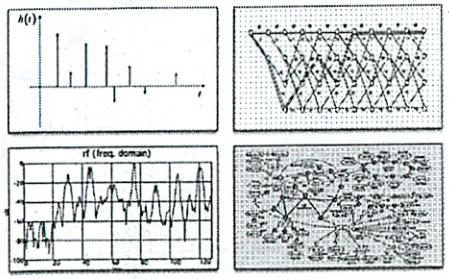
Rewatch 4/4



INTRODUCTION TO BECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Spring 2011
Lecture #15

- frequency response
- LTI systems with "zeros"
- filters

think in terms of freq

- not osilscope traces
- freq domain

Last time signals
this " systems - freq
pack diff transmissions in diff freqs

From Last Time

spectral coefficients ⇒ time

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \qquad \boxed{\text{Synthesis equation}}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n} \qquad \boxed{\text{Analysis equation}}$$

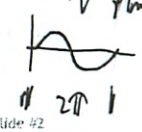
- x[n] and a_k are both periodic with period N
- 2π/N radians/sample is the fundamental frequency. Complex exponentials in Fourier series equations have frequencies which are some harmonic of 2π/N — 1 cycle
- If x[n] is real, $a_{-k} = a_k^*$ (i.e., they are complex conjugates)
- a_0 is the average of the x[n]

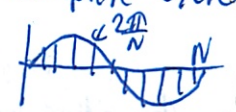
$k=0$

radians
time step

Use domain that is easiest for that qu

Will go through 1 complete cycle in N time





so $\frac{2\pi}{N}$ = freq

everything else is harmonic of this

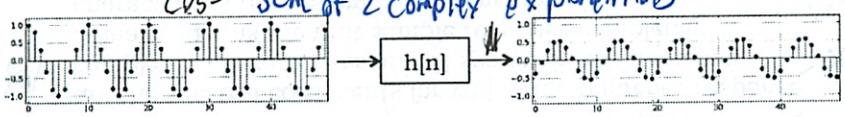
if k=2 - 2 time steps in period

- everything harmonic of freq

Sinusoidal

Complex Exponentials and LTI Systems

cos = sum of 2 complex exponentials



Frequency division multiplexing depends on an interesting property of LTI channels:

> if the channel input x[n] is a complex exponential of a given amplitude, frequency and phase, the response will be a complex exponential *at the same frequency*, although the amplitude and phase may be altered. As we'll see, the change in amplitude and phase will, in general, depend on the frequency of the input.

Let's prove this to be true...

in time domain - unit sample response

Frequency Response

amp freq LTI

$$Ae^{j\Omega n} \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

n samples

Using the convolution sum we can compute the system's response to a complex exponential as input:

$$\begin{aligned} y[n] &= \sum_m h[m]x[n-m] \\ &= \sum_m h[m]Ae^{j\Omega(n-m)} \\ &= Ae^{j\Omega n}\sum_m h[m]e^{-j\Omega m} \\ &= x[n]H(e^{j\Omega}) \end{aligned}$$

give it name

where we've defined the *frequency response* of a system as

$$\boxed{H(e^{j\Omega}) \equiv \sum_m h[m]e^{-j\Omega m}}$$

Completely characterizes sys in freq domain

calc for a certain Ω
get a complex #
phase shift + amp change

Reminds us it's the response for complex exponentials at freq Ω

If wrote program to duplicate figures in this lecture
You would ace quiz

Another Way to Characterize LTI Systems

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \longrightarrow \boxed{H(e^{j\Omega})} \longrightarrow y[n] = \sum_{k=\langle N\rangle} a_k H(e^{jk\frac{2\pi}{N}})\, e^{jk\frac{2\pi}{N}n}$$

how any arbitrary waveform
not just a single complex exponential

⊗ each component

looks like synthesis eq
- except a_k is modified by H

The frequency response tells us how the system will affect each of the spectral coefficients that determine the input. As you can see from the equation above

complex #

$$b_k = a_k H(e^{jk\frac{2\pi}{N}})$$

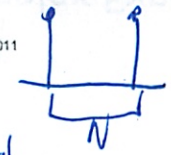
spectral coefficients here are a_k modified by H

are the spectral coefficients for y[n]. The frequency response completely characterizes an LTI system in the frequency domain, just as the unit sample response completely characterizes the system in the time domain.

- scale by freq response at that freq

know how amp changes
how phase changes

Spectral Coefficients



N

Unit Sample and Response

Try some examples

↓ unit sample, periodic w/ N timesteps

$$\delta[n] \longrightarrow \boxed{H(e^{j\Omega})} \longrightarrow h[n]$$

↑ unit sample response

We can compute the (periodic) spectral coefficients for the (periodic) unit sample:

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\underbrace{x[0]}_{\text{everything else}=0} e^{-jk\frac{2\pi}{N}0} = \frac{1}{N}$$

Now use our new formula for the (periodic) system response from the previous slide:

$$h[n] = \sum_{k=\langle N\rangle} \frac{1}{N} H(e^{jk\frac{2\pi}{N}})\, e^{jk\frac{2\pi}{N}n}$$

spectral coeff.

This is the synthesis equation!

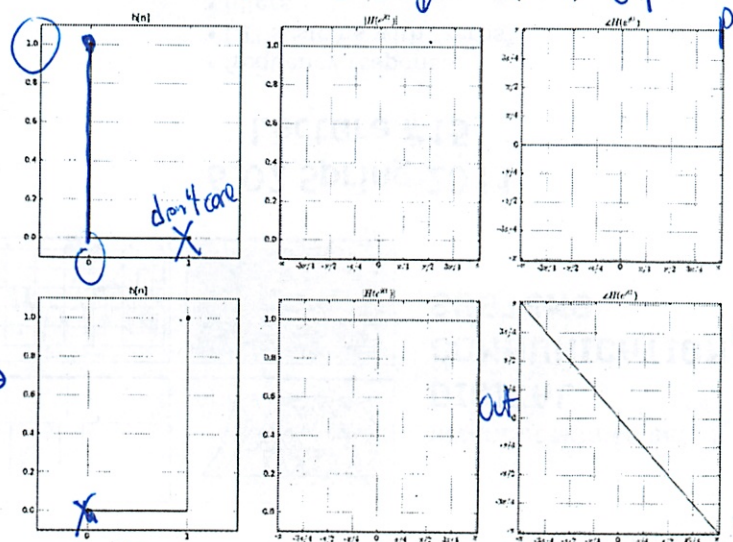
Compute unit circle w/ freq response

Or how to compute unit sample resp if have freq response
H → h

Example h[n] and H(e^{jΩ})

amplitude/magnitude

↓ phase angle/phase] polar



delay by 1 →

↑ will get y[n] = x[n]

↑ changing by each time step

↑ rad in

nothing changes about symbol

normal sin

↓ going out

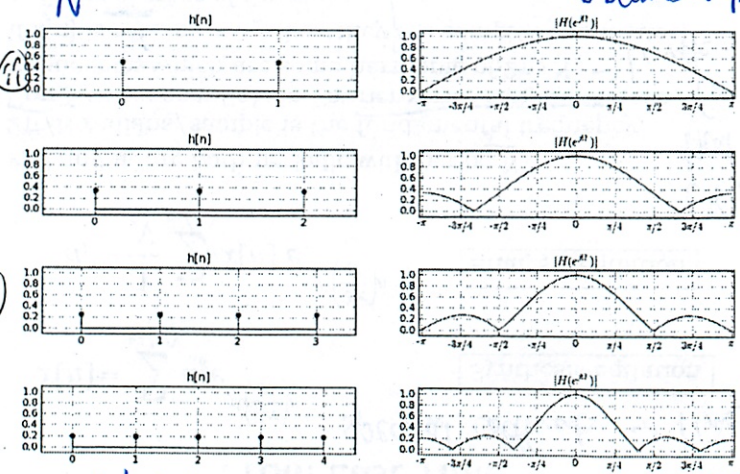
0,1,0,1,01

(these graphs always seem weird to me)
- so what would it look like?

Moving Averages

add / N

↓ becomes output



doing moving average

throws away part of edge
- high freq

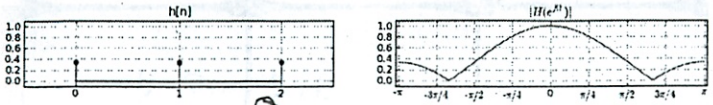
What is freq response of such a system?
- as avg over more points - avg out more + more of high freq
- the high freq!

Moving avg → nothing!

for 3 pts 0 happens at specific point

Can we mess w/ heights to get 0 anywhere?

$H(e^{j\Omega})$ with Zeros



$$H(e^{j\Omega}) = \sum_m h[m]e^{-j\Omega m} = h[0]e^{-j\Omega 0} + h[1]e^{-j\Omega 1} + h[2]e^{-j\Omega 2}$$ adding

$$= h[0] + h[1](e^{-j\Omega}) + h[2](e^{-j\Omega})^2$$ looks quadratic

Hmm. A quadratic equation with two roots at frequency ±φ:

$$H(e^{j\Omega}) = (e^{j\Omega} - e^{j\varphi})(e^{j\Omega} - e^{-j\varphi})$$ ← goes to 0 at ±φ

$$= (e^{j\Omega})^2 - (e^{j\varphi} + e^{-j\varphi})(e^{j\Omega}) + e^{j\varphi}e^{-j\varphi}$$

$$= (e^{j\Omega})^2 - 2\cos(\varphi)(e^{j\Omega}) + 1$$ ← ~~looks quadratic~~

Can you set roots where you want by setting φ

Matching terms in the two equations, we see that an LTI system would have a frequency response that went to zero at ±φ if

h[0]=1, h[1]=-2cos(φ) and h[2] = 1.

line up powers

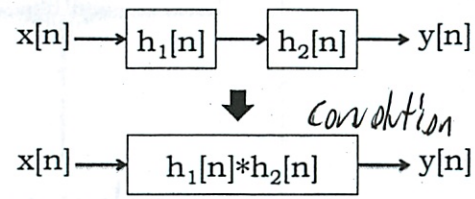
Unit sample response for this freq response

$h[n] = [1, \underset{\uparrow}{0}, 1]$

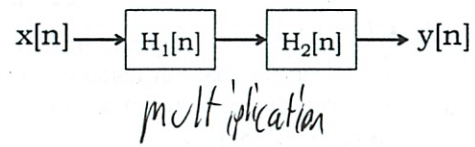
Want to be able to filter

Series Interconnection of LTI Systems

From Lecture 4:

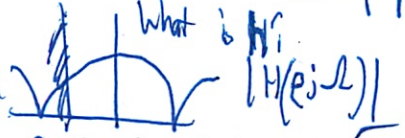


How about:



← has 1 0

Quiz problem

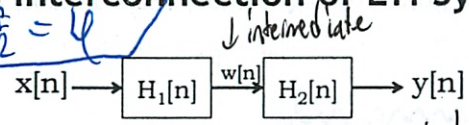


What is $|H(e^{j\Omega})|$

$\cos(\frac{\pi}{2}) = 0$

$-\frac{\pi}{2}$ $\frac{\pi}{2} = \varphi$

Series Interconnection of LTI Systems



↓ intermediate

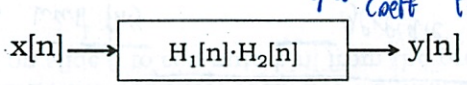
Spectral coefficient scaled

$$w[n] = \sum_{k=\langle N\rangle} a_k H_1(e^{jk\frac{2\pi}{N}})\, e^{jk\frac{2\pi}{N}n} \equiv \sum_{k=\langle N\rangle} b_k e^{jk\frac{2\pi}{N}n}$$

expand out

$$y[n] = \sum_{k=\langle N\rangle} b_k H_2(e^{jk\frac{2\pi}{N}})\, e^{jk\frac{2\pi}{N}n} = \sum_{k=\langle N\rangle} a_k H_1(e^{jk\frac{2\pi}{N}}) H_2(e^{jk\frac{2\pi}{N}})\, e^{jk\frac{2\pi}{N}n}$$

↑ spectral coeff ↑ freq response of two systems



Frequency response of two LTI systems in series is the term-by-term product of the individual frequency responses.

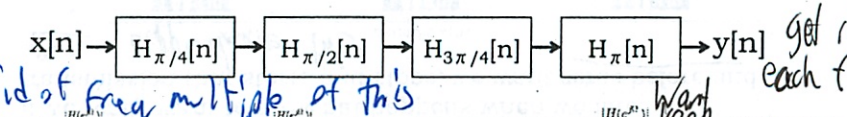
Convolution time domain → mult. in freq domain

get rid of all the freqs past a point

A Poor Man's Low-pass Filter

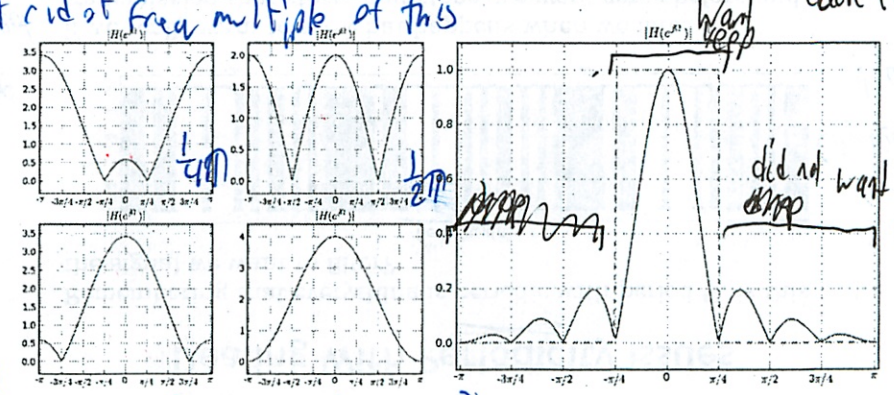
$-\pi$ $-\pi/4$ $\pi/4$ π

Suppose we wanted a low-pass filter with a cutoff frequency of π/4?



get rid of each freq

get rid of freq multiple of this



multiply all together composition

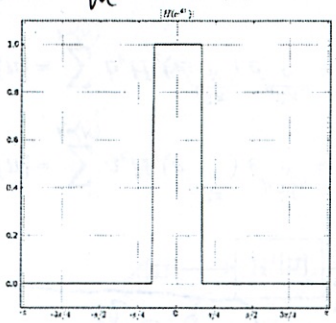
↑ approx

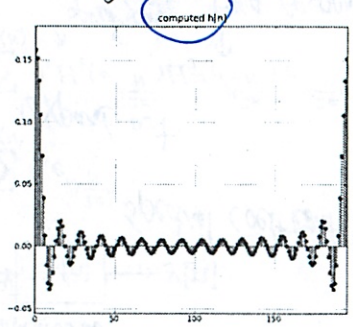
Wait! Maybe This Will Work Better...

Suppose we draw the frequency response we want and then use the equation on slide 6 to compute h[n] from the proposed $H(e^{j\Omega})$?

we want this

generate this





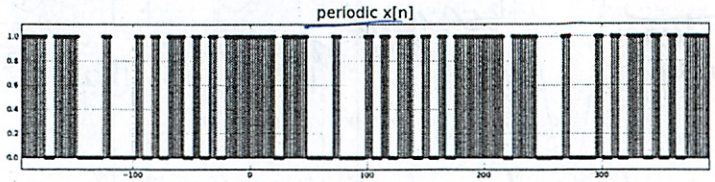
Is that the sample response for our perfect low pass filter

yes + no

Example from previous lecture: N=196, cutoff at ±k=15.

Dealing With Periodicity Issues

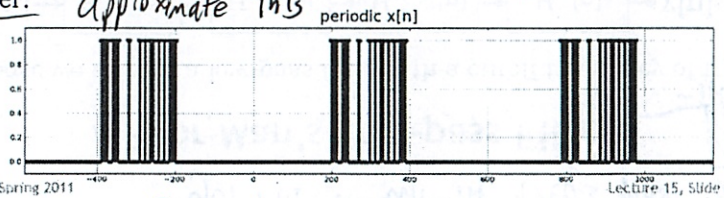
Remembering that everything is periodic with period N, is this the signal we want to filter?



copied over + over

If we really want to see what happens when we filter a transmission that starts and stops, we want zeros before and after:

approximate this



actually hopping to low pass filter this

approx by sticking in extra 0s

Updated Filtering Plan

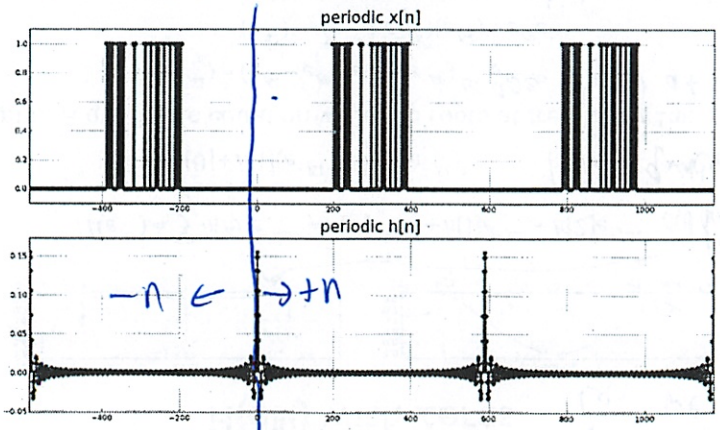
Now that N has grown because of the zero padding, we have to recompute our h[n] using the larger N:

non causal - if have pre-recorded signal -

causal - can't see future

can just shift all the indicies over

convolve



Non-causal h[n]!

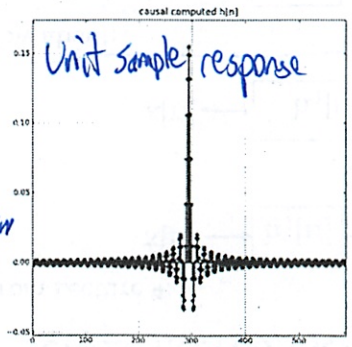
go back to diagram recompute w/ new N

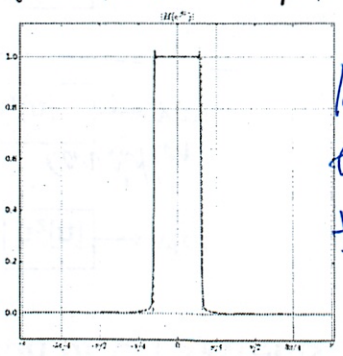
Computed h[n] for Low-pass Filter

now convolve (flip + slide)

If you're hung up on causality (which admittedly is useful for real-time signal processing), we can make a causal h[n] by adding an N/2 sample delay.

filter theory is complex





look what freq kept + thrown away

google "FIR filter"

does a good job stopping things

This freq response I want

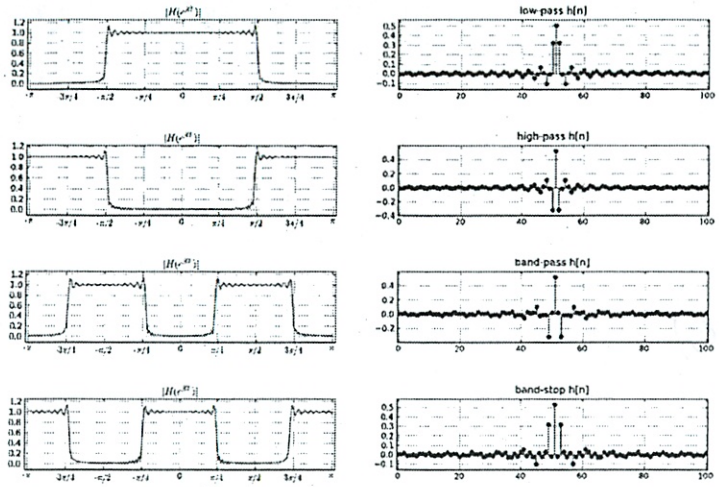
since replication - since periodic in time

- non causal

If access to whole thing – can jump back + forth
– would be non-causal

Useful Filters

lots of work exists on filters



called

do on way out

– both low + high pass

The Need for Speed

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \qquad a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Computing these series involves $O(N^2)$ operations – when N gets large, the computations get very s l o w....

