You may not use calculators, notes, textbooks, or personal electronic devices. As a courtesy to other students, please turn off all cell phones. Read each question carefully. Whenever possible, include justification for your reasoning and show your work. Answers without any work or explanation will receive little or no credit. There are 10 questions and a 3 hour time limit on this exam. Good luck.

Question	Score	Maximum
1	6	10
2	12	15
3	8	9
4	7+1	28
5	3	7
6	3	11
7	2	10
8	6	10
9	8	12
10	1.5	10
Total	57.5	126

 Name: Plasmeier

1. a) (5 pts) Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{2y-2}{x^2}$$

$$X^2 \frac{dy}{dx} = 2y - 2$$

$$\frac{1}{2y-2}\,dy = \frac{1}{x^2}\,dx$$

$$\int (2y-2)^{-1} \qquad \int$$

2

$$\ln(2y-2) \cdot \cancel{\frac{2y^2}{2} - 2y} = \frac{1}{x} + C$$

$$(2y-2)\cdot e^{y^2} \cdot e^{-2y} = \frac{1}{x} + C$$

implicit sol??

b) (5 pts) Use an integrating factor to find the solution (in terms of a definite integral) to the initial value problem

$$\frac{dy}{dx} + y = \frac{1}{1+x^2}, \qquad y(2) = 3$$

Still need

$$P(x) = 1 \qquad Q(x) = \frac{1}{1+x^2}$$

$$e^{\int P(x)dx} = e^{\int 1 dx} = e^x \checkmark$$

4

$$y \cdot e^x = \int_0^2 \frac{1}{1+x^2} e^x \, dx$$

$$y = \frac{1}{e^x}\left(e^x \ln(1+x^2)\frac{x^3}{3} - \int_0^2 \ln(1+x^2)\frac{x^3}{3} e^x\right)$$

$$\int u\,dv = uv - \int v\,du$$

$u = e^x$ $\quad dv = \frac{1}{1+x^2}$

$du = e^x$ $\quad v =$

$$\int (1+x^2)^{-1}$$

$$\ln(1+x^2) \cdot \frac{x^3}{3}$$

$$y = \ln(1+x^2)\frac{x^3}{3}$$

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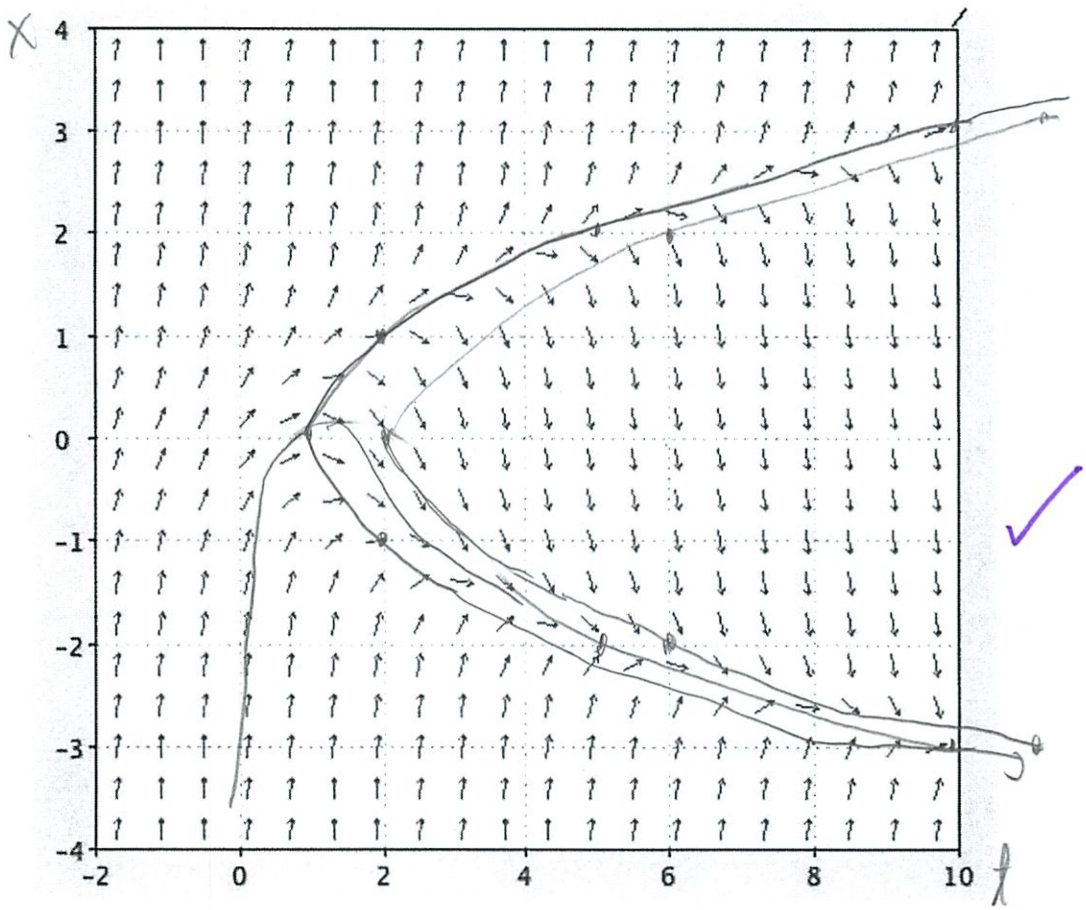
2. The differential equation $dx/dt = x^2 - t + 1$ has the following direction field:



(a) (4 pts) Write the equations for the 0 and -1 isoclines. Then sketch them on the direction field.

0-isocline

$0 = x^2 - t + 1$

When is this true?

$x^2 = t - 1$

$x = \pm\sqrt{t-1}$

$t > 1$

-1 isocline

$-1 = x^2 - t + 1$ ✓

$x^2 = t - 2$

$x = \pm\sqrt{t-2}$

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(b) (4 pts) Sketch the solution $x(t)$ passing through the point $(t, x) = (1, 0)$ on the direction field above. Does this solution have $x(100) < -50$? Explain your answer using fences.

Well we can see $x = \sqrt{t-1}$ is a lower fence
$x = -\sqrt{t-2}$ is an upper fence
So the answer is pretty much $-\sqrt{t}$ which is $-\sqrt{100} = -10$ not lest than -50

prove (-2)

→ Over

(c) (4 pts) Use Euler's method with step size 1/2 to approximate the value of $x(2)$ where $x(t)$ is the solution passing through $(t, x) = (1, 0)$.

$x^2 - t + 1$

1	0
1.5	$0 + \frac{1}{2} \cdot [(0)^2 - (1) + 1] = 0$
2	$0 + \frac{1}{2} \cdot [(0)^2 - (1.5) + 1] = -\frac{1}{4}$

✓

(d) (3 pts) Do you expect your answer in (c) is more or less than the actual value of $x(2)$? Explain.

My answer is more than actual sol
Since the function is concave down
which Euler does not fully reflect

prove (-1)

funnel check

$$U(t)' > v(U, U(x)$$

$$U = -(t-2)^{1/2}$$

$$U' = -\frac{1}{2}(t-2)^{-1/2} \cdot 1$$

$$= -\frac{1}{2\sqrt{t-2}}$$

$$-\frac{1}{2\sqrt{t-2}} > -1 \quad \text{for } t > 2$$

✓ yes always

$$L(t)' < L(L, L(x))$$

$$L = -(t-1)^{1/2}$$

$$L' = -\frac{1}{2}(t-1)^{-1/2} \cdot 1$$

$$= \frac{-1}{2\sqrt{t-1}}$$

$$-\frac{1}{2\sqrt{t-1}} < 0 \quad \text{for } t > 1$$

✓ yes

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3. A population of island turtles contracts a contagious, but curable virus. To model the fraction P of infected turtles, we use the simple differential equation

$$\frac{dP}{dt} = -cP + sP(1-P), \quad c\text{: cure rate, } s\text{: transmission rate.}$$

logistic

a) (5 pts) For $c = 1$ and $s = 2$, draw the phase line, labeling critical points as sinks, sources, or neither. Then sketch several representative solutions

$\frac{dP}{dt} = -P + 2P(1-P)$

Critical Pts - when is deriv $= 0$

$0 = -P + 2P - 2P^2$

$0 = P - 2P^2$

$\frac{-1 \pm \sqrt{1 - 4 \cdot (-2) 0}}{-2 \cdot 2}$

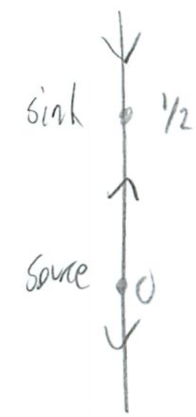
$0, \frac{1}{2}$

Now test various Ps

$P = 1$ $\quad 1 - 2(1)^2$ $\quad -1 \ominus$

$P = \frac{1}{4}$ $\quad \frac{1}{4} - 2(\frac{1}{4})^2$ $\quad \frac{1}{4} - \frac{1}{8} \oplus$

$P = -1$ $\quad -1 - 2(-1)^2$ $\quad -3 \ominus$



✓

b) (4 pts) Letting c and s be arbitrary positive constants, for what ratios c/s are all turtles eventually cured? Explain.

$\frac{c}{s}$ = population sustainable / constraint ant

this is bifurcation pts

Solve for P in terms of c, s

$-cP + sP - sP^2$

$-sP^2 + (s-c)P + 0$

$P = \frac{-(s-c) \pm \sqrt{(s-c)^2 - 4 \cdot (-s) \cdot 0}}{2 \cdot -s}$

$= \frac{-s + c \pm (s-c)}{-2s}$

$0, \frac{-2s + 2c}{-2s} = 1 + \frac{2c}{-2s}$

$P = -1$, $-s - s + c = -2s + c$

No - not what we want

Above $\frac{c}{s} = 1$

Since we want $1 - \frac{c}{s}$ to be above 0 No!

(3)

Read the problem!!!

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4. This question deals with various inputs $f(t)$ in the differential equation:

$$y'' + 2y' + 2y = f(t)$$

a) (5 pts) Find the general solution $y(t)$ to the homogeneous equation (i.e. $f(t) = 0$).

5

$$(s^2+2s+2)y=0$$

$$\frac{-2 \pm \sqrt{4-4\cdot 1\cdot 2}}{2}$$

$$-1 \pm i$$

$$e^{(-1\pm i)t} = e^{-t}e^{it}$$

$$= e^{-t}(\cos 1t + i \sin t)$$

$$y = Ae^{-t}\cos t + Be^{-t}\sin t$$ ✓

b) (5 pts) Find the general solution for the response to the input $f(t) = \cos t$.

2

Try to eliminate

$f = \cos t$

$D = -\sin t$

$D^2 = -\cos t$

So D^2+1

add as term

or guess $A\cos t + B\sin t$

$$\frac{-0 \pm \sqrt{0^2-4\cdot 1\cdot 1}}{2}$$

$$\pm i$$

$$e^{it} = \cos t + i \sin t$$

but repeated root?
no

$$y = Ae^{-t}\cos t + Be^{-t}\sin t + \underbrace{C\cos t + D\sin t}_{\text{determine them}}$$

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should have looked at more.. didn't review old exam think figured it

c) (5 pts) Solve the initial value problem with input $f(t) = e^{-t}\sin t$ and $y(0) = y'(0) = 0$.

$$y'' + 2y' + 2y = e^{-t}\sin t$$

Guess $A e^{-t}\cos t + B e^{-t}\sin t$

0

$$y' = +Ae^{-t}\sin t + -Be^{-t}\cos t$$

$$y'' = -Ae^{-t}\cos t + +Be^{-t}\sin t$$

$$(-Ae^{-t}\cos t + Be^{-t}\sin t) + 2(Ae^{-t}\sin t - Be^{-t}\cos t) + 2(Ae^{-t}\cos t + Be^{-t}\sin t) = e^{-t}\sin t$$

cos terms $-A - 2B + 2A = 0$

sin terms $B + 2A + 2B = 1$

$2A + 3B = 1$, $2A = 1 - 3B$, $A = \frac{1}{2} - \frac{3}{2}B$

$A - 2B = 0$, $(\frac{1}{2} - \frac{3}{2}B) - 2B = 0$, $-\frac{7}{2}B = -\frac{1}{2}$, $7B = 1$, $B = 1/7$

$A = \frac{1}{2} - \frac{3}{2}(\frac{1}{7}) = \frac{1}{2} - \frac{3}{14} = \frac{7}{14} - \frac{3}{14} = 5/14$

$y_p = \frac{5}{14} t e^{-t}\cos t + \frac{1}{7} t e^{-t}\sin t$

repeated roots?

over

d) (5 pts) Solve the initial value problem with input $f(t) = \delta(t-5)$ and $y(0) = 1, y'(0) = 2$.