~n^2

Happily, in 1965 Cooley and Tukey published a fast method for computing the Fourier transform (aka FFT, IFFT), rediscovering a technique known to Gauss. This method takes O(N log N) operations.

$N = 1000$, $N^2 = 1000000$ $N\log N \approx 10000$

60 total

Caveat: scaling is different for the FFT: the spectral coefficients *aren't* scaled by 1/N – that scaling happens on the inverse transform back to the time domain.

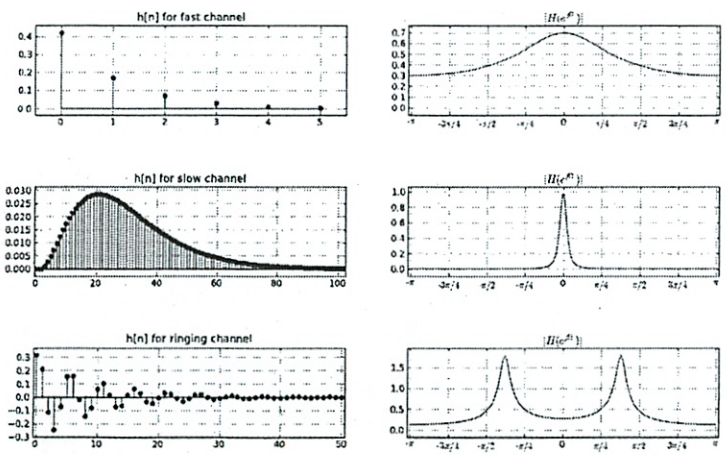
FFT right – puts scalling on left

Computes a_k Spectral coefficents

Frequency Response of Channels

our

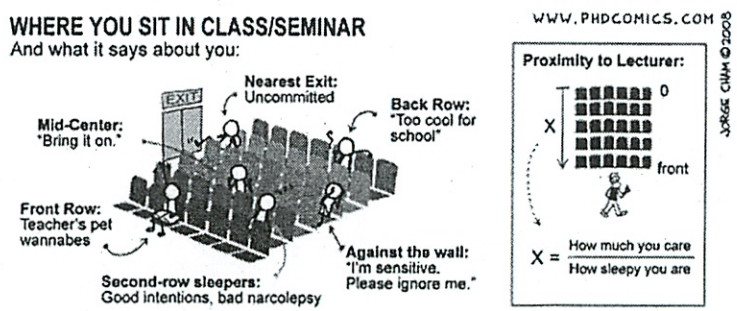
from before



– not losing high freq! protected

thats why expensive

– so not attenuated



$F = \frac{2\pi}{N}$ = fund freq

others are a harmonic of

if $x[n]$ is real $a_k = a_k^*$ (complex conjugates)

but are symetrical - so real

a_0 = avg of $x[n]$

6.02 Recitation

3/31

- all signals can be represented by harmonics
- time $\Leftrightarrow$ freq domain

$x[n]$ "periodic"

$$x[n] = \sum_k a_k e^{j\Omega_k n}$$ alt representation of seq

$\therefore N \ldots k \in \langle N \rangle$

$$\Omega_k = \frac{2\pi}{N} \cdot k$$

Why are we doing this?

LTI systems - if have signal w/ certain freq

$$\tilde{x}[n] = A \cdot \underbrace{e^{j\Omega n}}_{\text{just one harmonic}} \rightarrow \boxed{LTI} \rightarrow B \cdot e^{j\Omega}$$

$$\frac{B}{A} = H(e^{j\Omega})$$

↑ characteristic of freq response

$$\overset{\text{input}}{x\alpha} \rightarrow \square \rightarrow \overset{\text{output}}{y\alpha}$$

↓ LTI does something - constant

$$a_k e^{j\Omega_k n} \rightarrow \square \rightarrow a_k H(e^{j\Omega_k}) e^{j\Omega_k n}$$

(2)

Freq same
- only multiplier changes

Before we did convolving
Now we are doing this
} same thing! just how we are thinking about it changes

FCC rules: don't time split
each use assigned a freq
Ω belongs to that band
Can transmit all the time on that band

Our ears only hear certain freq
- So are like filters in this class

Channel

Only 1 channel is the world
- ~~the~~ radio just one channel
Each use on a diff freq

(3)

$$\tilde{x}[n] = A e^{j\Omega n}$$

$$\tilde{y}[n] = \sum_m h[m]\, x[n-m]$$

$$= \sum_m h[m]\, A\, e^{j\Omega(n-m)}$$

$$= \sum_m h[m]\, (A e^{j\Omega n}) \cdot e^{-j\Omega m}$$

$$= (A e^{j\Omega n}) \left(\sum_m h[m] e^{-j\Omega m} \right)$$

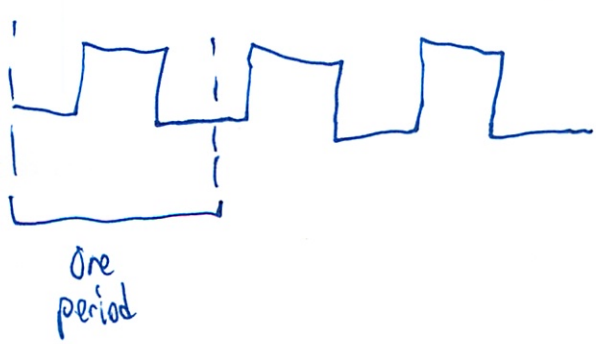
does not depend on time ← since LTI!

$$\tilde{y}[n] = \tilde{x}[n] \cdot \underbrace{H(e^{j\Omega})}_{\sum_m h[m] e^{-j\Omega m}}$$

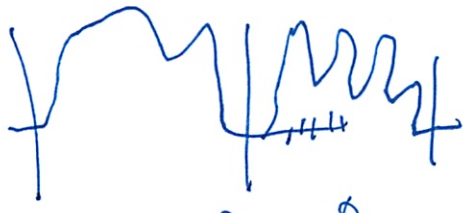
This is periodic signals

But real life is not periodic

- this lecture does not repeat over + over forever



(4)



have 0s in front + back

treat each seperatly

pretend each one is its own periodic signal
Only care for 1 step

1. Decide period $n = 100$
2. But diff message each slot
3. $0 - 5 = 0$ $\quad 44 - 44 = 0$
4. $0 - 99$ as if long periodic seq → period = 100
5. When recieve only use once
6. Repeat for each period

The 0s tapper of the effects

---

6. Can ya define H in terms of h
   - above
b. Given small hs - get H

$h[0] = 1$ $\quad h[2] = 1$
$h[1] = 0$ $\quad h[n] = 0$ for all other n values

(5)

Freq response of channel?

$$H(e^{j\Omega}) = h(0)\cdot 1 + h(1)\cdot e^{-j\Omega} + h(2)\cdot e^{-2j\Omega} + \ldots$$

$\cos(e^0)$

$\cos(0)$

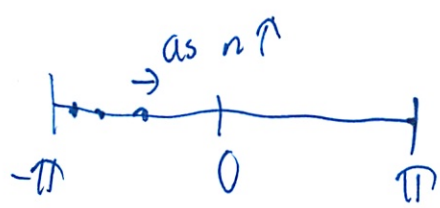
1

$$= 1 + e^{-2j\Omega}$$ function - nice polynomial dependent

$$\Omega_k = 2\frac{\pi}{N}$$

$$-\frac{N}{2}, \ldots, \frac{N}{2}-1$$

$$-\pi, \ldots, \pi$$



Should know H function for $[-\pi, \pi]$ domain

$$H[-\pi, \pi] \to \mathbb{C}$$

complex #

\mathbb{C}

$$\Omega \to H(e^{j\Omega})$$

⑥ other way to write

$$H(x) = 1 + x^{-2}$$

$$= \frac{x^2+1}{x^2}$$ ← way of describing filter

6c) Given this channel, told signal $x[n] = \cos(\varphi n) = \frac{1}{2}(e^{j\varphi n} + e^{-j\varphi n})$

complex exponential form

Want $y[n]=0$ for all n

- Sending signal w/ 2 freq components $\varphi, -\varphi$
- When eval H for that freq, should $=0$ for all n
- output = input · H
- have 2 freq - added up

$\cos(\varphi n)$ is not 0 for all n no matter what φ you choose

$\varphi = 0$ nope $\cos(0) = 1$

$\varphi = \pi$ - nope ... $= -1$

- So H for those freq must be 0
- Is it feasable for H to have 2 0s
- but will it have symetric 0s - for both $\theta, -\theta$

(5) Basically

$$H(e^{j\varphi}) = 0$$

$$H(e^{-j\varphi}) = 0$$

Is that possible?

$$|H(e^{j\Omega})| = |H(e^{-j\Omega})|$$ ← fact

- ~~modulus same~~
- symmetric around 0

$$\left(\sum_m \cos(-\Omega m)\, h(m)\right) + j\left(\sum_m \sin(-\Omega m)\, h(m)\right)$$

? complex # = $H(e^{j\Omega})$

$$\boxed{H(e^{-j\Omega m}) = \cos(-\Omega m) + j\sin(-\Omega m)}$$

$$\sqrt{(\quad)^2 + (\quad)^2}$$ ← will not change for $+, -\Omega$ since symetric

So modulus won't change

Now is there a Ω so that

~~$H(e^{j\Omega}) = 0$~~

do I mean -1?

$$0 = \cos(\Omega m) + j\sin(\Omega m)$$

$$\Omega = \frac{\pi}{2}$$

← tricky - be able to see tricks on the test

8

$$e^{-2j\Omega} = -1 \rightarrow \overbrace{\cos(-2\Omega)}^{\text{real}} + \overbrace{j\sin(-2\Omega)}^{\text{imag}} = -1$$

from part b (still same problem)

get rid since no imag part for other side

$$\cos(2\Omega) = -1$$

$$\Pi \rightarrow \Omega = \frac{\pi}{2}$$

---

d) Find max value of H

$$|H(e^{j\Omega})| = |1 + e^{-2j\Omega}|$$

$$= |1 + \cos(2\Omega) - j\sin(2\Omega)|$$

$$= \sqrt{(1+\cos(2\Omega))^2 + (\sin(2\Omega))^2}$$

$$= \sqrt{1 + 2\cos(2\Omega) + \underbrace{\cos^2(2\Omega) + \sin^2(2\Omega)}_{1}}$$

$$= \sqrt{2 + 2\cos(2\Omega)}$$

When is max of H

$$\overset{\max}{=} \sqrt{4}$$

$$\overset{\max}{=} 2$$

Checkoff 6

4/1

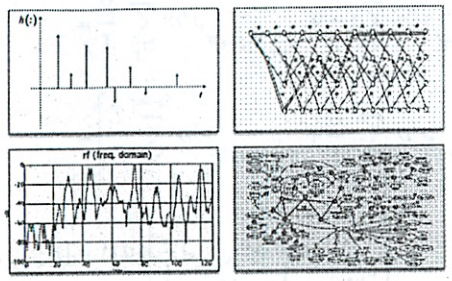
Doing this for ~~FDMA~~
freq multiplexing

- need to understand math first

Util = throughput - T_d = lenght of transmission

Understood wrong
- each person knows who is next
- has to pass token
- think of 1 cycle everyone has opp to talk
- not what I did where cycle was ~~this~~ lists of nos ~~then~~ till someone transmitted

TDMA is a simpler solution
- same thing just on 1 line



INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Spring 2011
Lecture #16

Quiz 2
4/12
26-100

- sharing the frequency spectrum
- modulation
- demodulation

freq division
multiplexing

- see website and old tutorial problems

Take signals + move around in spectrum

f_s, frequency, Ω and k

Various frequency specifications we'll use

- f_s, the sample frequency in samples/sec – radio tells you 102.1 FM
- f, the signal frequency in Hz = cycles/sec
  - $-f_s/2 \le f \le f_s/2$ limit
- Ω, the angular frequency in radians/sample
  - $-\pi \le \Omega \le \pi$ convert sample # to radians
- k, the spectral coefficient index
  - $-N/2 \le k \le N/2$

$$\boxed{\Omega = 2\pi\frac{f}{f_s} = 2\pi\frac{k_\Omega}{N}}$$

$>\pi$ wrap ⊖

$\frac{3\pi}{2} = -\frac{\pi}{2}$

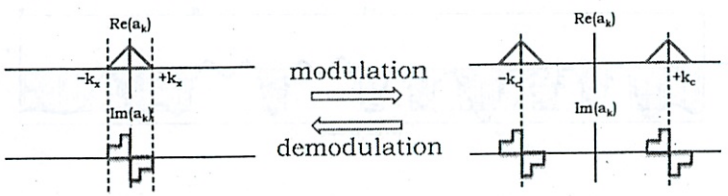
Examples: f_s=1e6 samples/sec, f = 10 kHz, N = 1000
so Ω = .02π and k_Ω = 10

k = 15, N = 100, f_s = 1e6
so Ω = .3π and f = 150 kHz

f, f_s don't need to be multiples

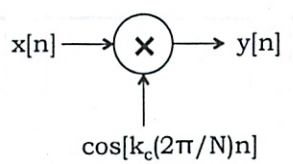
Using Some Piece of the Spectrum

- You have: a band-limited signal x[n] at *baseband* (i.e., centered around 0 frequency).
- You want: the same signal, but centered around some specific frequency $k_c(2\pi/N)$.
- Modulation: convert from baseband up to $k_c(2\pi/N)$
  Demodulation: convert from $k_c(2\pi/N)$ down to baseband



Signal centered at 0 — Signal centered at k_c

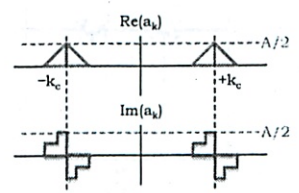
Modulation



For band-limited signal a_k are nonzero only for small range of ±k

$$y[n] = \left[\sum_{k=-k_x}^{k_x} a_k e^{jk\frac{2\pi}{N}n}\right]\left[\frac{1}{2}e^{jk_c\frac{2\pi}{N}n} + \frac{1}{2}e^{-jk_c\frac{2\pi}{N}n}\right]$$

$$= \frac{1}{2}\sum_{k=-k_x}^{k_x} a_k e^{j(k+k_c)\frac{2\pi}{N}n} + \frac{1}{2}\sum_{k=-k_x}^{k_x} a_k e^{j(k-k_c)\frac{2\pi}{N}n}$$



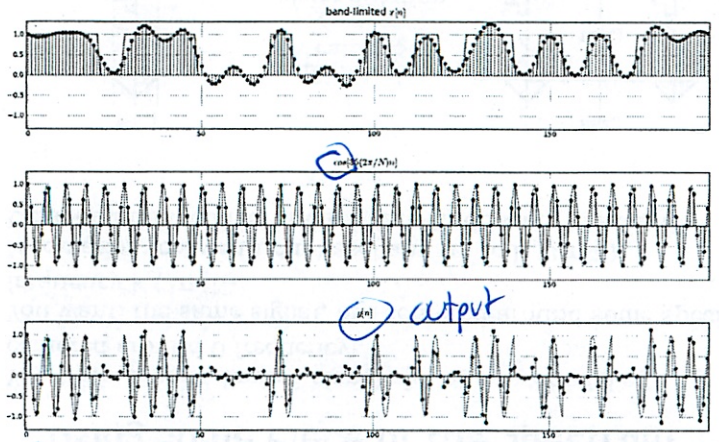
$y[n] = x[n] \cdot \cos\left(k_c \cdot \frac{2\pi}{N} n\right)$

$= \sum_{k=-k_x}^{k_x} a_k e^{j\frac{k2\pi}{nN}}$ express as

→ flip

$$= \left[\sum_{k=-k_x}^{k_x} a_k e^{j\frac{k2\pi}{N}n}\right]\left[\frac{1}{2}e^{jk_c\frac{2\pi}{N}n} + \frac{1}{2}e^{-jk_c\frac{2\pi}{N}n}\right]$$ Continued sep sheet

Example: Modulation (time)



= original

35 cycles of wave points in N

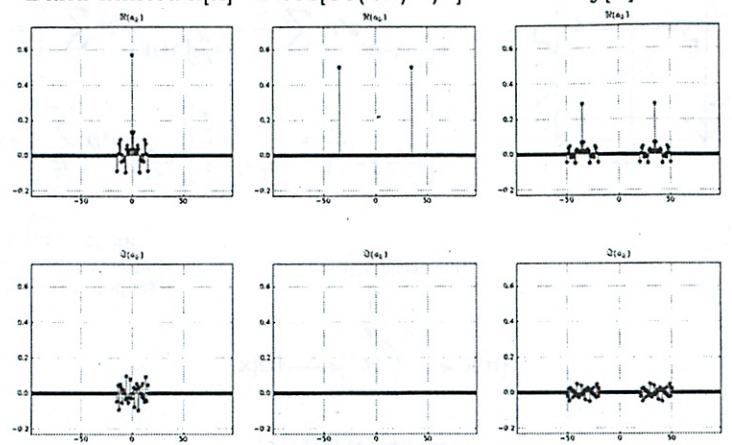
output

everytime in = 1 get a burst out of carrier freq

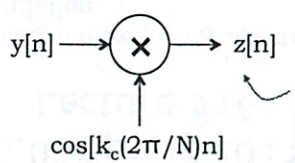
turn waveform into spectrial coeffs.