LaPlace easiest

1

$$y'' + 2y' + 2y = \delta(t-5)$$

$$s^2 + 2s + 2\, \mathcal{L}(y) - s(1) - 2 = \mathcal{L}(\delta(t-5))$$

$$\mathcal{L}(y) = \frac{(s+2) + \mathcal{L}(\delta(t-5))}{s^2 + 2s + 2}$$

factor $\frac{-2 \pm \sqrt{4 - 4\cdot 1\cdot 2}}{2}$

$-1 \pm i$

$$= \frac{\mathcal{L}(\delta(t-5)) + s + 2}{(s - (-1 \pm i))}$$ 4 complex

And now the homogeneous

$$Y_h = Ae^t \cos t + Be^t \sin t$$

$$0 = Ae^0 \cos(0) + Be^0 \sin 0$$

$$0 = A + 0$$

$$Y_h' = -Ae^t \sin t + Be^t \cos t$$

$$0 = -Ae^0 \sin 0 + Be^0 \cos 0$$

$$0 = -0 + B$$

$$0 = B$$

So together

$$Y = \underbrace{0e^t \cos t + 0e^t \sin t}_{\text{homo}} + \underbrace{\frac{5}{14} t e^{-t} \cos t + \frac{1}{7} t e^{-t} \sin t}_{\text{particular}}$$

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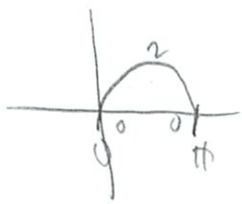
Boundry Value Problem

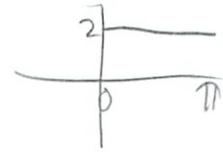
did not look at much either

e) (8 pts) Find a solution $y(t)$ to $f(t) = 2$ on the interval $[0, \pi]$ with $y(0) = y(\pi) = 0$.

- make periodic
- only look on boundry



or $2 \times$ square wave
Period 2π



$$Sq(t) = \frac{4}{\pi} \sum_{n\,odd} \frac{1}{n} \sin\left(\frac{n\pi t}{L}\right)$$

$L = 2\pi$

$$= \frac{4}{\pi} \sum_{n\,odd} \frac{1}{n} \sin\left(\frac{n\cancel{\pi} t}{2\cancel{\pi}}\right)$$

$$a_n = \frac{1}{2\pi} \int_{-L}^{L} \frac{4}{\pi} \frac{1}{n} \underbrace{\underbrace{\sin\left(\frac{nt}{2}\right)}_{odd} \underbrace{\cos\left(\frac{nt}{2}\right)}_{even}}_{odd}$$

0

$$\int_{-L}^{L} odd = 0$$

$$b_n = \frac{4}{2\pi^2} \int_{-\pi}^{\pi} \frac{1}{n} \sin^2\left(\frac{nt}{2}\right)$$

$\sin \cdot \sin = \pi$

$$\frac{4}{2\pi^2} \frac{1}{n} \pi$$

$$= \frac{4\cancel{\pi}}{2n\pi^{\cancel{2}}} = \frac{4}{2n\pi}$$

$$y = 0 + \sum_{n=1}^{\infty} \left(\frac{4}{2n\pi} \cos\left(\frac{nt}{2}\right) + 0 \sin\left(\frac{nt}{2}\right)\right)$$

I be meai

 Name: Placeholder

3

5. (7 pts) Find a <u>recursive relation</u> for the <u>general</u> power series solution $y(x)$ to the differential equation

$$y'' - xy' - x^2y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=0} a_{n-1}(n-1)x^n$$

$$y'' = \sum_{n=2}^{\infty} a_n (n+1) n x^{n-2} = \sum_{n=0}^{\infty} a_{n-2}(n-1)(n-2)x^n$$

$$\sum_{n=0}^{\infty} a_{n-2}(n-1)(n-2)x^n - x\sum_{n=1}^{\infty} a_n n x^{n-1} - x^2 \sum_{n=2}^{\infty} a_n(n+1)n x^{n-2} = 0$$

$$\sum_{n=0}^{\infty} a_{n-2}(n-1)(n-2)x^n - \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=2}^{\infty} a_n(n+1)n x^n = 0$$

$$\sum_{n=0}^{\infty} \left(a_{n-2}(n-1)(n-2) - a_n n - a_n(n+1)n\right)x^n = 0$$

$$a_{n-2} = \frac{a_n n + a_n(n+1)n}{(n-1)(n-2)}$$

$$= \frac{a_n n\left(1+(n+1)\right)}{(n-1)(n-2)}$$

a_B & a_2 ?? -2.5

← recurrence relationship no more work to do

backwards

-1.5

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6. Consider an LC circuit with $L = 1.0$ henries and $C = 0.1$ farads (and no resistance). We apply an external voltage $V(t)$ given by the period 4 function

$$V(t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi t}{2}\right)$$

darn...

← correction

a) (3 pts) Write down a second-order ODE for the current $I(t)$ in this circuit.

$$1.0\, I'' + 0 I' + 10\, I = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi t}{2}\right)$$

2/3

b) (5 pts) Find a periodic particular solution to your equation in part (a).

Fourier series?

$$Y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) + b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$a_n = \frac{1}{2}\int_{-2}^{2} \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos^2\left(\frac{n\pi x}{2}\right)$$

$$b_n = \frac{1}{2}\int_{-2}^{2}\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\underbrace{\cos()}_{e}\underbrace{\sin()}_{o}$$

$$\underbrace{e \cdot o}_{0}$$

$$\int \text{odd} = 0$$

$$a_n = \frac{4}{\pi^2}\sum\frac{(-1)^n}{n^2} 2\pi \qquad \cos\cdot\cos = 2\pi$$

homogenous $(s^2 + 10)$

$$\frac{-0 \pm \sqrt{0 - 4\cdot 1\cdot 10}}{2}$$

$$\pm\sqrt{10}\, i$$

$$Y = 0 + \sum_{n=1}^{\infty}\frac{4}{\pi}\frac{(-1)^n}{n^2} 2\pi \cos\left(\frac{n\pi x}{2}\right) + 0\sin\left(\frac{n\pi x}{2}\right)$$

$$Y = \sum_{n=1}^{\infty}\frac{8(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right)$$

✗ 1/5

$t = x$ notation switching?!

c) (3 pts) Your answer is a superposition of sinusoids, but one sinusoid dominates the behavior. What is its period? Explain.