Example: Modulation (freq)

Spectral coeffs



Demodulation



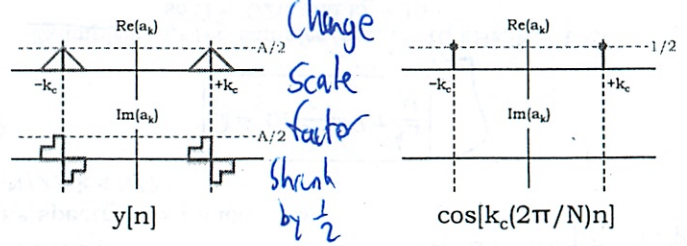
Hmm. So z[n] has what we want at baseband, but has signal we don't want at $\pm 2f_c$

$$z[n] = y[n]\left[\frac{1}{2}e^{jk_c\frac{2\pi}{N}n} + \frac{1}{2}e^{-jk_c\frac{2\pi}{N}n}\right]$$

$$= \left[\frac{1}{2}\sum_{k=-k_x}^{k_x} a_k e^{j(k+k_c)\frac{2\pi}{N}n} + \frac{1}{2}\sum_{k=-k_x}^{k_x} a_k e^{j(k-k_c)\frac{2\pi}{N}n}\right]\left[\frac{1}{2}e^{jk_c\frac{2\pi}{N}n} + \frac{1}{2}e^{-jk_c\frac{2\pi}{N}n}\right]$$

$$= \frac{1}{4}\sum_{k=-k_x}^{k_x} a_k e^{j(k+2k_c)\frac{2\pi}{N}n} + \underbrace{\frac{1}{2}\sum_{k=-k_x}^{k_x} a_k e^{jk\frac{2\pi}{N}n}}_{\text{What we want}} + \frac{1}{4}\sum_{k=-k_x}^{k_x} a_k e^{j(k-2k_c)\frac{2\pi}{N}n}$$

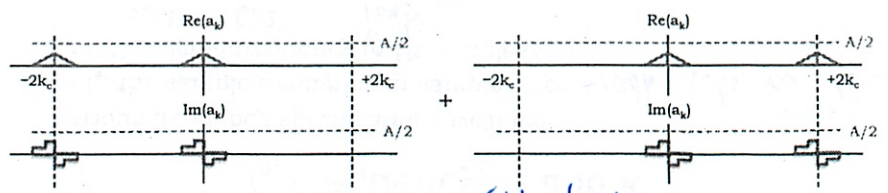
Demodulation Frequency Diagram



Change Scale factor shrink by $\frac{1}{2}$

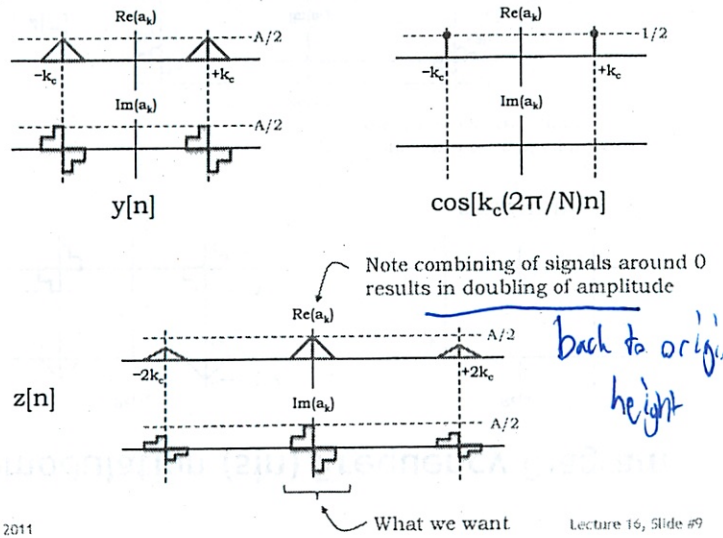
z[n]

Center around k_c



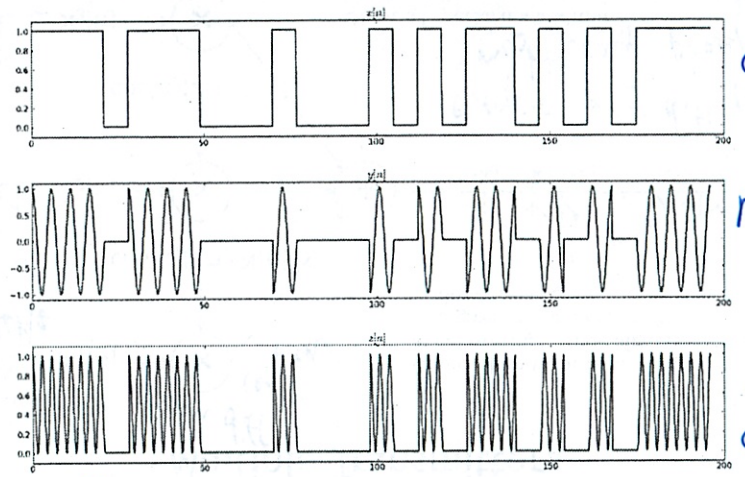
Add together for final spectrum

Demodulation Frequency Diagram



back to original height

Example: Demodulation (time)



original

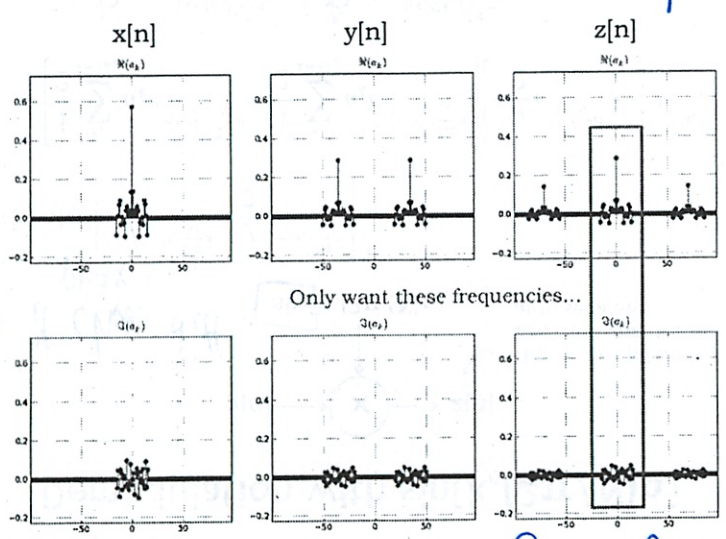
modulated up to 35

demod using 35

Showing idealized signals

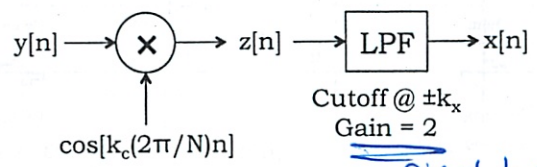
filled in waveform!

Example: Demodulation (freq) Spectrum involved

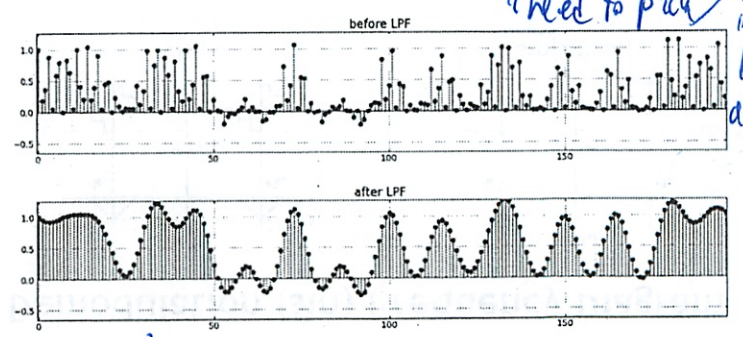


get rid of this - w/ low pass filter

Demodulation + LPF



need to pick

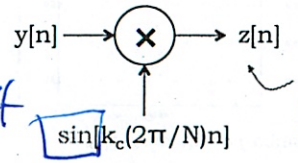


inc both low freq want and high freq

pick freq want to "tune in" demodulate to base band - filter to only signal you want

decoded w/ same phase

Demodulation with $\sin[k_c(2\pi/N)n]$



Hmm. So z[n] no longer has the signal we want at baseband!

what if chose diff phase?

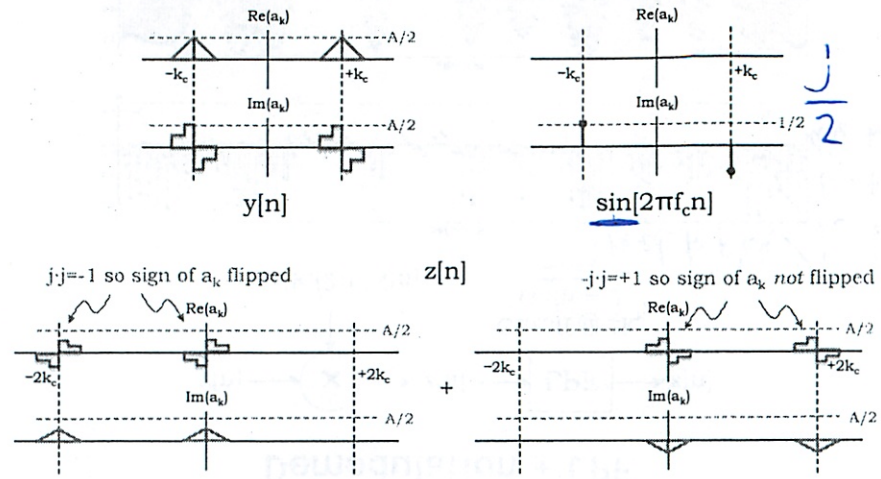
$$z[n] = y[n]\left[-\frac{j}{2}e^{jk_c\frac{2\pi}{N}n} + \frac{j}{2}e^{-jk_c\frac{2\pi}{N}n}\right]$$

$$= \left[\frac{1}{2}\sum_{k=-k_x}^{k_x} a_k e^{j(k+k_c)\frac{2\pi}{N}n} + \frac{1}{2}\sum_{k=-k_x}^{k_x} a_k e^{j(k-k_c)\frac{2\pi}{N}n}\right]\left[-\frac{j}{2}e^{jk_c\frac{2\pi}{N}n} + \frac{j}{2}e^{-jk_c\frac{2\pi}{N}n}\right]$$

$$= -\frac{j}{4}\sum_{k=-k_x}^{k_x} a_k e^{j(k+2k_c)\frac{2\pi}{N}n} + \frac{j}{4}\sum_{k=-k_x}^{k_x} a_k e^{j(k-2k_c)\frac{2\pi}{N}n}$$

Oops, no baseband signal!

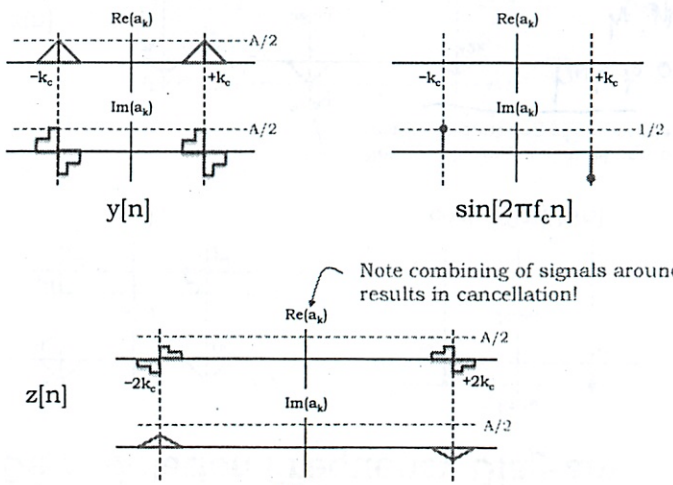
Demodulation (sin) Frequency Diagram



$\frac{j}{2}$

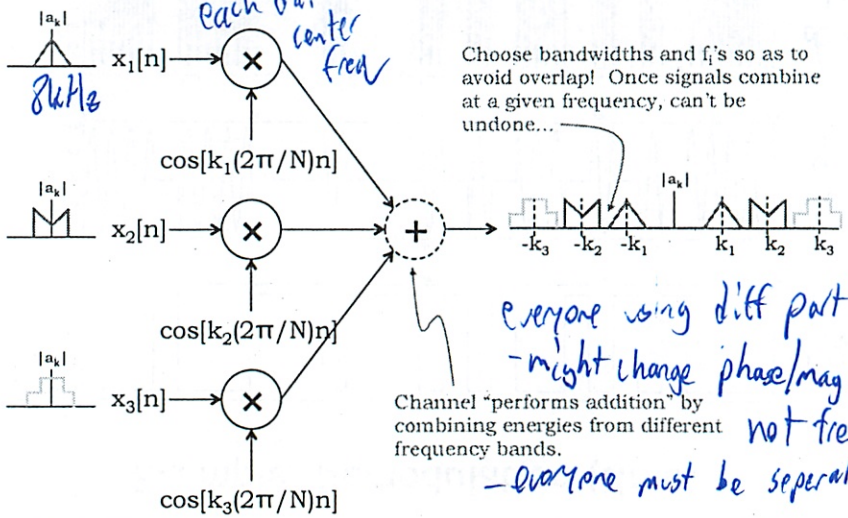
all real → img
img → real flipped (-1)

Demodulation (sin) Frequency Diagram



signal Cancles!

Multiple Transmitters



each diff center freq

Choose bandwidths and f_i's so as to avoid overlap! Once signals combine at a given frequency, can't be undone...

Channel "performs addition" by combining energies from different frequency bands.

everyone using diff part of sys
- might change phase/mag - but not freq
- everyone must be seperate

↓ solved wed

everyone would need to agree when $t=0$! But this is hard to be in phase

Lecture 16 Addendum

$$= \frac{1}{2}\sum_k a_k e^{j(k+k_c)\frac{2\pi}{N}n} + \frac{1}{2}\sum_k a_k e^{j(k-k_c)\frac{2\pi}{N}n}$$

Look at some examples

 is original

---

<u>Demo</u>

Peaks are diff radio stations

"Tune" to one

- pull at base band signal

Need to be exactly in center

Sounds weird if b/w stations

All of this is happening in real time

6.02 Review

WP **Carrier wave**
- modulated w/ an input signal to carry info
- you say its center freq

Cutoff freq - boundry in freq response
- energy begins to be ↓

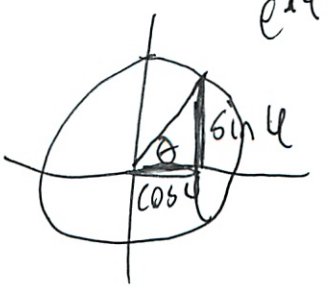
Freq response - measure of system's output spectrum in response to input signal
depends on the item
magnitude + phase vs freq
w/ test signal

Fourier series
- decompose periodic fn into sum of (possibly ∞) simple oscillating functions named sines + cosines (or complex exponentials)

Complex Exponential = Euler's formula

$$e^{ix} = \cos x + i \sin x$$

$e^{i\varphi} = \cos\varphi + i\sin\varphi$

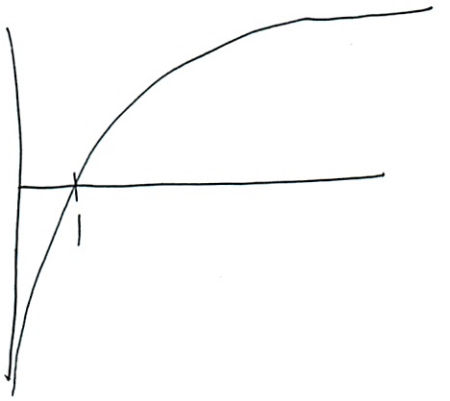


* **i = imaginary unit** *

e = base of natural log

②

<u>Natural log</u> = area under curve $y = \frac{1}{x}$ from 1 to e



$$= \int_1^a \frac{1}{x}\,dx$$

$$e^{\ln x} = x$$

$$\ln(e^x) = x$$

<u>Complex logs</u>

- multiplicitive property still works

<u>Imaginary #s</u> - number that gives ⊖ result when squared

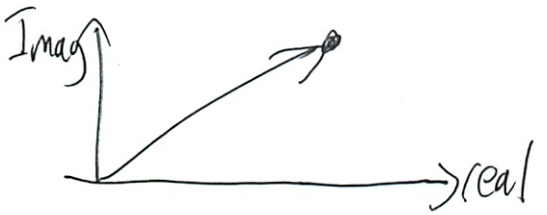
i = imaginary unit

$i^2 = -1$

$\underbrace{a}_{\text{real part}} + \underbrace{b}_{\text{imag part}} i$ ← complex #

③

<u>Complex #</u> - real part and imag part



So can solve $x^2+1=0$

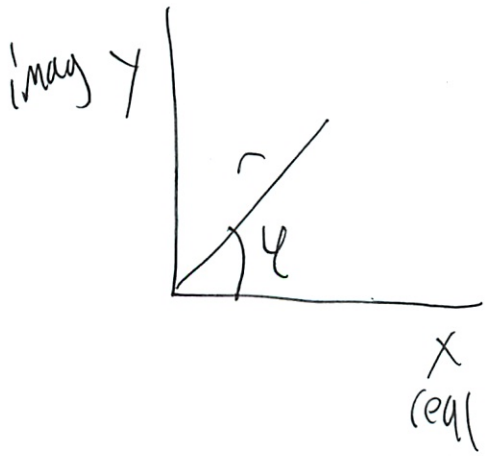
$x^2=-1$

$x=\pm i$

since $i^2=-1$

Can add geometrically

Can express in polar form



$z=x+yi$

so

$r=|z|=\sqrt{x^2+y^2}$

$\varphi=\arctan\left(\frac{y}{x}\right)$ if $x>0$

$=$ " $+\pi$ if $x<0$ and $y\geq 0$

$=$ " $-\pi$ if $x<0$ and $y<0$

$=\frac{\pi}{2}$ if $x=0$ and $y>0$

$=-\frac{\pi}{2}$ if $x=0$ and $y<0$

$=$ Und if $x=0$ $y=0$

$z=r(\cos\varphi+i\sin\varphi)$

(4)

Euler's Formula

$$\frac{1}{1+x^2} = \frac{1}{2}\left(\frac{1}{1-ix} + \frac{1}{1+ix}\right)$$

Fourier Series

- Study off = Fourier analysis
- partial diff eq

Diff eq

- unknow fn for 1 or several variables that relate function of itself + deriv
- (so solving simultaneously?)

Partial Diff Eq

- w/ partial derivitives

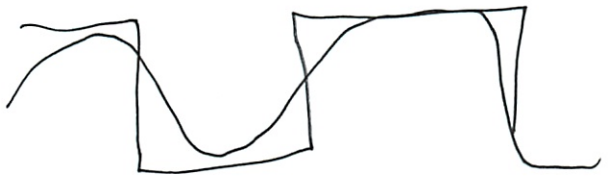
Partial deriv

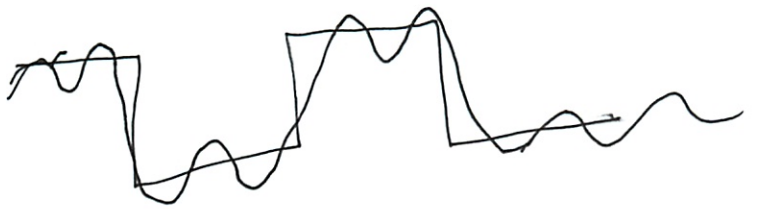
- deriv w/ respect to one variable
- others held constant

∂

(5) <u>Fourier series</u> - originally heat
- but anything sinusodial

So like an approx for square wave





<u>Fourier analysis</u> - general fns represented as sum of simpler trig functions

- <u>analysis</u> - breaking up
- <u>synthesis</u> - putting together

<u>Forier transforms</u> - is a list of possible transforms

<u>Discrete Fourier Transform</u> - transforms 1 fn to another
- called the freq domain representation

<u>Freq domain</u> - just chart in terms of freq, not time

(6)

Discrete F. Trans.