N smallest dominates

Smallest is $n=1$ or $\cos\left(\frac{\pi t}{2}\right)$

Period for all is 4

✗

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7. a) (5 pts) Express the convolution $1 * 1 * 1 * 1 * 1$ in terms of a more familiar function using the Laplace transform.

0

$$\mathcal{L}^{-1}(\delta(t)) * \mathcal{L}^{-1}(\delta(t)) * \mathcal{L}^{-1}(\delta(t)) * \mathcal{L}^{-1}(\delta(t)) * \mathcal{L}^{-1}(\delta(t))$$

$$\mathcal{L}^{-1}(\delta(t) + \delta(t) + \delta(t) + \delta(t) + \delta(t))$$

X

$$\mathcal{L}^{-1}(5\delta(t))$$

$$5\mathcal{L}^{-1}(\delta(t))$$

$$5e^{-0s}$$

b) (5 pts) Given any function $f(t)$ in the differential equation

$$x'' + 9x = f(t)$$

(s^2+9)

with rest conditions $x(0) = x'(0) = 0$, find a solution $x(t)$ as an integral expression involving $f(t)$.

? w/ LaPlace

2

$$\mathcal{L}(x)(s^2+9) = \mathcal{L}(f(t))$$

$$\mathcal{L}(x) = \frac{\mathcal{L}(f(t))}{\mathcal{L}(s^2+9)}$$

$$= \frac{\mathcal{L}(f(t))}{\frac{1}{3}\sin(3t)}$$

mult. not divide.

$$x(t) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(f(t))}{\frac{1}{3}\sin(3t)}\right)$$

dido 4 practice

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8. Consider the system of differential equations

$$\begin{aligned} x_1' &= 2x_1 + x_2 \\ x_2' &= 4x_1 - x_2 \end{aligned}$$

a) (6 pts) Find the general solution to this system of equations.

6

$x' = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} x$

$\begin{pmatrix} 2-\lambda & 1 \\ 4 & -1-\lambda \end{pmatrix} \to (2-\lambda)(-1-\lambda) - 4$

$\boxed{\lambda^2 - \lambda - 6 = 0}$

$\lambda = \frac{+1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-6)}}{2}$

$= \frac{+1 \pm 5}{2} = 3, -2$ ✓

$\lambda = 3$, $\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}$

$-V_1 + V_2 = 0$

$V_2 = V_1$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -2$, $\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}$

$4V_1 + V_2 = 0$

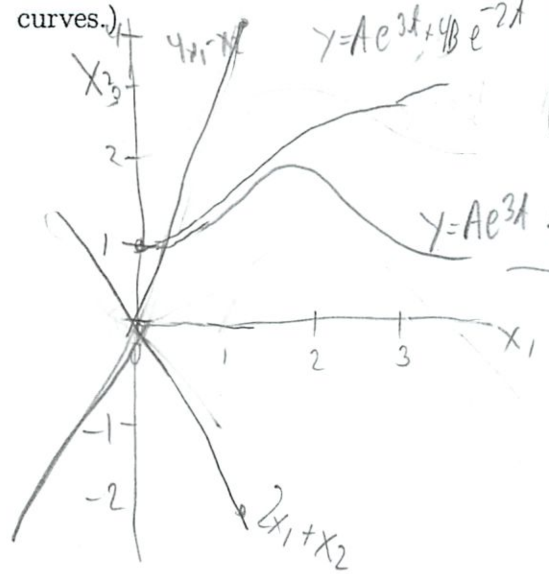
$4V_1 = -V_2$

$V_1 = -\frac{V_2}{4}$ $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

$y = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + B\begin{pmatrix} -1 \\ 4 \end{pmatrix} e^{-2t}$ ✓

b) (4 pts) Sketch the two straight line solutions on the phase plane, together with the solution passing through $(x_1, x_2) = (0, 1)$. (A rough sketch is fine, but be sure to indicate the positive direction of time with arrows on your solution curves.)

0



So $\frac{dx_1}{dt} = 2x_1 + x_2$

0,0

$-\frac{1}{2}$, 1

2, -4

1, -2

$\frac{dx_2}{dt} = 4x_1 - 1x_2$

where does it remain 0?

0,0 1,4

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9. a) (6 pts) Find the general solution to the system

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix}$$

$\begin{pmatrix} -3-\lambda & -1 \\ 2 & -1-\lambda \end{pmatrix}$ $(-3-\lambda)(-1-\lambda)+2$

$\lambda^2+4\lambda+5$

$\frac{-4 \pm \sqrt{16-4\cdot1\cdot5}}{2}$

$-2 \pm i$

don't need other one due to symmetry

$\begin{pmatrix} -1 \\ 1+i \end{pmatrix} e^{(-2+i)t} = \begin{pmatrix} -1 \\ 1+i \end{pmatrix} e^{-2t}(\cos t + i\sin t)$

$= -e^{-2t}\cos t - ie^{-2t}\sin t$

$e^{-2t}\cos t + ie^{-2t}\cos t + i\sin t e^{-2t} - \sin t e^{-2t}$

$Re = \begin{pmatrix} -e^{-2t}\cos t \\ e^{-2t}(\cos t - \sin t) \end{pmatrix}$

$Im = \begin{pmatrix} -e^{-2t}\sin t \\ e^{-2t}(\cos t + \sin t) \end{pmatrix}$

$= A\begin{pmatrix} -\cos t \\ \cos t - \sin t \end{pmatrix} e^{-2t} + B\begin{pmatrix} -\sin t \\ \cos t + \sin t \end{pmatrix} e^{-2t}$

$\lambda = -2+i$ $\begin{pmatrix} -1-i & -1 \\ 2 & 1-i \end{pmatrix}$ $(-1-i)v_1 - v_2 = 0$ $(-1-i)v_1 = v_2$

$2v_1 + (1-i)v_2 = 0$ $2v_1 + (1-i)(-1-i)v_1 = 0$

$2v_1 + [(-1)+0i-1]v_1 = 0$ $2v_1 - 2v_1 = 0$

$6+v$ $-(-1-i)$ $-(1+i)$

$\begin{pmatrix} -1 \\ 1+i \end{pmatrix}$, $-2+(1-i)(1+i)$

$1-(-1)$

✓ works

b) (4 pts) Determine the exponential matrix e^{At} with A as above.

$e^{At} =$

Often hard to find

but you can find it w/ some tricks...

w/ the fundamental matrix...

0.

-4.

c) (2 pts) Express the solution to the initial value problem $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ in terms of the exponential matrix. (You need not substitute in the expression for e^{At}. You can leave it as a general formula.)

looks like I chose 3

forgot

did study

$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = e^{At}$

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10. Our population of (now healthy) island turtles T is attacked by seahawks S (both measured in hundreds of animals). Suppose that these populations obey a system of differential equations

$$\begin{aligned} T' &= aT - ST - T^2 \\ S' &= -bS + ST \end{aligned}$$

last day

for some constants $a > b > 0$.

a) (2 pts) Find the critical point with both populations S and T positive. ; set each = to 0?

; solve system 1st?

.5

$0 = -T^2 + (a-S)T$

$\frac{-(a-S) + \sqrt{(a-S)^2 - 4 \cdot -1 \cdot 0}}{2 \cdot -1}$

$\frac{S-a+a-S}{-2} = 0$

$0 = -bS + ST$

$= S(-b+T)$

$S=0$ then T is anything ???

b) (8 pts) Suppose that $a = 5$ and $b = 2$. Linearize the system at the critical point with S, T positive and determine its eigenvalues. Use your answer to describe qualitatively the trajectory of solutions nearby this critical point. (Terms like saddle, spiral, node, sink, source, etc. are welcome.)

1

$T^2 \sim T \;\alpha$

$\begin{pmatrix} & \end{pmatrix} \begin{matrix} \leftarrow T' \\ \leftarrow S' \end{matrix}$

T's S

$T' = 5T - ST - T$

$S' = -2 + ST$

$\begin{pmatrix} 5-S & -1 \\ S & -5 \end{pmatrix}$ $\begin{pmatrix} (5-S)-\lambda & -1 \\ -2 & S-\lambda \end{pmatrix}$ $\lambda^2 - (S)(5-S)\lambda + (5S-S^2) + 2$

$\lambda^2 - 5S - S^2\lambda + 5S - S^2 + 2$

18.06 Professor Strang Final Exam May 23rd, 2012

Michael

Your PRINTED name is: Plasmeier

Grading

1 ~~6~~ 6
2 9
3 ~~6~~ 8
4 4
5 4
6 ~~0~~ 2
7 6
8 8

~~43~~ 47.

Please circle your recitation:

r01	T 11	4-159	Ailsa Keating	ailsa
r02	T 11	36-153	Rune Haugseng	haugseng
r03	T 12	4-159	Jennifer Park	jmypark
r04	T 12	36-153	Rune Haugseng	haugseng
r05	T 1	4-153	Dimiter Ostrev	ostrev
r06	T 1	4-159	Uhi Rinn Suh	ursuh
r07	T 1	66-144	Ailsa Keating	ailsa
r08	T 2	66-144	Niels Martin Moller	moller
r09	T 2	4-153	Dimiter Ostrev	ostrev
r10	ESG		Gabrielle Stoy	gstoy

$$15 - 20\lambda + 5\lambda^2 - 3\lambda + 4\lambda^2 - \lambda^3 = 0$$

$$-\lambda^3 + 9\lambda^2 - 23\lambda + 15 = 0$$

$$\lambda(-\lambda^2 + 9\lambda - 23) = -15$$

$$\frac{-9 \pm \sqrt{81 - 4 \cdot -1 \cdot -23}}{-2}$$

$$\frac{-9 \pm \sqrt{11}\, i}{-2}$$

$$\begin{array}{r} 23 \\ \underline{4} \\ 92 \end{array}$$

$81 - 92 =$

$\lambda_1 \lambda_2 \lambda_3 = 3$

$3 + 1 + 5 = 9 = \lambda_1 + \lambda_2 + \lambda_3$

$-1, -3$

$t \quad x$
$9 \quad 3$

-3
-3

1 (12 pts.)

(a) - Find the eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{bmatrix}$$

$(3-\lambda)(1-\lambda)(5-\lambda) = 0$

-1
-3

$(3-4\lambda+\lambda^2)(5-\lambda) = 0$

Seems too long, do alt way

$l = \frac{1}{1}$ row 3 = row 3 − row 2

$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

pivots = 3, 1

det = $\prod$ pivots = $\prod \lambda$
trace = $\sum$ diag = $\sum \lambda$

$3 = \lambda_1 \lambda_2 \lambda_3$
$9 = \lambda_1 + \lambda_2 + \lambda_3$
$\lambda = 0, 3, 6$

still doesn't fit

$\lambda = 2$: $\begin{bmatrix} 1 & 1 & 4 \\ 0 & -1 & 5 \\ 0 & 1 & 3 \end{bmatrix}$ $y = 5z$, $y + 3z = 0$, $5z + 3z = 0$, $8z = 0$

$\lambda = -1$: $\begin{bmatrix} 4 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 1 & 6 \end{bmatrix}$ $2y + 5z = 0$, $1y + 6z = 0$, $y = -6z$, $2(-6z) + 5z = 0$, $-12z + 5z = 0$, $-7z = 0$

$\lambda = 6$: $\begin{bmatrix} -3 & 1 & 4 \\ 0 & -5 & 5 \\ 0 & 1 & -1 \end{bmatrix}$ $-5y + 5z = 0$, $y - z = 0$, $y = z$, $-3x + 5 = 0$, $-3x = -5$, $x = \frac{5}{3}$

$\begin{bmatrix} 5/3 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 3$: $\begin{bmatrix} 0 & 1 & 4 \\ 0 & -2 & 5 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2y + 5z = 0$
$y + 2z = 0$
$y = -2z$
$-2(-2z) + 5z = 0$
$4z + 5z = 0$ $9z = 0$

$\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ ×

−2

$\lambda = 0$: $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{bmatrix}$

$y - 5z = 0$
$y = 5z$

$\begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$ ×

$3x + 5 + 4 = 0$
$3x = -9$
$x = -3$

(b) - Write the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of eigenvectors of A.