- Sample discretely

$$N_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N} kn}$$

did in recitation

Freq ω = # occilations per time

$\exists A$

1 occilation

phase - where in cycle it begins

$\leftrightarrow$ move left + right

6.02 Recitation

4/5

Quiz - 1 week from today
Lectures 6-15
& last wed

Modulation + Filter Design

Last time LTI channel

- in terms $h[\,]$ - time domain

$$x[\,] \to h[\,] \to y[\,]$$

↑ h convolved w/ x

but we are writing
- for each freq

$$x[\,] = \sum a_k e^{j\Omega k}$$

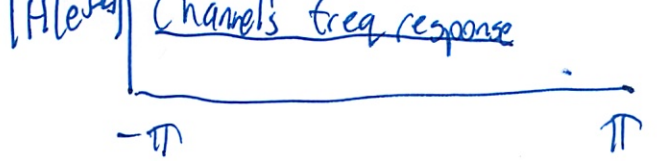
$$\downarrow$$

$$y[\,] = \sum_k a_k H(e^{j\Omega k}) \cdot e^{j\Omega k}$$

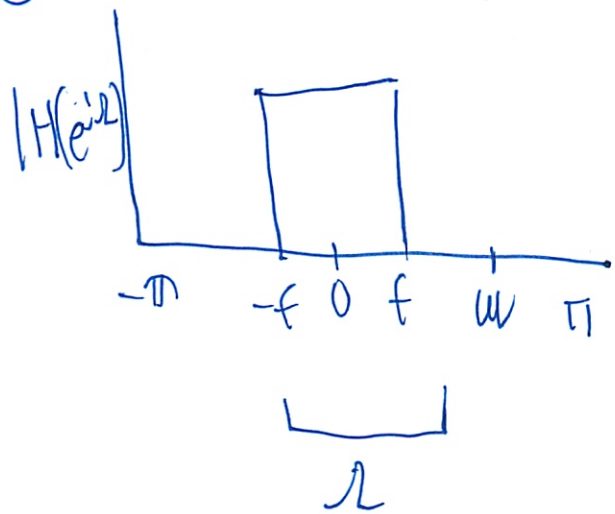
↑ channel freq response

$$\sum_{m=-\infty}^{\infty} h(m)\, e^{-j\Omega m}$$

Can we design a good channel? Filter design



②



$$x = \sum_{\Omega \in [-f, f]} a_\Omega e^{j\Omega n}$$

But some noise gets added
- high freq

$$z(n) = x(n) + N(n)$$

what recieve — transmitted — noise $b_\omega \cdot e^{j\omega n}$

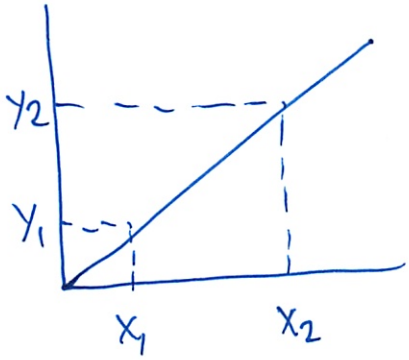
$$= \sum_{\Omega \in [-f, f]} a_\Omega e^{j\Omega n} + b_\omega e^{j\omega n}$$

throw this out w/ a filter

Can design a filter as an LTI channel
- choose $h()$ so that its freq response has a certain behavior

(3)

In HW, we did curve fitting of a line



$$y = Ax + B$$

$$\left.\begin{aligned} y_1 &= Ax_1 + B \\ y_2 &= Ax_2 + B \end{aligned}\right\} \text{Solve } A, B$$

Will do pattern matching for $h()$

$$H(e^{j\Omega k}) = \sum_{m=-\infty}^{\infty} h(m)\, e^{-j\Omega k m}$$

$$H(e^{j\Omega}) = \sum h(m) (e^{-j\Omega})^m$$

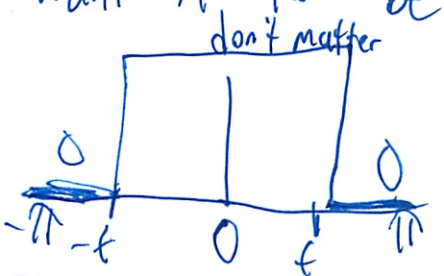
$$H(x) = \sum h(m)\, x^m$$

$$\left\{ x = e^{-j\Omega} \right. \quad \text{) its a polynomial!}$$

Want to fit a polinomial to get a certain behavior

Can use numpy fitting function

- want it to be 0 at certain points



want it to have roots here

many points

(4)

Could also do a more constructive, simple manner
- like in lecture

If had a quadratic, Want 0 at A, B. Real

$$g(x) = \text{quadratic} = \alpha x^2 + \beta x + \gamma$$

Want $g(a) = 0 \quad g(b) = 0$

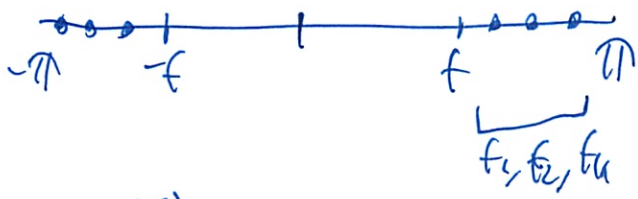
One such one is

$$(x-a)(x-b)$$

Now suppose want $g(a) = g(b) = g(c) = 0$

$$(x-a)(x-b)(x-c)$$

Pick at a few values



$$\underbrace{(x - e^{-jf})(x - e^{jf})}_{\text{1st filter}} \underbrace{(x - e^{-jf_1})(x - e^{jf_2})}_{\text{2nd filter}} \cdots (x - e^{-jf_k})(x - e^{jf_k})$$

e^{-jk}

The more points you have, the better the approx.

5

When put filter in series → multiply

$$\boxed{(x-e^{-jf})(x-e^{jf})} \rightarrow \boxed{(x-e^{-jf_1})(x-e^{jf_1})} \rightarrow \cdots$$

$$\boxed{(x-e^{-jf_k})(x-e^{jf_k})}$$

So this is same as previous filter

What should its small hs be?

quadratic expansion

$$x^2 - (e^{jf} - e^{-jf})x + 1$$

$h(2) = 1$

$h(1) = -2\cos(f)$

$h(0) = 1$

Only powers that are there read off coefficients

$H(z) = \sum_{k=0}^{3} h(k) z^{-k}$

$z = e^{-j2\pi f}$

Often this seems so technical
- need to step back + see bigger pic

$$e^{jx} = \cos x + j\sin x$$

$$e^{-jx} = \cos(-x) + j\sin(-x)$$

$$e^{jx} + e^{-jx} = 2\cos(x)$$

Could do in time domain
- Convolve 1 by 1
- too much convolution!

Theme: using polynomials to find a fit algebraically for a particular response

Add 0 padding to overcome periodicity

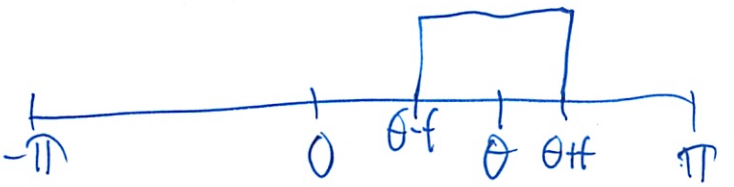
Modulation

If all you care about a certain freq band $-f \to f$

- Use a filter

But what if want to tune into 2 things?

Related to modulation



First design normal filter, centered around 0

Then modulate over $\to$ to be centered around θ

$$x[n] = a_\Omega e^{j\Omega}$$

$$\Omega \in [-f, f]$$

Multiply by $\cos(\theta n) = \frac{1}{2}(e^{j\theta n} + e^{-j\theta n})$

7

Resulting output

$$y[n] = x(n) \cdot \cos(\theta n)$$

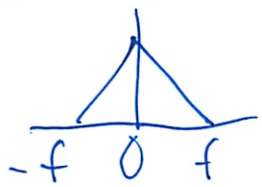
$$= \left(\sum_{\Omega} a_{\Omega} e^{j\Omega n}\right)\left(e^{j\theta n} + e^{-j\theta n}\right) \cdot \frac{1}{2}$$

Ω shifted by $\pm\theta$

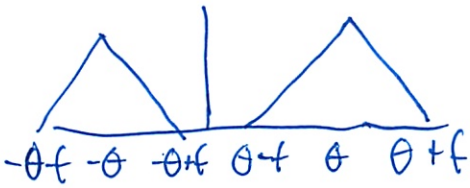
$$= \frac{1}{2}\sum_{\Omega} a_{\Omega} e^{j(\Omega+\theta)n} + \frac{1}{2}\sum_{\Omega} a_{\Omega} e^{j(\Omega-\theta)n}$$

Shifted in 2 parts
(split into 2 pieces)

started w/



$\cdot \cos(\,)$



both showed up

- real part and imag part
  ↑ focusing on real part

So shifting

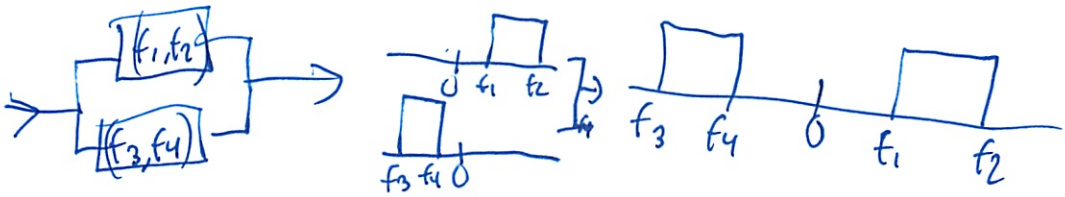
$\cdot$ by $e^{-j\theta n}$

then use a normal filter

8

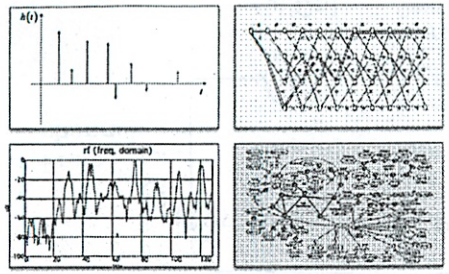
Modulation / filter design kinda the same thing

Filters in parallel



They are like gatekeepers

If don't get through one, try the other!



INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Spring 2011
Lecture #17

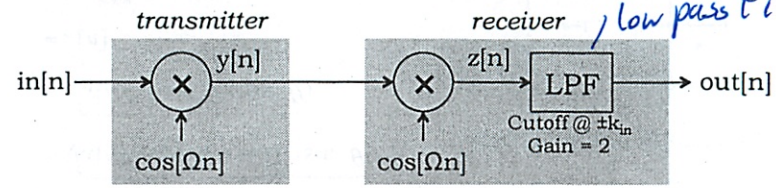
- mismatch of rcvr's freq & phase
- quadrature demod
- BPSK, QPSK, DQPSK

Quiz Tue
7:30-9:30
26-100

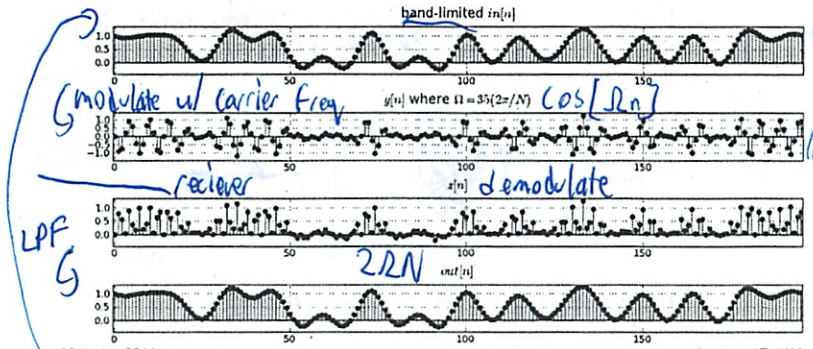
No lecture mon

Mon's lecture

Ideal Modulation/Demodulation



low pass filter



modulate w/ carrier freq cos[Ωn]

out everything else

discretized

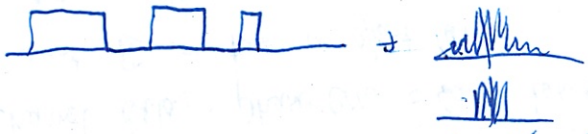
reciever demodulate

LPF

2Ω

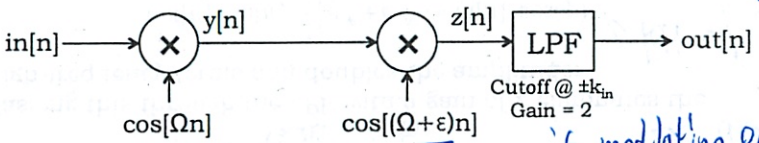
just like x[n]

x[n] = in[x]

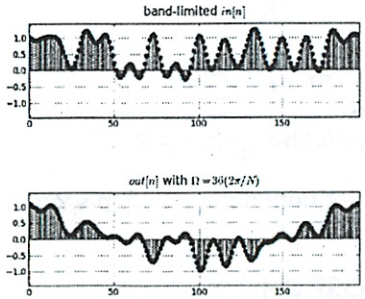


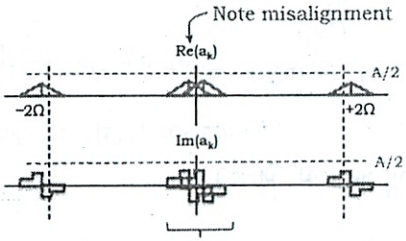
baseband - very low freq
- only I can use
- huge antenna

Frequency Error in Demodulator



if modulation error



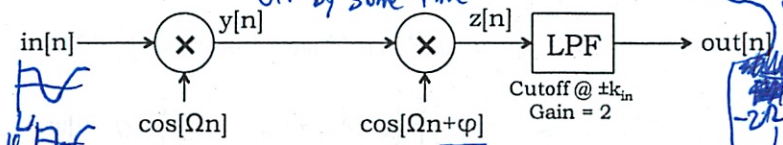


Baseband k's not correct
Combine amplitudes from different k's!

demodulate not where modulate was

Made copies in weird places

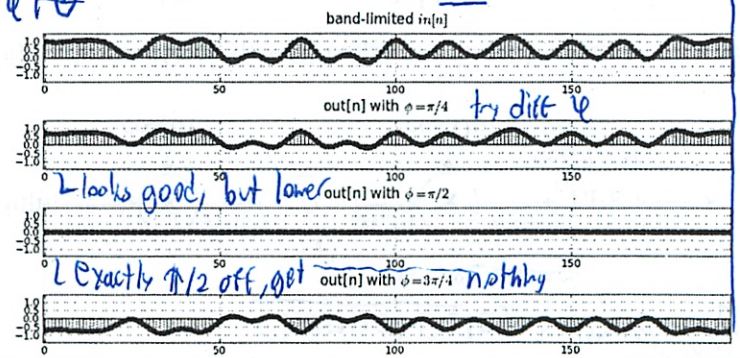
Phase Error in Demodulator



off by some time

modulation & reciever

-2Ω LPF 2Ω discard



try diff φ

looks good, but lower

exactly π/2 off, get nothing

Phase Error Math

Let's derive an equation for z[n]:

could do math in cos

$$z[n] = y[n]\cdot\cos[\Omega n+\varphi] = in[n]\cdot\cos[\Omega n]\cdot\cos[\Omega n+\varphi]$$

$$= in[n]\cdot\frac{1}{2}\left[e^{j\Omega n}+e^{-j\Omega n}\right]\cdot\frac{1}{2}\left[e^{j\varphi}e^{j\Omega n}+e^{-j\varphi}e^{-j\Omega n}\right]$$

easier to use complex exp

$$= in[n]\cdot\frac{1}{4}\left[e^{j\varphi}e^{j2\Omega n}+e^{j\varphi}+e^{-j\varphi}+e^{-j\varphi}e^{-j2\Omega n}\right]$$

scaling factor

2's disappear

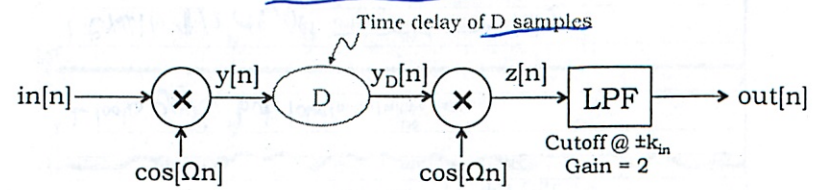
Passing this through the LPF with a gain of 2 eliminates the high-frequency terms and doubles the amplitude:

$$out[n] = in[n]\cdot\frac{2}{4}\left[e^{j\varphi}+e^{-j\varphi}\right] = in[n]\cdot\cos(\varphi)$$

left w/ just

So a phase error of φ results in amplitude scaling of cos(φ).

Channel Delay = time of flight



$$z[n] = y_D[n]\cdot\cos[\Omega n] = y[n-D]\cdot\cos[\Omega n] = in[n-D]\cdot\cos[\Omega(n-D)]\cdot\cos[\Omega n]$$

$$= in[n-D]\cdot\frac{1}{2}\left[e^{-j\Omega D}e^{j\Omega n}+e^{j\Omega D}e^{-j\Omega n}\right]\cdot\frac{1}{2}\left[e^{j\Omega n}+e^{-j\Omega n}\right]$$

$$= in[n]\cdot\frac{1}{4}\left[e^{-j\Omega D}e^{j2\Omega n}+e^{-j\Omega D}+e^{j\Omega D}+e^{j\Omega D}e^{-j2\Omega n}\right]$$

Passing this through the LPF:

Looks like a phase error

$$out[n] = in[n]\cdot\frac{2}{4}\left[e^{-j\Omega D}+e^{j\Omega D}\right] = in[n]\cdot\cos(\Omega D)$$

Channel delay + phase error = same issue

just don't want $\frac{\pi}{2}$ – take a giant step

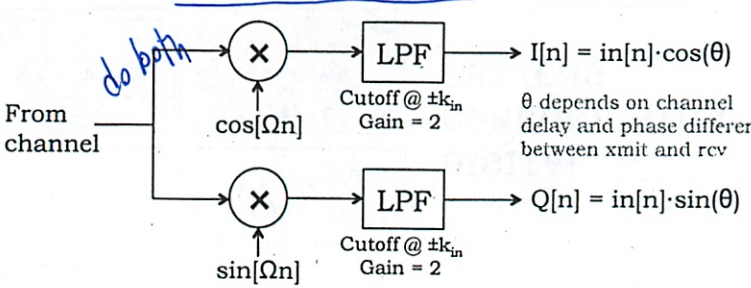
had to do on old radios

Fixing Phase Problems in the Receiver

So phase errors and channel delay both result in a scaling of the output amplitude, where the magnitude of the scaling can't necessarily be determined at system design time:

- channel delay varies on mobile devices
- phase difference between transmitter and receiver is arbitrary

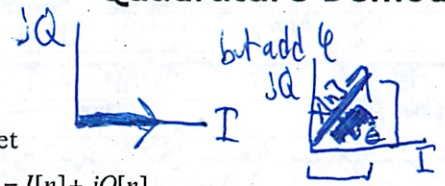
One solution: *quadrature demodulation* modern radios use

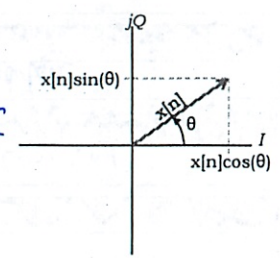


? I will be non 0

could compensate for delays

Quadrature Demodulation





If we let

$$w[n] = I[n] + jQ[n]$$

then

$$|w[n]| = \sqrt{I[n]^2+Q[n]^2}$$

so add

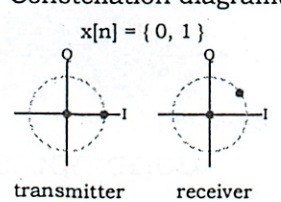
$$= \sqrt{x[n]^2\cdot\cos^2\theta + x[n]^2\cdot\sin^2\theta}$$

$$= \sqrt{x[n]^2\cdot(\cos^2\theta+\sin^2\theta)}$$

$$= x[n]$$

belt + suspenders approach

Constellation diagrams:



Other types of modulation systems

Phase Modulation

A sinusoid is characterized by its frequency, amplitude and phase – one can modulate anyone of these using x[n], which represents the message to be transmitted.

- Amplitude modulation (AM) – what we've done so far – times when signal goes away
- Frequency modulation (FM)
- Phase modulation – our next topic detect changes in phase

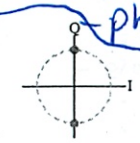
Using AM the signal can have zero amplitude, indistinguishable from no signal at all, which can confound the circuitry that "fine tunes" the amplitude and frequency at the receiver.

Using phase modulation, aka phase-shift keying (PSK), the transmitted signal has constant amplitude; information is encoded in the phase of the carrier sinusoid.

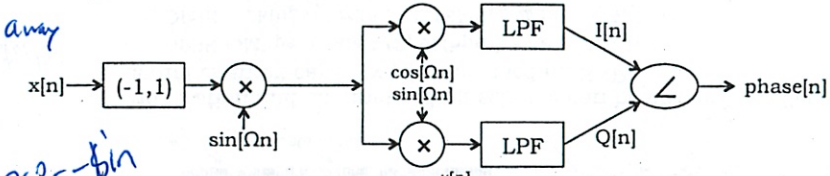
always getting cos in

Need to tell 3 things about waves
- mag
- freq
- phase

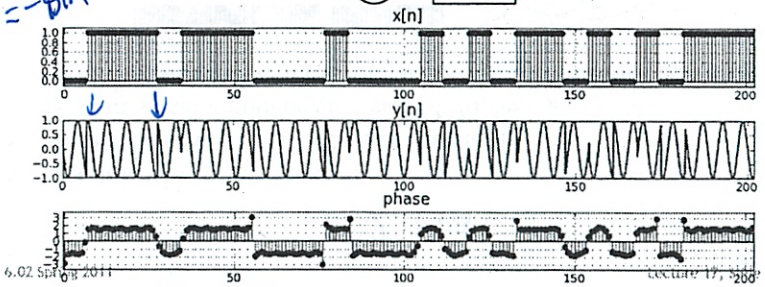
BPSK



In binary phase-shift keying (BPSK), the message bit selects one of two phases for the carrier, e.g., π/2 and –π/2.



5π –180° = –π/2



everytime chang it changes phase 180°

calc phase angle – can't sample, get x[n]

Dealing With Phase Ambiguity

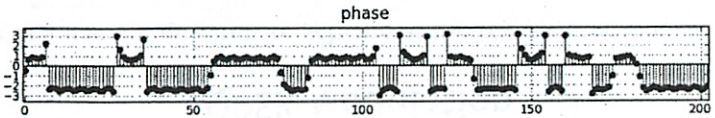
apparent phase change at reciever

reciever going back and forth b/w

BPSK is also subject phase changes introduced by channel delays or phase difference between xmit and rcv: the received constellation will be rotated with respect to the transmitter's constellation. Which phase corresponds to which bit?

was it 0→0 or 0→1

The fix? Think of the phase encoding as *differential*, not absolute: a change in phase corresponds to a change in bit value. Assume that, by convention, messages start with a single 0 bit, i.e., prepend a 0 to each to message. Then the first phase change represents a 0→1 transition, the second phase change a 1→0 transition, and so on.



don't know which was 1 or 0 – put know change in phase

Can't tell

Differential PSK

many variations on differential means

- Approach 1: encode bits
  - DBPSK – process one message bit at a time
    - "0": phase change is 0
    - "1": phase change is π
  - DQPSK – process two message bits at a time
    - "00": phase change is 0
    - "01": phase change is π/2
    - "10": phase change is π
    - "11": phase change is 3π/2
- Approach 2: encode transitions
  - DBPSK
    - No transition: phase change is 0
    - Transition: phase change is π
  - DQPSK: assume cyclic order is 00, 01, 11, 10
    - No advance in order: phase change is 0
    - Advance one position in order: phase change is π/2
    - And so on...

What about 3 bits at time, i.e., 8 constellation points?

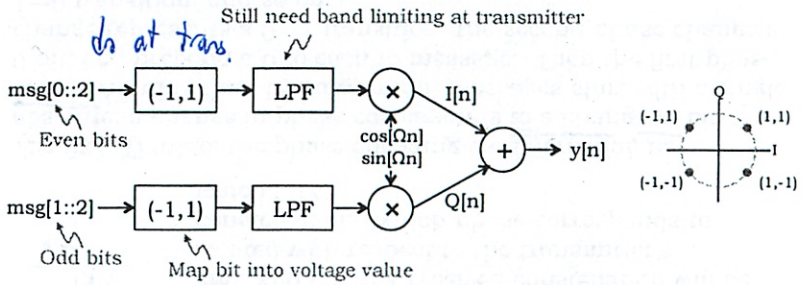
Could encode in > 2 phases

Noise gives clarity of data – cuts into noise margins

be flexible!

QPSK Modulation

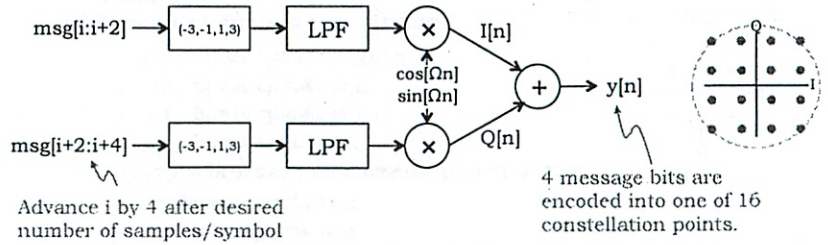
We can use the quadrature scheme at the transmitter too:



When mapping bits to voltage values, we should choose the values so that the maximum amplitude of y[n] is 1. For QPSK (also referred to as QAM-4) that would mean $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=(.707,.707)$

Quadrature Amplitude Modulation (QAM)

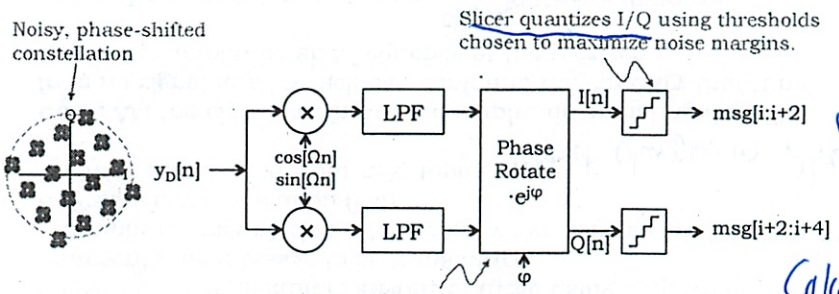
Using more message bits at a time, we can generate larger constellations using quadrature amplitude modulation. Here's a diagram of a QAM-16 system with 16 constellation points:



Larger constellations mean points are closer together, so there's more sensitivity to noise. Some systems adapt the size of the constellation to the noise level (QAM-4, QAM-16, QAM-64, ...), i.e., use 2, 4, 6, ... bits/symbol.

QAM Receiver

Here's a simplified diagram of a QAM-16 receiver:



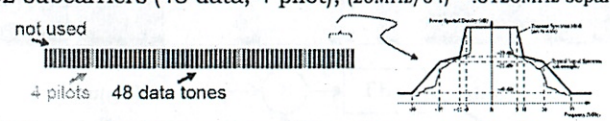
did satellight chips when it first started

Calculated φ

— so constellations come out right

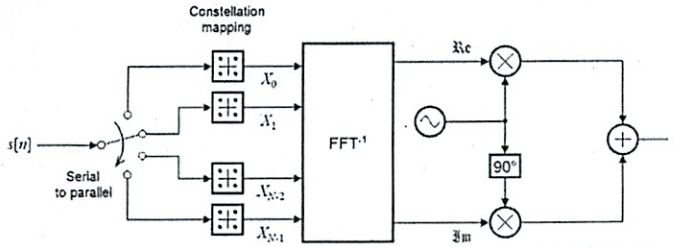
802.11a

- 12 channels in 5GHz band
- 20MHz bandwidth (16.6MHz occupied)
  - Orthogonal Frequency Division Multiplexing (OFDM)
  - 52 subcarriers (48 data, 4 pilot), (20MHz/64) = .3125MHz separation



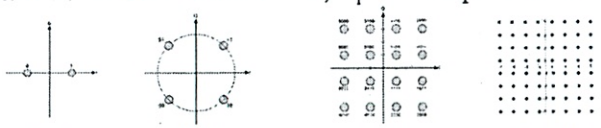
  - Modulation and channel coding scheme can be chosen to reflect actual channel capacity (choice based on SNR):
    - BPSK (6Mbps @ rate 1/2, 9Mbps @ rate 3/4, 1 bit)
    - QPSK (12Mbps @ rate 1/2, 18Mbps @ rate 3/4, 2 bits)
    - 16-QAM (24Mbps @ rate 1/2, 36Mbps @ rate 3/4, 4 bits)
    - 64-QAM (48Mbps @ rate 2/3, 54Mbps @ rate 3/4, 6 bits)
  - Symbol duration 4us (includes guard interval of 0.8us)
    - ~250k combined symbols per second, 48 data channels
    - ~Data rate = (48)(250k)(# bits/symbol)(code rate)
      = (12M)(# bits/symbol)(code rate)

OFDM Transmitter



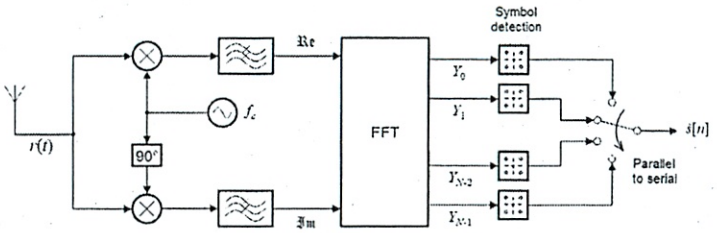
http://en.wikipedia.org/wiki/OFDM

- Signal is interleaved across multiple subchannels
  - For bidirectional links, can use some subchannels for each direction
- Constellation mapping can be chosen separately for each subchannel: for other than BPSK, X_i are complex values



BPSK QPSK 16-QAM 64-QAM

OFDM Receiver



- Need good frequency synchronization at receiver to keep subchannels orthogonal, need good gain control to keep amplitudes correct for slicing.
- Low-pass filter selects demodulated baseband signal
- Symbol detection for each subchannel is matched to modulation scheme selected by transmitter

To save your work, click the SAVE button at the bottom of this page. You can revisit this page, revise your answers and SAVE as often as you like.

To submit the assignment, click the SUBMIT button at the bottom of this page. YOU CAN SUBMIT ONLY ONCE. Once the assignment has been submitted, you can continue to view this page but will no longer be able to make any changes to your answers.

6.02 Spring 2011: Plasmeier,Michael E.

PSet PS7

Dates & Deadlines

4/5

issued: Mar-30-2011 at 00:00
due: Apr-07-2011 at 06:00
checkoff due: Apr-12-2011 at 06:00

Help is available from the staff in the 6.02 lab (38-530) during lab hours -- for the staffing schedule please see the Lab Hours page on the course website. We recommend coming to the lab if you want help debugging your code.

For other questions, please try the 6.02 on-line Q&A forum at Piazzza.

Your answers will be graded by actual human beings, so your answers aren't limited to machine-gradable responses. Some of the questions ask for explanations and it's always good to provide a short explanation of your answer.

Problem 1.

I have no clue how to do these type of problems

Consider an LTI system consisting of two filters hooked in series. The operation of the system is described by the equations below. `x[n]` is the input to the system, `w[n]` is the output of the first filter and input to the second, and `y[n]` is the output of the second filter.

```
w[n] = x[n] - sqrt(2)*x[n-1] + x[n-2]
y[n] = w[n] - sqrt(3)*w[n-1] + w[n-2]
```

A. Suppose `x[n]` = $e^{j\varphi n}$ for all `n`, For what positive value of φ is `w[n] = 0` for all `n`, i.e., for which φ is `H(e^{jφ})` = 0? Hint: for the first filter derive a formula for $H(e^{j\Omega})$ from the given h[n]. Then determine for what positive value of Ω your formula evaluates to 0.

So $w[n] = e^{j\varphi n} - \sqrt{2}\, e^{j\varphi n} + e^{j\varphi n} = 0$

$\cos\varphi + j\sin\varphi$

want value for φ

$\frac{\pi}{4}$ See other sheet

(points: 0.25)

B. Again suppose $x[n] = e^{j\varphi n}$ for all n. For what two positive values of φ is $y[n] = 0$ for all n?

$\frac{\pi}{6}$ See other sheet

(points: 0.25)

C. Suppose $x[n] = e^{j\pi n} = (-1)^n$ for all n, i.e., x[n] is a complex exponential with frequency π. Therefore $w[n] = H(e^{j\pi})(-1)^n$. What is the magnitude of $H(e^{j\pi})$?

$= H() \cdot x[n]$

Same

(points: 0.25)

D. $y[n]$ and $x[n]$ are related by the unit sample reponse of the combined system. In this case the unit sample response has 5 non-zero points:

$$y[n] = \Sigma_{(m = 0 \text{ to } 4)} h[m]x[n-m]$$

Please determine the values for $h[0]$ through $h[4]$.

1 $-\sqrt{3}-\sqrt{2}$

$-\sqrt{3}-\sqrt{2}$ 1

$2+\sqrt{6}$

(points: 0.25)

Problem 2.

Consider sequences y and x related by

$$y[n] = \Sigma_{(m = 0 \text{ to } 99)} x[n-m]$$

Suppose `x[n]` = $e^{j\varphi n}$. Note that x[n] is non-zero for negative indices. What is the smallest positive value of φ such that `y[n] = 0` for all `n`?

Hint: to determine the value of `y[n]` we need to sum the past 100 values of `x` starting at sample `n`. If `x` is a sinusoid, a sequence of samples of `x` will sum to zero only if the sequence represents some number of complete cycles of the sinusoid (since in a complete cycle all the positive samples of the sinusoid will exactly offset the negative samples). So the question is really asking what value of φ will ensure that a sequence of 100 samples from `x` will capture one complete cycle of the sinusoid.

$\frac{2\pi}{100}$

(points: 1)

Problem 3.

Consider sequences `x` and `y` related by

```
y[n] = x[n] + Cx[n-1] + x[n-2]
```

A. Suppose `x[n]` = $e^{j(\pi/4)n}$ for all n, If `|y[n]| = sqrt(2) + 0.5` for all `n`, what is the value of `C`?

C = -.006 + 1.35i

C = 92.56°

(points: 0.5)

B. If `C > 0` and `x[n]` = $e^{j\varphi n}$ for all n, for what value of φ will `|y[n]|` be largest?

4=0 y=3

(points: 0.5)

Problem 4.

Let `x[n]` be a sequence that is periodic with period N=11. For each of the following `x[n]`, please give the corresponding spectral coefficients for the discrete-time Fourier series for

`x[n]`, i.e., a_k for $-5 \le k \le 5$.

A. Determine a_k for `x[0:11] = [1,0,0,0,0,0,0,0,0,0,0]`. ← unit sample

$a_0 = 1$

(points: 0.2)

B. Determine a_k for `x[n] = 1`.

← same as before

$a_0 = 1$

(points: 0.2)

C. Determine a_k for `x[n] = cos(p(2π/11)n)` for some integer $-5 \le p \le 5$. How does your answer change if `p` could be any integer?

$a_p = \frac{1}{2}$

$0 \le p \le (N-1)$

(points: 0.2)

D. Determine a_k for `x[n] = cos(p(2π/11)n + φ)` for some integer $-5 \le p \le 5$ and phase offset φ.

$a_p = e^{j\varphi} \cdot \frac{1}{2}$

(points: 0.2)

E. Redo part (D) assuming $\varphi = -\pi/2$, i.e., `x[n] = sin(p(2π/11)n)`.

$-j/2, \quad -\frac{3j}{2}$

This next 3 pgs likly wrong

A1 cont

$0 = \cos\varphi + j\sin\varphi - \sqrt{2}(\cos\varphi + j\sin\varphi) + \cos\varphi + j\sin\varphi$

$0 = 2\cos\varphi + 2j\sin\varphi - \sqrt{2}\cos\varphi - \sqrt{2}j\sin\varphi$

Solve for φ

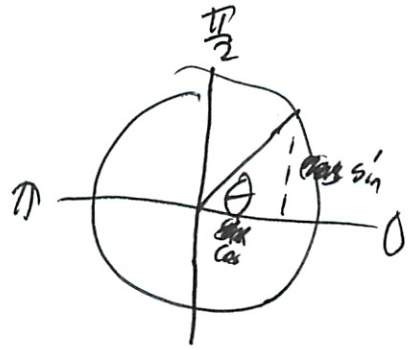
¿Guess + check?