- Find the vector $A^{10}\mathbf{v}$.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \underline{?} \begin{bmatrix} 5/3 \\ 1 \\ 1 \end{bmatrix} + \underline{?} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + \underline{?} \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$$

−2

$A^{10} = S \Lambda^{10} S^{-1}$

$S = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \lambda_1^{10} & & \\ & \lambda_2^{10} & \\ & & \lambda_3^{10} \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{-1}$

$\begin{bmatrix} 5/3 & 0 & -3 \\ 1 & -2 & 5 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6^{10} & & \\ & 3^{10} & \\ & & 0 \end{bmatrix} \begin{bmatrix} 5/3 & 0 & -3 \\ 1 & -2 & 5 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$

(c) If you solve $\frac{d\mathbf{u}}{dt} = -A\mathbf{u}$ (notice the minus sign), with $\mathbf{u}(0)$ a given vector, then as $t \to \infty$ the solution $\mathbf{u}(t)$ will always approach a multiple of a certain vector $\mathbf{w}$.

- Find this steady-state vector $\mathbf{w}$.

$$u(t) = -Ce^{\lambda_1 t}\begin{bmatrix} x_1 \end{bmatrix}$$

$$u(0) = -C\begin{bmatrix} x_1 \end{bmatrix} + -D\begin{bmatrix} x_2 \end{bmatrix}$$

$$u(0) = -C\begin{bmatrix} 5 \\ 3 \\ 1 \\ 1 \end{bmatrix} - D\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} - E\begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$$

its also

$$A^{\infty} = S\Lambda^{\infty}S^{-1}$$

X -2

can't be 0

can't be 3

2 (12 pts.)

Suppose A has rank 1, and B has rank 2 (A and B are both 3×3 matrices).

(a) - What are the possible ranks of $A + B$?

1, 2, 3

(b) - Give an example of each possibility you had in (a).

1

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

2

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

0 can't be

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

~~3 can't be~~

4

$1 = \text{rank } 1$

$B = \text{rank } 2$

(c) - What are the possible ranks of AB?

0, 1

- Give an example of each possibility.

1

$$\begin{bmatrix} 1 & 1 & 1 \\ & & \\ & & \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

0

$$\begin{bmatrix} 1 & 1 & 1 \\ & & \\ & & \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ -1 & & +1 \\ & & \end{bmatrix} = \begin{bmatrix} 1+-1 & 0 & 1+-1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

not possible ?

2

$$\begin{bmatrix} 1 & 1 & 1 \\ & & \\ & & \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \times$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & & \\ 2 & & 1 \\ & & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \times$$

not possible ?

prod [illegible] rule

No.

5

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3 (12 pts.)

(a) - Find the three pivots and the determinant of A.

$$A = \begin{array}{ccc} + & - & + \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

Row ex $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$l = \frac{-1}{1}$

row 2 = row 2 + row 1

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

$l = \frac{1}{1}$

row 3 = row 3 − row 2 $\quad 1 - -1$

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

pivots = $1, \cancel{1}, 2$

det is the product of the pivots

$(-1) \cdot 1 \cdot 1 \cdot 2 = -2$

~~for row ex~~ You didn't do a row exchange!

Verify w/ cofactors

ad + bc

$+1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + -1 \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}$

$1(-1) + -1(+1)$

-2

3/4

Should have looked at more ad−bc
ad+bc

(b) - The rank of $A - I$ is 2, so that $\lambda =$ 1 is an eigenvalue. of A correction

what can we conclude
- Markov
- entries add to abs 1

- The remaining two eigenvalues of A are $\lambda =$ 2, -1.

- These eigenvalues are all ~~positive~~ pos def, because $A^T = A$.

Symmetric

$$A - I = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

2/5

3

$(1-\lambda)(1-\lambda)(0-\lambda)$

$\det = \prod \text{pivots} = \prod \text{eig}$

$-2 = 2 \cdot 1 \cdot 1 = 1 \cdot \lambda_2 \cdot \lambda_3$

hmm

$\text{trace} = \sum \text{diag} = \sum \text{eig}$

$1 + 1 + 0 = 1 + \lambda_2 + \lambda_3$

$2 = \lambda_2 \cdot \lambda_3$

$1 = \lambda_2 + \lambda_3$

not possible - or did something wrong

$\lambda_2 = 2$

$\lambda_3 = 1.$

(c) The unit eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ will be orthonormal.

- Prove that:

$$A = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T + \lambda_3 \mathbf{x}_3 \mathbf{x}_3^T.$$

$x_3^T x_3$;
- dot prod

You may compute the $\mathbf{x}_i$'s and use numbers. Or, without numbers, you may show that the right side has the correct eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

$\lambda = 2$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x - z = 0$

$-y + z = 0$

$-x + y - 2z = 0$

$z = y$

$-x - y = 0$

$-x = y = z$

$y + y - 2y = 0$

$0 = 0$ ~~on banded~~ or set

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-z = 0$

$z = 0$

$-x + y - z = 0$

z must be 0

$$\begin{bmatrix} +1 \\ +1 \\ 0 \end{bmatrix}$$

Something w/ a projection

Since this is a projection, by def, it will be orthonormal

$$A = 2\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

$$= 2\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

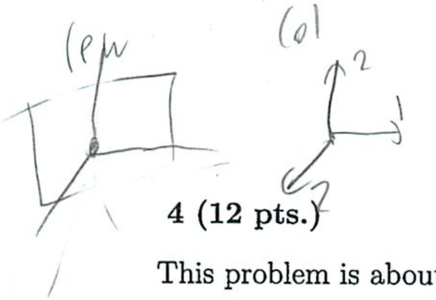
$$= \begin{bmatrix} 4 & 0 & -2 \\ 0 & 4 & 2 \\ -2 & 2 & 2 \end{bmatrix}$$

2 1/3

not quite fix,

8/12

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$



$\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$

orthogonal to

4 (12 pts.)

This problem is about $x + 2y + 2z = 0$, which is the equation of a plane through $\mathbf{0}$ in $\mathbb{R}^3$.

(a) - That plane is the nullspace of what matrix A?

$A = \begin{bmatrix} -4 & 1 & 1 \\ -4 & 1 & 1 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ 0/6   − 2.

- Find an orthonormal basis for that nullspace (that plane).

Gram Schmidt

$A = a$ $\cancel{\begin{bmatrix} -4 \\ -4 \\ -4 \end{bmatrix}}$

need more than 1

$B = b - \frac{A^T b}{A^T A} A$ how? - just normalize

$\sqrt{1^2+2^2+2^2}$

$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$ $\sqrt{9}$

3

$\cancel{B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{[-4\ -4\ -4] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{[-4\ -4\ -4] \begin{bmatrix} -4 \\ -4 \\ -4 \end{bmatrix}} \begin{bmatrix} -4 \\ -4 \\ -4 \end{bmatrix}}$

$\frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

No. 2d space

2/5.