$\varphi = 0$ $\quad 1 + i0$

$\varphi = \frac{\pi}{4}$ $\quad \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$\varphi = \frac{\pi}{2}$ $\quad 0 + i$

$\varphi = \pi$ $\quad 1 + i0$



$\varphi = \frac{\pi}{6}$ $\quad \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$\varphi = \frac{\pi}{3}$ $\quad \frac{1}{2} + \frac{\sqrt{3}}{2}i$

Try $\varphi = \frac{\pi}{4}$ $\qquad i = j$

$2\frac{\sqrt{2}}{2} + 2\frac{\sqrt{2}}{2}i - \sqrt{2}\frac{\sqrt{2}}{2} - \sqrt{2}\frac{\sqrt{2}}{2}i$

$\sqrt{2} + \sqrt{2}i - 1 - i$

$\sqrt{2} - 1 + (\sqrt{2} - 1)i$

Not $= 0$

(2)

Try $\varphi = \frac{\pi}{6}$

$$\cancel{2 \cdot \frac{\sqrt{3}}{2} + 2\frac{\sqrt{3}}{2} i - \sqrt{2}\frac{1}{2} - \sqrt{2}\frac{1}{2} i}$$

$$\cancel{\sqrt{3} + \sqrt{3} i - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i}$$

~~No~~

did wrong

~~$\varphi = \frac{\pi}{3}$~~

2.

$$2 \cdot \frac{\sqrt{3}}{2} + 2 \cdot \frac{1}{2} i - \frac{\sqrt{3}}{2}\sqrt{2} - \sqrt{2}\frac{1}{2} i$$

$$\sqrt{3} + i - \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2} i$$

No

Try $\varphi = \frac{\pi}{3}$

$$2 \cdot \frac{1}{2} + 2\frac{\sqrt{3}}{2} i - \sqrt{2} \cdot \frac{1}{2} - \sqrt{2} \cdot \frac{\sqrt{3}}{2} i$$

$$1 + \sqrt{3} i - \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2} i$$

No

$\varphi = 1$ $\quad 2 \cdot 1 + 0i - \sqrt{2} \cdot 1 + 0i$

$2 - \sqrt{2}$ ⊗

③

$u = \frac{\pi}{2}$

$2 \cdot 0 + 2 \cdot 1 i \quad - \sqrt{2} \cdot 0 - \sqrt{2} \cdot i$

$0 + (2 - \sqrt{2}) i$

? is that ok ???

Oh have to apply n

? Or do I?

and forgot last part

$+ x[n-2]$

?? same $x[n]$

Oh ho I did 2. - taken care of

So $u = \frac{\pi}{2}$

$0 + (2 + 2 - \sqrt{2}) i$

Let me try alt method: derive formula for $H(\ )$

- no I did that

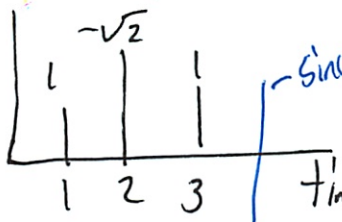
Is 0 real w/ some imag work?

TA help / In Lab

impulse response

$h[n] = \delta[n] - \sqrt{2}\,\delta[n-1] + \delta[n-2]$.

$1 - \sqrt{2}e^{-j\Omega}$



since delay

$H(e^{j\Omega}) = \sum_n x[n]e^{-j\Omega n}$.

2 cause time so just look at last term - more important

$H(e^{j\Omega}) = 1 - \sqrt{2}\boxed{e^{-j\Omega}} + e^{-2j\Omega} = 0.$

x_1 x^2 $e^{-0} = 1.$ since

$= 1 - \sqrt{2}\cos\varphi - \sqrt{2}\sin\varphi\, i + \cos 2\varphi + \sin 2\varphi\, i = 0$

method too complex

2x since was squared

$= 1 - \sqrt{2}e^{-j\Omega} + e^{-2j\Omega} = 0$

do this instead → Like $1 - \sqrt{2}x + x^2 = 0$

Solve for x

$\boxed{\text{See Lec 15 slide 9}}$

$x = \frac{1-i}{\sqrt{2}}, \frac{1+i}{\sqrt{2}}$

So $\frac{1+i}{\sqrt{2}} = \cos\varphi + \sin\varphi\, i$

Wolfram alpha $\varphi = -\frac{\pi}{4}$ which is also $\frac{7\pi}{4}$

(2)

Let me check

So $\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4}\, i$

$= -\frac{1-i}{\sqrt{2}}$

So $1 - \sqrt{2}\left(-\frac{1-i}{\sqrt{2}}\right) + \left(-\frac{1-i}{\sqrt{2}}\right)^2$

$= 0$

So works

---

b) Now do for y

- can we take w as granted

Well $w[n] = 0$

So can do fresh:

$h[n] = \delta[n] - \sqrt{3}\,\delta[n-1] + \delta[n-2]$

Unit sample response

Unit sample ~~response~~!

$H(e^{j\varphi n}) = e^{-j\varphi\cdot 0} - \sqrt{3}\, e^{-j\varphi\cdot 1} + e^{-j\varphi\cdot 2}$

$= 1 - \sqrt{3}\, e^{-j\varphi} + e^{-2j\varphi}$

(3)

$$= 1 - \sqrt{3}\cos -\varphi - \sqrt{3}\sin -\varphi i + \cos -2\varphi + \sin -2\varphi = 0$$

Or easy way

$$1 - \sqrt{3}x + x^2 = 0$$

$$x = \frac{\sqrt{3} - i}{2}, \frac{\sqrt{3} + i}{2}$$

$$\frac{\sqrt{3} - i}{2} = \cos\varphi + \sin\varphi i$$

$$\varphi = ?$$

~~$\varphi = 2\pi$?? wolfram alpha not clear~~

Just look at unit circle

$$\left(\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$$

$$\frac{\pi}{6}$$

Actually revisit a $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\uparrow \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Could also be $\frac{\pi}{4}$

(4)

c) $x[n] = e^{jn\pi} = (-1)^n$ for all n

complex ~~exp~~ exp w/ freq π

So $w[n] = H(e^{j\pi})(-1)^n$

What is mag $H(e^{j\pi})$?

Can I answer this w/ info I already know?

$x[n]$ = freq response? input?

- no was our complex exponential
- and now depends on n

So

$H(e^{j\pi n}) = H((-1)^n)$

$H[(-1)^n] =$? same ~~w~~ as above?

No $w[n] = H(e^{j\pi})(-1)^n$

$(-1)^0 + \sqrt{2}(-1)^{-1} + (-1)^{-2} =$ ___ No

- Same as above, except $\pi = \Omega$, solve

5

So TA said

$$1 - \sqrt{2}\, e^{-j\Omega} + e^{-2j\Omega}$$

$$\Omega = \pi$$

$$1 - \sqrt{2}\, e^{-j\pi} + e^{-2\pi j} = \text{enter in} \rightarrow 2+\sqrt{2}$$

"But should be same as what I had"

$$1 - 1\cdot\sqrt{2} + 1$$

So $2 + \sqrt{2}$ So we were both right!

still all real

---

d) Unit sample response

- 5 non 0 points

$$y[n] = \sum_{m=0}^{4} h[m]\, x[n-m]$$

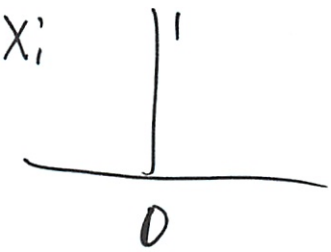
Determine values

- This is a non freq question.

(6)

Got something like wrong on quiz

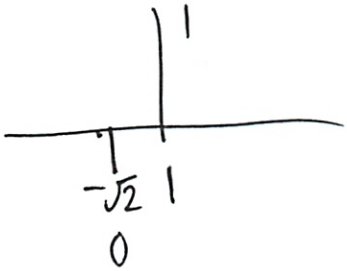
if put in 1 (unit sample) x_i



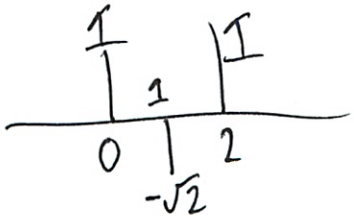
$w[0]$

$w[1]$

no look at $n-1$



$w[2]$



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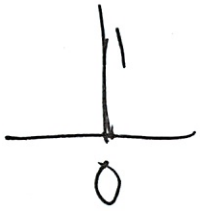
Now this is fed in next

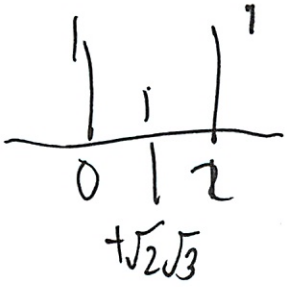
$w[0] \rightarrow y[n]$



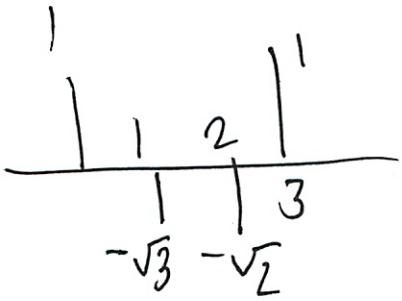
$y[1]$ now have 2 inputs

just convolve!

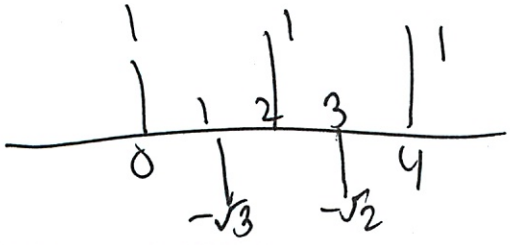
$y[2]$



$y[3]$



$y[4]$



¿Is that it?

8

Let me confirm some how

Lecture 4 multiplication

$y[n] = x[n] \cdot (h_1[n] \cdot h_2[n])$

$y[0] = x[0] \cdot (1 \cdot 1)$

$1 \cdot 1$

$y[1] = x[1] \cdot (\quad)$

0 this is what confused me

What is right way to do?

Got wrong on quiz - but forgot how Chris explained it

Convolve

$1 \quad -\sqrt{2} \quad 1$

$1 \quad -\sqrt{3} \quad 1$

At $t = 1$

$1 \quad -\sqrt{2} \quad 1$

$1 \quad -\sqrt{3} \quad 1$

$= 1$

(9)

$n=1$

$$1 \quad -\sqrt{2} \quad 1$$

$$1 \quad -\sqrt{3} \quad 1$$

$$-\sqrt{3} + -\sqrt{2} \quad \text{sum} \quad -\sqrt{3}-\sqrt{2}$$

$n=2$

$$1 \quad -\sqrt{2} \quad 1$$

$$1 \quad -\sqrt{3} \quad 1$$

$$1 \quad +\sqrt{2}\sqrt{3} \quad 1 \quad = \quad 2 + \underbrace{\sqrt{2}\sqrt{3}}_{=\sqrt{6}}$$

$n=3$

$$1 \quad -\sqrt{2} \quad 1$$

$$1 \quad -\sqrt{3} \quad 1$$

$$-\sqrt{2} \quad -\sqrt{3} \quad = \quad -\sqrt{3}-\sqrt{2}$$

$n=4$

$$1 \quad -\sqrt{2} \quad 1$$

$$1 \quad -\sqrt{3} \quad 1$$

$$= 1$$

So

$h[0]$	$h[1]$	$h[2]$	$h[3]$	$h[4]$
1	$-\sqrt{3}-\sqrt{2}$	$2+\sqrt{2}\sqrt{3}$	$-\sqrt{3}-\sqrt{2}$	1

(10)

Watch formula - use it

sub w[] in to get y[]

What does formula say

$$y[n] = \sum_{m=0}^{4} h[m]\, x[n-m]$$

but we want this

We don't have test pts

- perhaps she was wrong

Well let me try + check

$$y[0] = h[0]\, x[0-0] + h[1]\, x[0-1] + \dots$$

0 anyway

So $y[0] = h[0]\, x[0]$

$y[1] = h[0]\, x[1] + h[1]\, x[0]$

what is supposed to happen!

So I think I got it

(11)

2. $y[n] = \sum_{m=0}^{99} x[n-m]$

Suppose $x[n] = e^{j\omega n}$

negitive indicies?

What is smallest (+) value of ω so $y[n] = 0$ for all n?

closest to 0

"Same as previous"

- Sum to 0 only if some complete #

So ω should be that 100 samples is 1 complete cycle

- But what does ω control?

$y = e^{j\omega n}$

But last 100

$y[0] = x[0]$ ... rest 0

$y[1] = x[1] + x[0]$

$y[2] = x[2] + x[1] + x[0]$

etc to 99

(12)

$y[0] = e^{j0\varphi} = 1$

$y[1] = e^{j1\varphi} + 1$

$y[2] = e^{2j\varphi} + e^{j\varphi} + 1$

$y[3] = e^{3j\varphi} + e^{2j\varphi} + e^{j\varphi} + 1$

Should $= 0$

what is φ so is true

But what does φ have to be so ~~x~~ 100 samples

- a harmonic of 100
  period

$(k \bmod 100)(2\pi/100)$

It could be 1 thing through ~~x~~ 100 points

the fund freq

$\frac{2\pi}{N}$ $\frac{\text{radians}}{\text{sample}}$

$f = \frac{2\pi}{100}$

But how does that connect to φ?

(13) Try this

$$x^{99} + x^{98} + \dots + x^0 = 0$$

Solve wolfram

Computation time out

How does it relate to 4?

$\frac{2\pi}{\omega} = \phi$

They say elsewhere

$$\sum e^{jk \frac{2\pi}{N} n}$$

so that is like 4

So checking

$$\sum_{m=0}^{99} \left(\frac{2\pi}{100}\right)^m = 1.06704$$

Not exactly 0

Unless k. something

Computation timeout

k = .05 ~~AND~~ 1.03

No value of k

k = .0000001 = 1

k = 5 = 2-3

(19)

Ω = multiple $\frac{2\pi}{100}$

But must also find k

$$\boxed{\begin{gathered}|H(e^{j\Omega})| = 0 \\ y[n] = \sum_{0}^{n} h[n] \cdot x[n-m]\end{gathered}}$$

So I was right

- entered in WA wrong

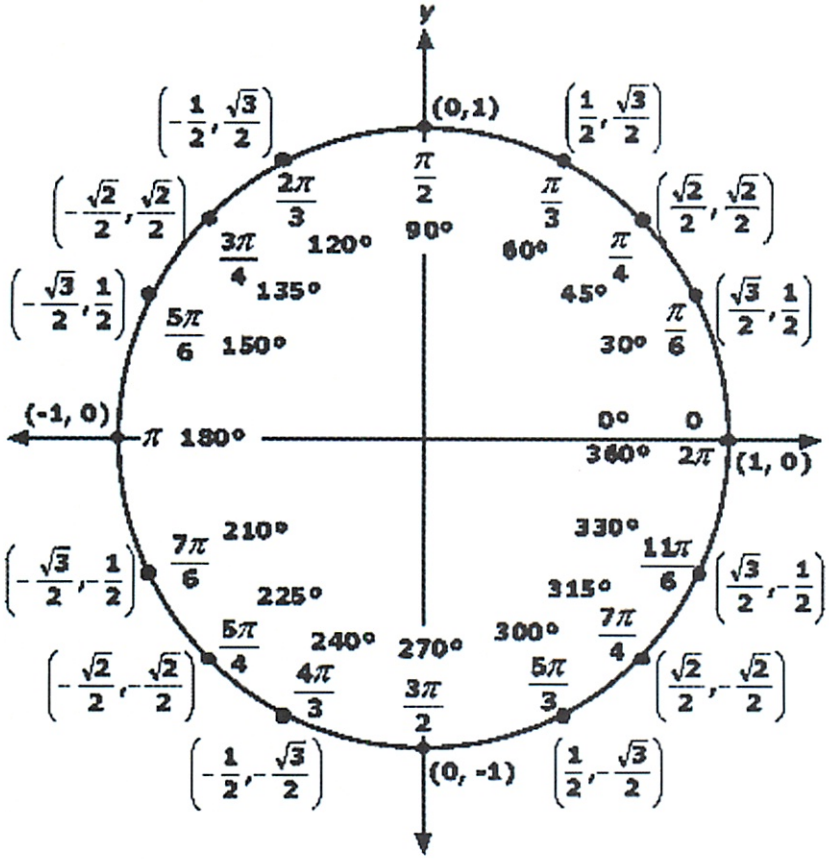
$$\sum_{m=0}^{99} e^{\frac{2\pi}{100} m i} = 0$$

not $\sum_{m=0}^{99} \frac{2\pi}{100} = 1.03$

Try some other value

✓ is something else

15



LA help on 2

$y(n) = \sum_{0}^{N} h(m)\, x(n-m)$

$|H(e^{j\omega})| = 0$

$y(n) = \sum_{0}^{N} h(m) \cdot x(n-m)$

$0 = \sum_{n=0}^{N} h(n) e^{j\omega n}$

$h(n) = -1$

$h(n) = 0$ $n \neq -1$

$1 \quad n = 0$

$1 + e^{2d}$

(16)

3. $y[n] = x[n] + Cx[n-1] + x[n-2]$

$x[n] = e^{j\pi/4\, n}$ for all n

So $\Omega = \frac{\pi}{4}$

if $y[n] = \sqrt{2} + .5$

What is C

So

$$H[e^{j\pi/4\, n}] = e^{0\cdot\pi/4\, j} + Ce^{-1\cdot\pi/4\, j} + e^{-2\pi/4\, j}$$

$$\overset{\text{set}}{=} \sqrt{2} + .5$$

Solve for C

$$1 + Ce^{-\pi/4\, j} + e^{-2\pi/4\, j} = \sqrt{2} + .5 + 0i$$

$C = -.06 + 1.35i$

I am guessing it should be nicer

Or $C = 1.91 = \sqrt{2} + \frac{1}{2}$ ← but that is input

(17)

But should c not be nicer

Try abs

- same

- unless did something wrong

b) If $c > 0$ $x = e^{i\varphi n}$ - what value φ will $y(n)$ be largest

Say $c = 1$

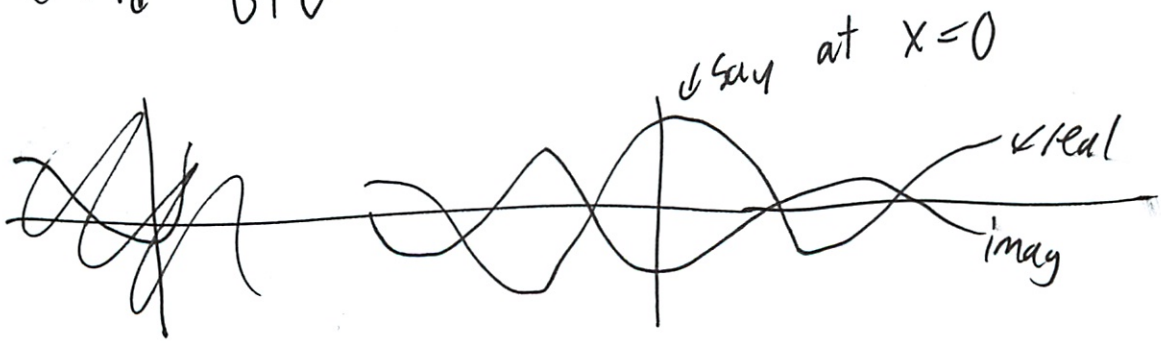
Maximize over φ

Graph

$$1 + e^{-xi} + e^{2xi} = y$$

Does not work

Could G+V



At $\varphi = 0$ $y = 3$

$\varphi = \frac{\pi}{4}$ $y = 1.77 - 1.77i$

$\varphi = \frac{\pi}{2}$ $y = 0 - 1i$

(18)

Back to a at $\frac{\pi}{2}$ it would be $-\frac{\pi}{4}$

sqrt(2) + .5 = 1.914

$$\frac{1.914}{2\pi} = 3.00 = 172.3$$

---

4. $x[n]$ seq periodic period $N = 11$

For each give spectral coefficients

a_k for $-5 \le k \le 5$

? is limit like this since $N = 11$?