$\cancel{= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{(-4\ -4\ -4)}{(16+16+16)} \begin{bmatrix} -4 \\ -4 \\ -4 \end{bmatrix}}$

$\cancel{\frac{-12}{48}}$

$\cancel{= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - -\frac{1}{4} \begin{bmatrix} -4 \\ -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$

your really got whole space here

(b) That plane is the column space of many matrices B.

- Give two examples of B.

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 3 & 6 & 6 \end{bmatrix} \; 0/0$$

0/0

−2.

(c) - How would you compute the projection matrix P onto that plane? (A formula is enough)

- What is the rank of P?

$$P = \frac{aa^T}{a^Ta} \quad 0/0 \quad \text{No.}$$

1/4.

−3.

P is an orthonormal basis – so rank is ~~full = 3~~

$\underline{r = 2}$

BS though

5 (12 pts.)

Suppose $\mathbf{v}$ is any unit vector in $\mathbb{R}^3$. This question is about the matrix H.

$$H = I - 2\mathbf{v}\mathbf{v}^T.$$

(a) - Multiply H times H to show that $H^2 = I$. Show is a projection

$$(I - 2vv^T)(I - 2vv^T)$$

Can't distribute - would change order

$$I - (2vv^T I) - 2vv^T$$

$$I - 2vv^T - 2vv^T$$

$$I$$

(b) - Show that H passes the tests for being a symmetric matrix <u>and</u> an orthogonal matrix.

Sym $A^T = A$

$$H^T = I^T - 2\underbrace{v^T v}_{\text{zero if orthogonal (dot prod)}}$$

$$= I - 2 \cdot 0$$

$$= I$$

$$I^T - 2v^T v = I - 2vv^T$$

$$-2v^T v = -2vv^T$$

$$v^T v = vv^T$$

~~only if I~~

Or shows that it is the same - no matter the v

(c) - What are the eigenvalues of H?

You have enough information to answer for any unit vector $\mathbf{v}$, but you can choose one $\mathbf{v}$ and compute the λ's.

$(A - \lambda I) = 0$ solve for λ

$I - \lambda I$

$I - 2vv^T - \lambda I = 0$

$I(1-\lambda) = 2vv^T$

$I(1-\lambda)$

do w/ this

see

If $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ← unit vector

$H = I - 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

$= I - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

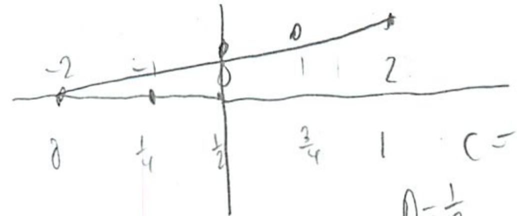
4

$= \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

$\lambda = (-1-\lambda)(1-\lambda)(1-\lambda) = 0$

$\Pi \lambda = \Pi \text{ pivots} = 1 \cdot 1 \cdot -1 = -1$ $\quad \Sigma \lambda = \Sigma \text{ diag}$

$\lambda = 1, +1, -1$ $\quad 1 + 1 - 1 = 1$



$C = \frac{\text{rise}}{\text{run}} = \frac{1/2}{2} = \frac{1}{4}$

$D = \frac{1}{2}$

6 (12 pts.)

(a) - Find the closest straight line $y = Ct + D$ to the 5 points:

$$(t, y) = (-2, 0),\ (-1, 0),\ (0, 1),\ (1, 1),\ (2, 1).$$

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \quad \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{result}} \qquad C\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + D = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{b}$$

$C - 2D = 0$
$C - D = 0$
$C = 1$
$C + D = 1$
$C + 2D = 1$

or a more matrixy way:
$x = A^{-1} b$ — not square...

1

$C = \frac{1}{4}$
$D = \frac{1}{2}$

not really matrix

I have it!

(b) - The word "closest" means that you minimized which quantity to find your line?

The error

1

(c) - If $A^T A$ is invertible, what do you know about its eigenvalues and eigenvectors? (Technical point: Assume that the eigenvalues are distinct – no eigenvalues are repeated).

(seperate

~~Dot product (?)~~

~~(should = 0)~~

~~Not $A^T = A^{-1}$ that means orthonormal~~

~~part of SVD~~

~~if A 1x3~~

~~(3x1)~~

0

invertable = full set of unique e values and e vectors

(that's all)

7 (12 pts.)

This symmetric Hadamard matrix has orthogonal columns:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad \text{and} \quad H^2 = 4I.$$

(a) What is the determinant of H?

$l = \frac{1}{1}$ row 2 = row 2 − row 1

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

row ex

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & -2 & 0 & -2 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$l = 1/1$ 2 = row 2 − row 1

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

row ex

2

$l = \frac{1}{1}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$\frac{-2}{-2}$ row 3 = row 3 − row 2

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$\frac{-2}{2}$ row 4 = row 4 + row 3

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

pivots $1, -2, 2, -4$

3 row ex = −

+16

3

(b) What are the eigenvalues of H? (Use $H^2 = 4I$ and the trace of H).

det $= \prod$ pivots $= \prod \lambda$ also $4+4+4+4 =$ trace of H^2

trace $= \sum$ diag $= \sum \lambda$ so $\sqrt{H^2} = 4$

$1 - 2 + 2 - 4 = -3$

3

$-16 = \lambda_1 \lambda_2 \lambda_3 \lambda_4$

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 4$

$\lambda = 2, -2, -2, 2$

(c) What are the singular values of H?

? not invertable ?
but that is matrix - not specific values

0

? nothing

? but how are
certain
values singular?

8 (16 pts.)

In this TRUE/FALSE problem, you should *circle* your answer to each question.

TA: True means always true

(a) Suppose you have 101 vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{101} \in \mathbb{R}^{100}$.

- Each v_i is a combination of the other 100 vectors: TRUE – FALSE

- Three of the v_i's are in the same 2-dimensional plane: TRUE – FALSE

One of them repeats – correct?

(b) Suppose a matrix A has repeated eigenvalues 7, 7, 7, so $\det(A - \lambda I) = (7 - \lambda)^3$.