So what was it - one for each

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \cos\left(k \frac{2\pi}{N} n\right) e^{-jk\left(\frac{2\pi}{N}\right)n}$$

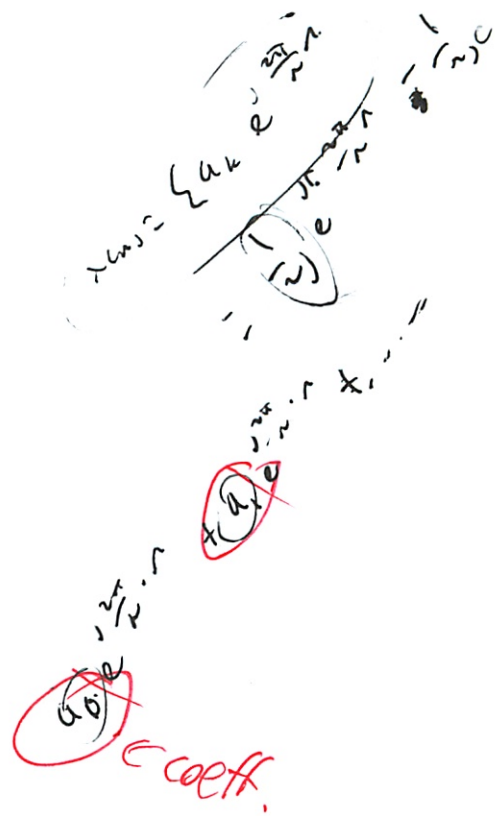
- But what was the simple way?
- "by inspection"

(19)

Convert to complex exponential

$e^{jk\frac{2\pi}{N}n}$

$\frac{1}{2}j$



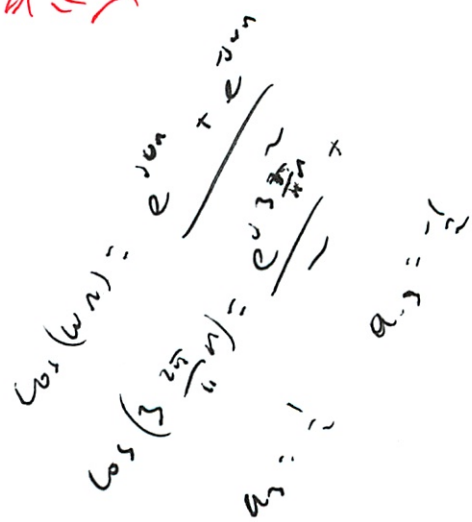
$\in$ coeff.

$a_{\pm 0} = 1$

$a_{\pm 1} = 1$

0 elsewhere

when $k = r$



Think I got it now

should have written down
– was thinking that

(20)

a_k for $e^{j\varphi \cdot 0}$

$|$

$a_0 = 1$

b) So now $e^{j\varphi \cdot \phi}$ still 0 since not given

- same as before ;)

c) $x[n] = \cos\left(p\left(\frac{2\pi}{11}\right)n\right)$ for integer $-5 \le p \le 5$

So this is $\cos(\varphi) = \frac{1}{2}e^{j\varphi} + \frac{1}{2}e^{-j\varphi}$

$a_p = 1 \cdot \frac{1}{2}$; so its always 1/2 ;

* $0 \le p \le N-1$

d) Find a_k for $\cos\left(p\frac{2\pi}{11}n + \varphi\right)$ for $-5 \le p \le 5$

and some phase offset

What does phase offset do?

(21)

$$e^{jp\frac{2\pi}{N}n + \varphi}$$

$$= e^{jp\frac{2\pi}{N}n} \cdot e^{\varphi}$$

in summation bring before

$$e^{\varphi} + \sum \ldots$$

So $a_p = e^{\varphi} \cdot \frac{1}{2}$

~~So when~~

so when fill in

$$e^{\varphi} \cdot \frac{1}{2} a_p$$

– no must be other way

---

e) $\varphi = -\frac{\pi}{2}$ $\sin\left(p \frac{2\pi}{11} n\right)$

So w/ sin

$$a_p = \frac{j}{2} + e^{-\pi/2} = \frac{j}{2} + \cos -\frac{\pi}{2} + \sin -\frac{\pi}{2} j$$

$$a_{-p} = -\frac{j}{2} + e^{-\pi/2} = -\frac{j}{2} + \ldots$$

(22)

$\cos -\frac{\pi}{2} = 0$

$\sin -\frac{\pi}{2} = -1$

So

$= \frac{j}{2} - 1j = -\frac{j}{2}$

$-\frac{j}{2} - j = -\frac{3j}{2}$

Now into code!

(points: 0.2)

Python Task #1: Spectral coeffients

Useful download links:

PS7_tests.py -- test jigs for this assignment
PS7_1.py -- template file for this task

Please write Python functions that implement coversion to/from the frequency domain:

fseq = frequency_domain(*tseq*)

This function takes a sequence representing the samples of a real signal in the time domain (*tseq*) and returns a sequence representing the spectral coeffients of the time-domain waveform (*fseq*). The spectral coefficients, which are complex numbers, should be computed using the analysis equation presented in the lecture on the discrete-time Fourier series. The length of *tseq* determines N, the period of the time-domain waveform. The elements of *fseq* should begin with k = -N/2 corresponding to the frequency (-N/2)*(2π/N).

?? what does that mean

tseq = time_domain(*fseq*)

This function takes a sequence representing the spectral coefficients of a signal in the frequency domain (*fseq*) and returns a sequence representing the samples of the signal in the time-domain waveform (*tseq*). The samples should be computed using the synthesis equation presented in the lecture on the discrete-time Fourier series. The length of *fseq* determines N, the period of the time-domain waveform. The elements of *tseq* should begin with n = 0. You can assume the first element of fseq corresponds to k = -N/2.

Note that although the values computed by the synthesis equation are, in general, complex, we'll only be considering real signals, so it will be convenient if you convert each of the samples to a real number before returning the sequence (i.e., take the real part of each number computed by the synthesis equation).

Python supports complex numbers and the cmath module provides useful functions that take complex arguments. The simplest way to create the complex constant A+jB in Python is to type complex(A,B). If x is a complex number, x.real returns the real part as a regular Python number and x.imag returns the imaginary part.

Complex exponentials in Python are easy too. For example, to evalute the expression $e^{-jk(2\pi/N)n}$, you could use

```
import cmath
x = cmath.exp(complex(0,-k*2*cmath.pi*n/N))
```

PS7_1.py is the template file for this task. The test code does some simply sanity checks, then generates some random sequences to check that the conversion to and then from the frequency domain results in the original time domain waveform.

When you're ready, please upload your code and the figure of the spectral coefficients generated by the tests:

Upload code for Task 1: Browse...

(points: 4)

Upload figure for Task 1: Browse...

(points: 1)

Python Task #2: Low-pass filter

Useful download link:

PS7_2.py -- template file for this task

Please write a Python functions that returns the unit sample response for an approximate low-pass filter:

`hseq = low_pass(freq)`

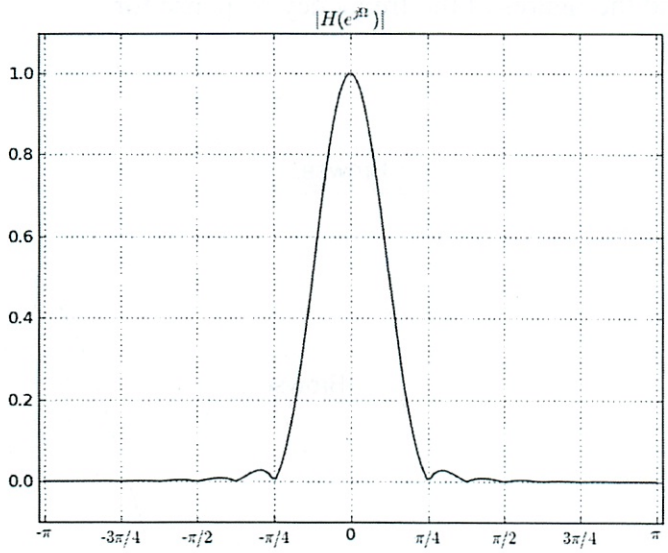
Return a sequence representing the causal unit sample response `h[n]` for an LTI system that approximates a low-pass filter with a cutoff frequency *freq*, where $0 \le freq \le \pi$. Please scale the elements of `h[n]` so that the step response is exactly 1 in the steady state. ← ? what does that mean

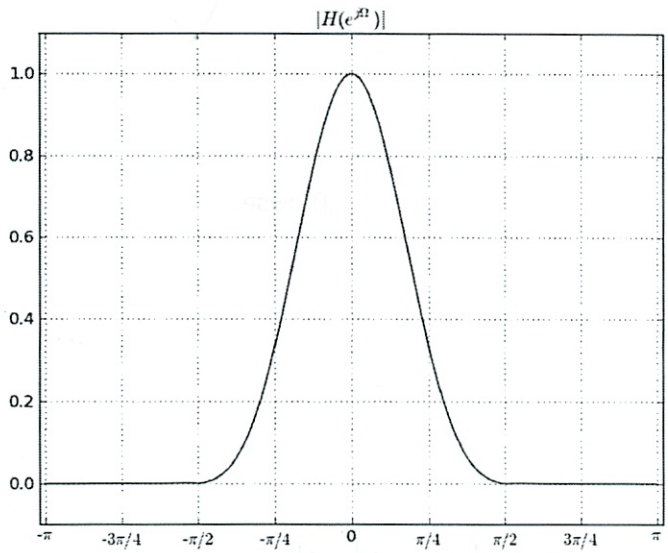
$-\pi$ $-f$ 0 f π

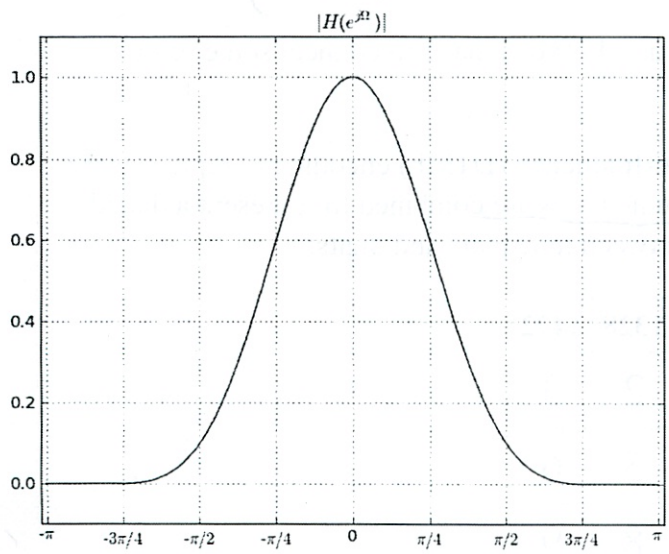
You may find it useful to write a helper function `one_zero(freq)` that returns `h[n]` for an LTI system that has a frequency response with zeros at `±freq`. You can then hook several of these smaller LTI systems in series, where you've chosen the zeros to cover the frequency range above the cutoff frequency. Do you remember how to compute the unit sample response of LTI systems hooked in series? You might find `numpy.convolve(h1,h2)` to be useful.

PS7_2.py is the template file for this task. The test code generates three low-pass filters with cutoff frequencies of $\pi/4$, $\pi/2$ and $3\pi/4$, plotting the frequency response of each filter.

It can take some experimenting to devise a strategy for figuring how many one-zero systems to use and where to place their zeros. Here are my plots for the frequency response of the three filters; your milage may vary!


$|H(e^{j\Omega})|$
1.0
0.8
0.6
0.4
0.2
0.0
$-\pi$
$-3\pi/4$
$-\pi/2$
$-\pi/4$
0
$\pi/4$
$\pi/2$
$3\pi/4$
π


$|H(e^{j\Omega})|$
1.0
0.8
0.6
0.4
0.2
0.0
$-\pi$
$-3\pi/4$
$-\pi/2$
$-\pi/4$
0
$\pi/4$
$\pi/2$
$3\pi/4$
π


$|H(e^{j\Omega})|$
1.0
0.8
0.6
0.4
0.2
0.0
$-\pi$
$-3\pi/4$
$-\pi/2$
$-\pi/4$
0
$\pi/4$
$\pi/2$
$3\pi/4$
π

Freq

When you're ready, please upload your code and the figures of the frequency response for filters as generated by the tests:

Upload code for Task 2: Browse...

(points: 2)

Upload figure for $\pi/4$ filter: Browse...

(points: 0.33)

Upload figure for $\pi/2$ filter: Browse...

(points: 0.33)

Upload figure for $3\pi/4$ filter: Browse...

(points: 0.34)

Python Task #3: Decoding dialing tones

Useful download link:

PS7_3.py -- template file for this task
PS7_dtmf.wav -- .wav file with dialed number

C #s dialing

Note: this task involves less hand-holding than usual -- you'll have to connect some of the dots yourself!

The analog phone system uses a dual-tone multi-frequency (DTMF) encoding to represent the digits you dial -- aka touch-tone dialing. Two audio tones are combined to represent a dialed digit. The table below shows the relationship between tone pairs and digits:

Hz	1200	1328	1472
688	1	2	3
768	4	5	6
848	7	8	9

928	*	0	#

These frequencies vary slightly from the official standard so that they are all exact harmonics of our fundamental frequency of 16 Hz. Note that the fundamental frequency is all one needs to know to convert frequencies in Hz to the corresponding `k` for the spectral coefficients.

Your task is to write a program that decodes the audio tones found in PS7_dtmf.wav and prints the dialed digits it detects. You can play this .wav file on your computer to hear what it sounds like. For each dialed digit, there are 500 samples of dual-tone followed by 500 samples of silence.

You'll find that the spectral coefficients computed from the samples have a little bit noise introduced by the quantization that happened when converting the audio samples to the .wav file format. So you shouldn't expect to see exactly 0 for the magnitude of a spectral coefficient that you'd expect to be zero in the perfect world. Your code should be forgiving about this and should accept frequencies with ±1 of the nominal k.

PS7_3.py gets you started by reading the samples from the wav file and splitting the samples into shorter sequences for each dialed digit. It then calls `decode_dialed_digit(samples)` to determine which digit the tones represent. Your code only needs to detect the digits 0 through 9 and does not have decode "*" and "#".

the chart

When you're ready, please upload your code. You'll only get credit for your code if the dialed number entered in the box below is correct.

What was the dialed number?

(points: 0)

Upload code for Task 3: Browse...

(points: 5)

You can save your work at any time by clicking the Save button below. You can revisit this page, revise your answers and SAVE as often as you like.

Save

To submit the assignment, click on the Submit button below. YOU CAN SUBMIT ONLY ONCE after which you will not be able to make any further changes to your answers.

Once an assignment is submitted, solutions will be visible after the due date and the graders will have access to your answers. When the grading is complete, points and grader comments will be shown on this page.

Submit

1. <u>Coding</u> 4/6

So how exactly do you convert?

Use analysis eq

TA

a_k - for certain freq $\frac{2\pi k}{N}$

'consecutive a_k determine

N consec values for a_k

Start $-\frac{N}{2} + 1$ maybe

integer

for each a_k - over all n time

add 1 till N times

return list a_k

---

b) The other way, code of a_k reversed

Cool! - Once I figured it out - built in 5 min!

Start -12 not -13

Do - floor ( )

not floor(-)

and forgot + a_{kn} to summation

②

Much better now

Test passed!

---

for b return just real

And all ~0 now

+ Xn as well

0 issue fixed

What is n the time index?

- Yeah

The k is diff now

Last should be 1.286

-.953 1.43

-1.33 2.29

.... -.08

I am close - but sign off?

Try starting k at $-\frac{N}{2}$

Should have no real portion

And python [index] is 0 to 24) do seperate

while k should be -12 → 0 → 12

③ Done! Took ~1 hr 7:40

#2 We can't just chop!
So one-zero we need to write

Casual

Poor-man

Conv in time is x of freq

So poor man's filter is mult of 4 filters in freq

~~Ok~~
data is then multiplied

like #2 c_1 in time domain
Whee c is changes freq

Want $x[n] = 0$ at freq 4 (given). Find c

~~Convert c freq~~ Return all the h

$x[n] + c\,x[n-1] + x[n+2]$

Return coeff $[1, c, 1]$

(4)

Convolve them

When add sum $= 1 = 1 + C + 1$

The ones give our ours

We should do sub filters

Then normalize

So basically find C

How do we know filter is 3 time periods?

Set $0 = t_0$

$$0 = 1 + C e^{-\Omega j} + e^{-2\Omega j}$$

Ω is freq given

$$C = \frac{1 + e^{-2\Omega j}}{-e^{-\Omega j}}$$

Will be convoled longer - since more complex

but need to scale

$1 + 2 + 1 = 4$

So divide all by 4 $\quad \frac{1}{4} + \frac{2}{4} + \frac{1}{4}$

(5)

What about imag?

- keep I guess

1, 2 work

3 looks weird

Need to filter π in all cases

↑ 0 actually is what want

So looks good

But I think I should do flatter

Like a bunch more

Like for every $\frac{\pi}{16}$ larger

I had − sign error

And do π not 0

↑ which was error from π

✓ Done 8PM ~1.5 hrs

(6) Hints

k·16 = freq

graph

2 spikes - but reflected so 4

4 spikes add to 1 = .25 ± noise

Use table to convert freq back

Makes graphs w/o changes

I should actually play into phone

So our chart is just showing the 4 values

So iterate over return for ~~1/40~~ < .20

Look for times - only keep last 2

Now need a_k when that happens

But how to know <u>which</u> freq at now

$-\frac{N}{2}$ to $\frac{N}{2}$

$-92 \quad -43 \quad 43 \quad 92$

? multiply by something? 16?