- Then A certainly cannot be diagonalized ($A = S\Lambda S^{-1}$): TRUE – FALSE

? must be unique

- The Jordan form of A must be $\mathcal{J} = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix}$: TRUE – FALSE

Jordan form hard to calc ← we don't know how to find

(c) Suppose A and B are 3×5.

earlier problem

- Then $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$: TRUE – FALSE

(d) Suppose A and B are 4×4.

- Then $\det(A + B) \leq \det(A) + \det(B)$: TRUE – FALSE

No idea

(e) Suppose $\mathbf{u}$ and $\mathbf{v}$ are orthonormal, and call the vector $\mathbf{b} = 3\mathbf{u} + \mathbf{v}$. Take V to be the line of all multiples of $\mathbf{u} + \mathbf{v}$.

- The orthogonal projection of $\mathbf{b}$ onto V is $2\mathbf{u} + 2\mathbf{v}$: TRUE – FALSE

? where do you get that

(f) Consider the transformation $T(x) = \int_{-x}^{x} f(t)dt$, for a fixed function f. The input is x, the output is $T(x)$.

? not really on exam?

8/16

- Then T is always a linear transformation: TRUE – FALSE

$\int x = \frac{x^2}{2} \cdot 2 = x^2$

$\int 2x = 2\frac{x^2}{2} = x^2$

$\frac{x^2}{2} + \frac{x^2}{2} = x^2$

$\int x^3 = \frac{x^4}{4}$

$\int 2x^3 = \frac{x^4}{4} \cdot 2$

$\int x^3 + x^3$ can split $\int x^3 + \int x^3$

$-x$ x

$\int_{-x}^{x} f(d_1) + \int_{-x}^{x} f(d_2) \stackrel{?}{=} \int_{-x}^{x} f(d_1 + d_2)$

$c\int f(d) = \int f(cd)$

take constants out of integrals

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This is the end of 18.06. Hope you enjoyed learning Linear Algebra!

18.06 Debrief

All about eig valus

Not good at finding by hand

But a lot of stuff not on
- FFT
- Transform in depth

Also vector spaces

Didn't know what it meant

But mostly confused on finding λ by hand

Should have praticed more

Talked w/ TA

She hadn't looked at

Thought trace was on original

But says I should not worry

Not in danger of failing

Email her for grade in evening

2

Rereading book

- perhaps did better than I thought

$\lambda = 0$ is valid

and det $= 0$ means one $\lambda = 0$

(which I think I included)

~~[illegible]~~ $\lambda = 6, 3, 0$

trace is on original of A

↳ should have been clearer on

Oh singular value $= \sigma$

darn

Ok hopefully ended up well

↳ since there was no qr I was sure I got

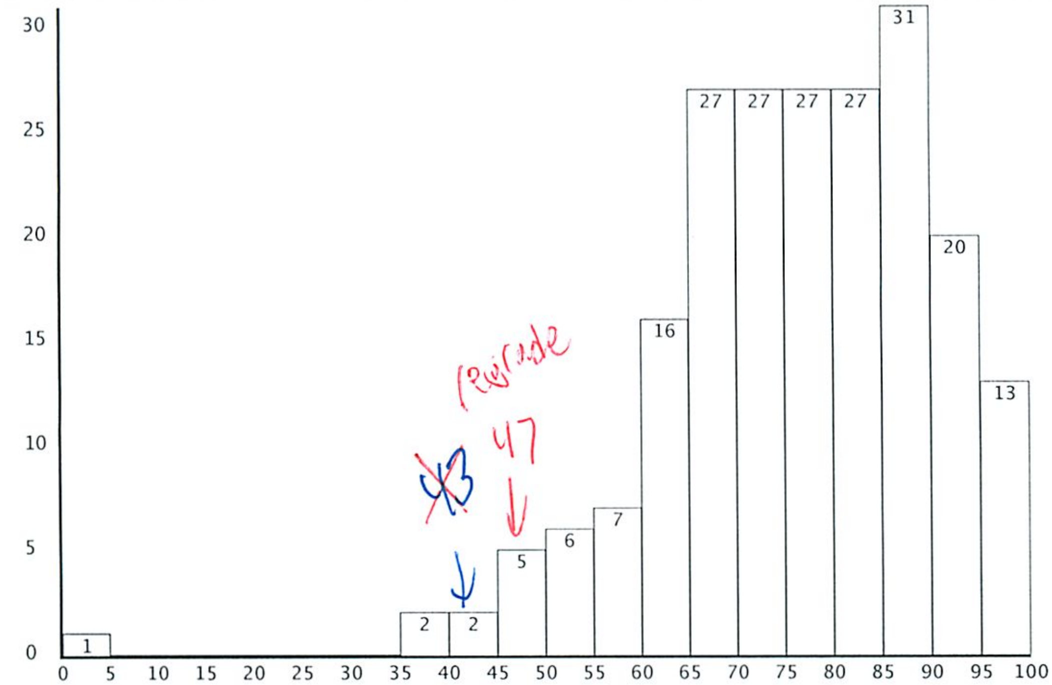
- usually how is on finals

Prob lost my B

18.06 Linear Algebra

Dashboard | Students | Assignments

Grading Summary for Final Exam



Number of Scores: 211
Average: 75.68
Standard Deviation: 14.19

Ouch!!!

18.06 Linear Algebra

Grade Report

Grade Report for Michael E. Plasmeier

Assignment/Exam Name	Graph	Due Date	Points	Max Pts	Weight
Homework 1		02.16.2012	83.00	100.00	1.88%
Homework 2		02.23.2012		100.00	1.88%
Homework 3		03.01.2012	77.00	100.00	1.88%
Homework 4		03.08.2012	77.00	100.00	1.88%
Exam 1		03.09.2012	69.00	100.00	15.00%
Homework 5		03.22.2012	83.00	100.00	1.88%
Homework 6		04.05.2012	85.00	100.00	1.88%
Exam 2		04.11.2012	56.00	100.00	15.00%
Homework 7		04.19.2012	94.00	100.00	1.88%
Homework 8		04.26.2012	86.00	100.00	1.88%
Homework 9		05.03.2012	91.00	100.00	1.88%
Exam 3		05.08.2012	65.00	100.00	15.00%
Final Exam		05.23.2012	43.00	100.00	40.00%
		CUMULATIVE SCORE	**58.41 / 100.04 = 58.39%**		

Instructor's Comments