(7)

688
1472

✓ Yes Need 16

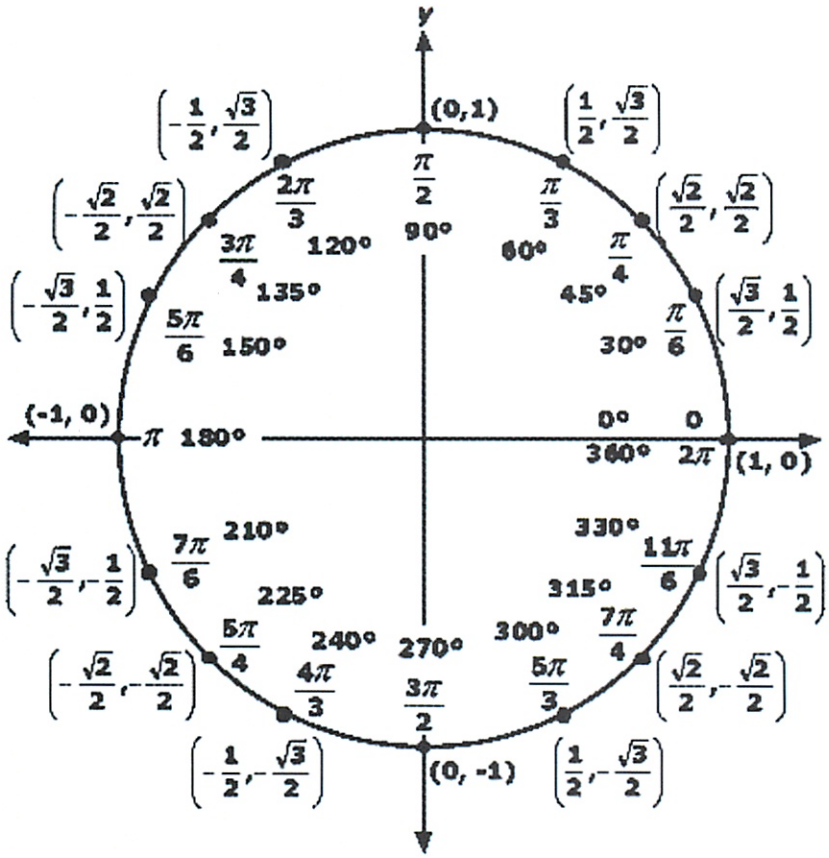
How build chart?

manually?

314 592 6708

9:40 done

Now 6.042!


y
(0,1)
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
(-1, 0)
(1, 0)
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
(0, -1)
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$ 90°
$\frac{\pi}{3}$ 60°
$\frac{\pi}{4}$ 45°
$\frac{\pi}{6}$ 30°
0° 0
360° 2π
$\frac{2\pi}{3}$ 120°
$\frac{3\pi}{4}$ 135°
$\frac{5\pi}{6}$ 150°
π 180°
$\frac{7\pi}{6}$ 210°
$\frac{5\pi}{4}$ 225°
$\frac{4\pi}{3}$ 240°
$\frac{3\pi}{2}$ 270°
$\frac{5\pi}{3}$ 300°
$\frac{7\pi}{4}$ 315°
$\frac{11\pi}{6}$ 330°

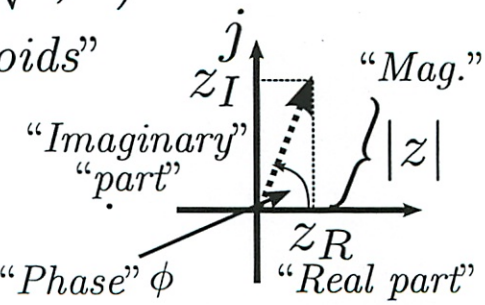
	Time	Frequency
Must know	*"Decomposition"*	*"Get Freq. coefs."*
	$x[n]=\sum_{<k>} a_k\, e^{j(\frac{2\pi k}{N})n}$	$a_k=\sum_{<n>} x[n]\, e^{-j(\frac{2\pi k}{N})n}$
	$x[n]$ has N samples /periodic in N	$a_k = a_{k+N}$ "periodic in N"
	$x_1[n] + x_2[n]$	$a_k = b_k + c_k$ *"Linear"*
		$x_1[n] \leftrightarrow b_k,\ x_2[n] \leftrightarrow c_k$
	$\alpha \cdot x[n]$	$\alpha \cdot a_k$ *"Linear"*
Properties	$x[n]$ is real	$a_k = a^*_{-k}$ *"Conjugate Sym."*
	$x[n]$ real and $x[n]=x[-n]$	a_k's are real
	$x[n-\alpha]$	$a_k \cdot e^{-j(\frac{2\pi k}{N})\alpha}$ *"Time Shift"*
	$x[n] \cdot e^{j(\frac{2\pi k}{N})\alpha}$	$a_{k-\alpha}$ *"Freq. Shift"*
Examples	$x[n] = 1$	$a_k = \delta[k]$ *"DC-constant"*
	$x[n] = \delta[n]$	$a_k = \frac{1}{N}$ *"unit sample"*
	$x[n] = \cos\left((\frac{2\pi\ell}{N})n\right)$	$a_k = \frac{1}{2}\delta[\ell] + \frac{1}{2}\delta[-\ell]$
	$x[n] = \sin\left((\frac{2\pi\ell}{N})n\right)$	$a_k = \frac{1}{2j}\delta[\ell] - \frac{1}{2j}\delta[-\ell]$

$$e^{j\underbrace{(\frac{2\pi k}{N})}_{\text{"Freq."}}n} = \cos\left((\frac{2\pi k}{N})n\right) + j\sin\left((\frac{2\pi k}{N})n\right)$$ *"Euler's"*

"Freq." (*fraction of* 2π) *"discrete-time sinusoids"*

"A complex number has 4 parts"

$$z = \underbrace{z_R}_{|z|\cos(\phi)} + j\underbrace{z_I}_{|z|\sin(\phi)} = |z|e^{\phi}$$



$\lvert z\rvert^2 = z_R^2 + z_I^2$	$\phi = \tan^{-1}(\frac{z_I}{z_R})$

"Conjugation"

$$z^* = z_R - jz_I = |z|e^{-\phi}$$

Massachusetts Institute of Technology
Dept. of Electrical Engineering and Computer Science
Fall Semester, 2006
6.082 Introduction to EECS 2

Modulation and Demodulation

Introduction

A communication system that sends information between two locations consists of a *transmitter*, *channel*, and *receiver* as illustrated in Figure 1. The channel refers to the physical medium carrying the *information signal* (voice, video, data etc.) from one location to another. The physical medium can be free space or a variety of waveguides (wires, optical fibers, etc.) that direct the energy across the channel to the receiver. The *transmitted signal* carrying the information through the channel can be electromagnetic, optical, acoustic, or other forms of energy radiation. Cell phones and wireless networks send information across free space using electromagnetic waves.

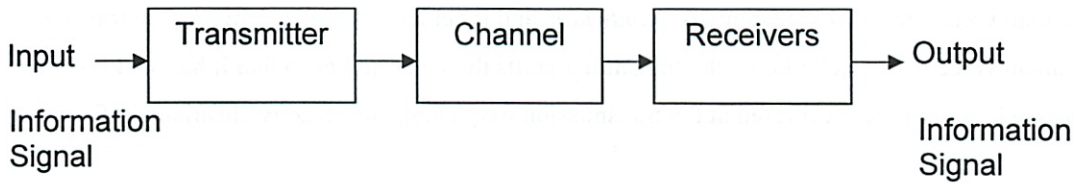


Figure 1: Communication System Block Diagram

In order to send these electromagnetic waves across free space the frequency of the transmitted signal must be quite high compared to the frequency of the information signal. For example the information signal in a cell phone is a voice signal with a bandwidth on the order of 4kHz. The typical frequency of the transmitted and received signal is on the order of 900MHz. Figure 2 illustrates how the spectrum of voice signals falls outside the frequency range of the transmission channel; the figure is not drawn to scale.

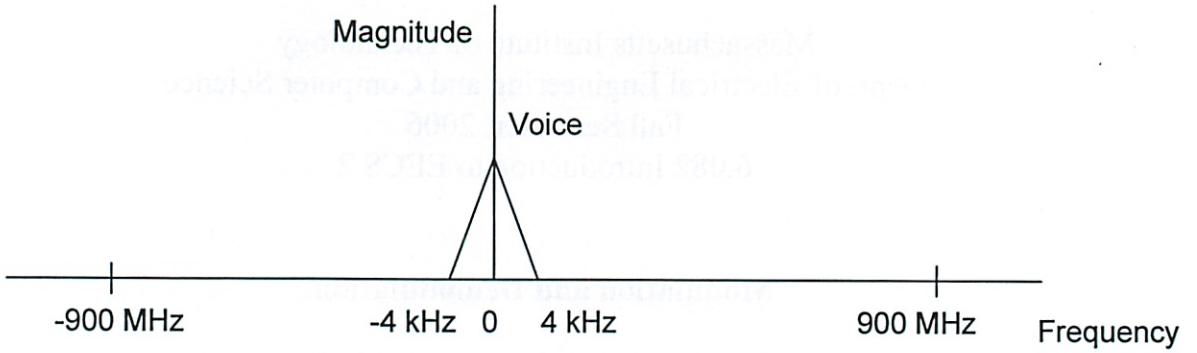


Figure 2

The main reason the transmission frequency is so high is that the wavelength of the electromagnetic wave is proportional to the reciprocal of the frequency. For example the wavelength of a 1 GHz electromagnetic wave in free space is 30cm whereas a 1kHz electromagnetic wave is one million times larger or 30km. As you will see when we study antennas, the size of the antenna and other components are related to the wavelength. For small portable devices, higher frequency transmission is a requirement.

To transmit signals with frequencies required by the communication channel, *the transmitter centers the spectrum of the information signal at the transmission frequency*. This process of shifting the frequency spectrum of a signal is called *modulation*. As an example human voice spans a 4 kHz range or *bandwidth*, and is centered at 0 kHz. In order to transmit human voice over a cell phone, the transmitter shifts the voice signal so that it has a 4 kHz bandwidth but is now centered at the transmission frequency, 900MHz as illustrated in Figure 3.

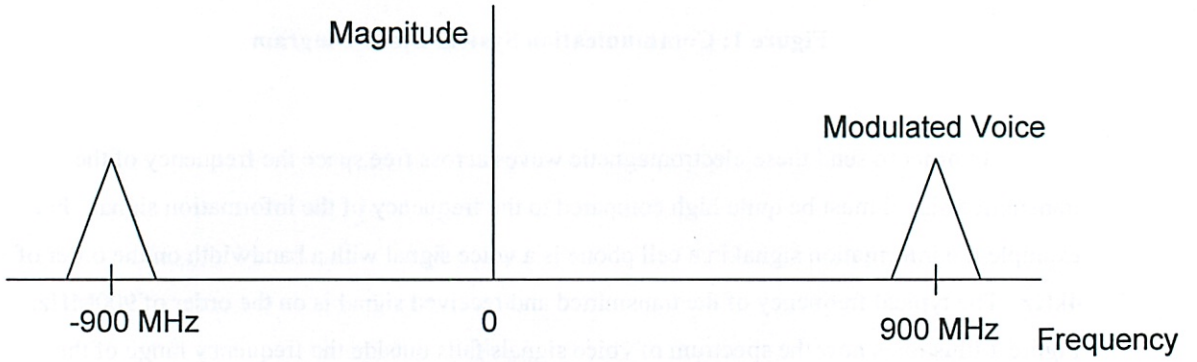


Figure 3

In the receiver the reverse process takes place. The receiver *centers the spectrum of the received signal at the original center frequency of the information signal*; we refer to this process as *demodulation*. In the case of human voice transmission, the receiver shifts the spectrum of the received signal (the received signal had a spectrum centered at 900 MHz) so that it is centered at 0 kHz as shown in Figure 4.

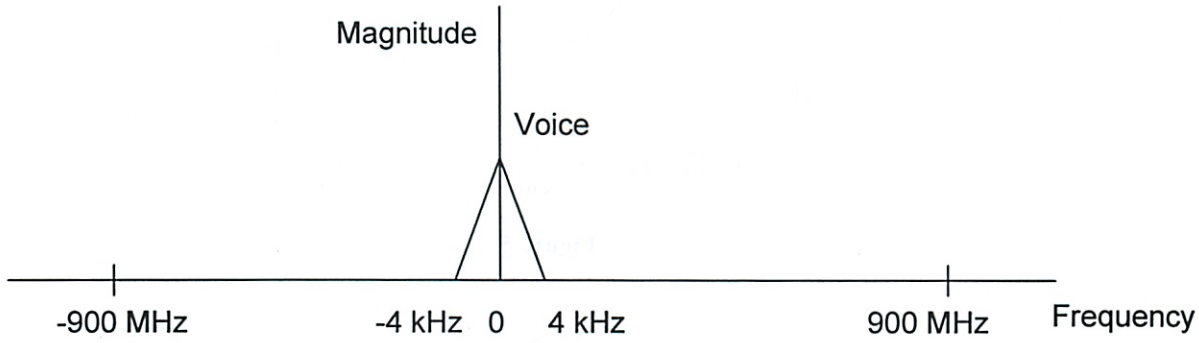


Figure 4

Modulation

We take advantage of the trigonometric identity shown below in the implementation of signal modulation. The identity shows that the product of cosines with frequencies f_1 and f_2 results in cosines with frequencies f_1+f_2 and f_1-f_2. *In other words, multiplication by the f_2 cosine shifts or modulates the f_1 cosine to the new frequencies f_1+f_2 and f_1-f_2.*

$$\cos(2\pi f_1 t) * \cos(2\pi f_2 t) = \frac{\cos\{2\pi(f_1 + f_2)t\} + \cos\{2\pi(f_1 - f_2)t\}}{2}$$

Let's study the effect of modulation in the time and frequency domain; assume f_1 = 1 Hz and f_2= 10 Hz. Figure 5 and 6 show the time-domain plots of the 1 Hz and 10 Hz cosines, and Figure 7 shows the time-domain plot of the product of these two cosines. Notice how the 1 Hz cosine appears as the *envelope* that shapes the 10 Hz cosine.

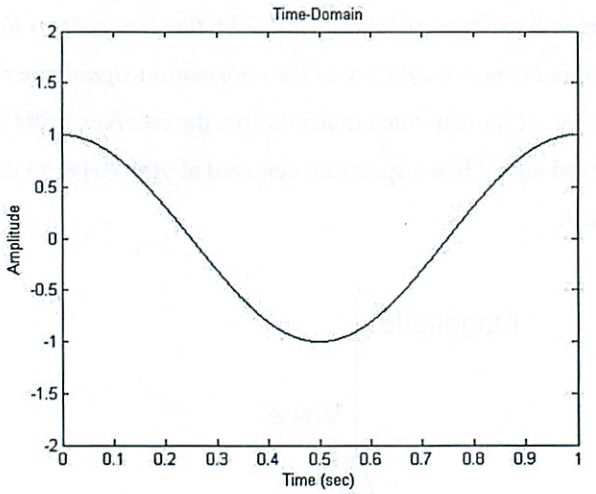


Figure 5

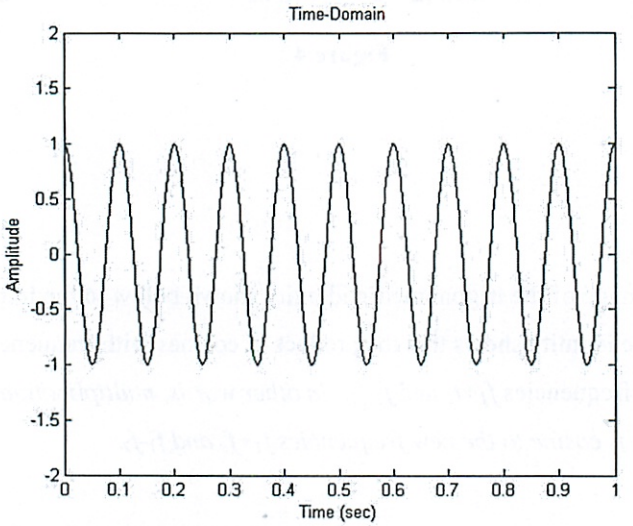


Figure 6

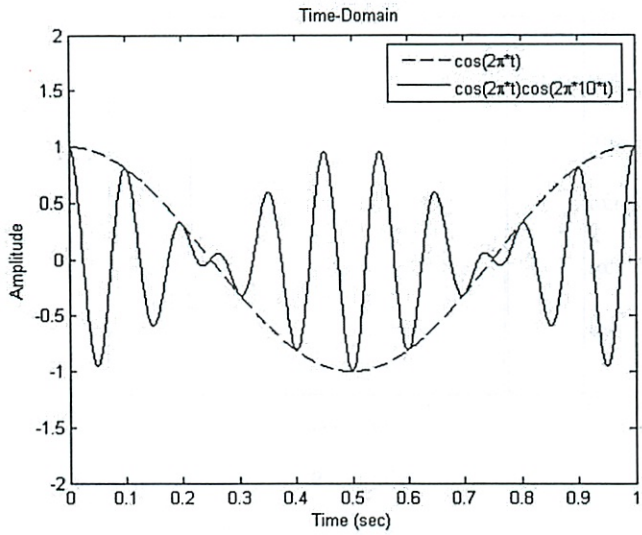


Figure 7

In the frequency domain the 1 Hz and 10 Hz cosines appear as illustrated in Figures 8 and 9 respectively; recall that spectrum of a real signal has even magnitude, which is why you see spectral peaks at +/- 1 Hz and +/- 10 Hz.

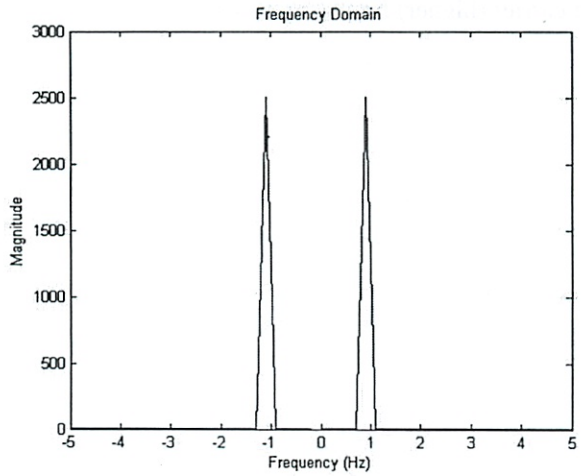


Figure 8

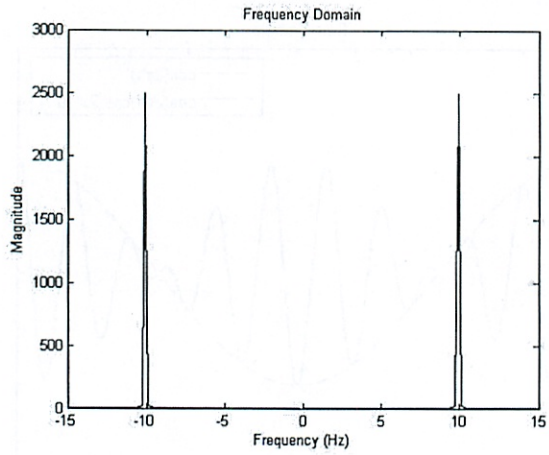


Figure 9

Figure 10 illustrates the spectrum of the product of the 1 and 10 Hz cosines; note how the spectrum has four peaks. The two spectral peaks at +/- 11 Hz correspond to the f_1+f_2 cosine while the two peaks at +/- 9Hz correspond to the f_1-f_2 cosine. At this point, we have modulated a 1 Hz cosine up to 9 and 11 Hz. Of course we could have also said that we modulated the 10 Hz signal to 9 and 11 Hz, but it is customary to think of the lower frequency signal as the information signal that we modulate up to the carrier (higher) frequency signal.

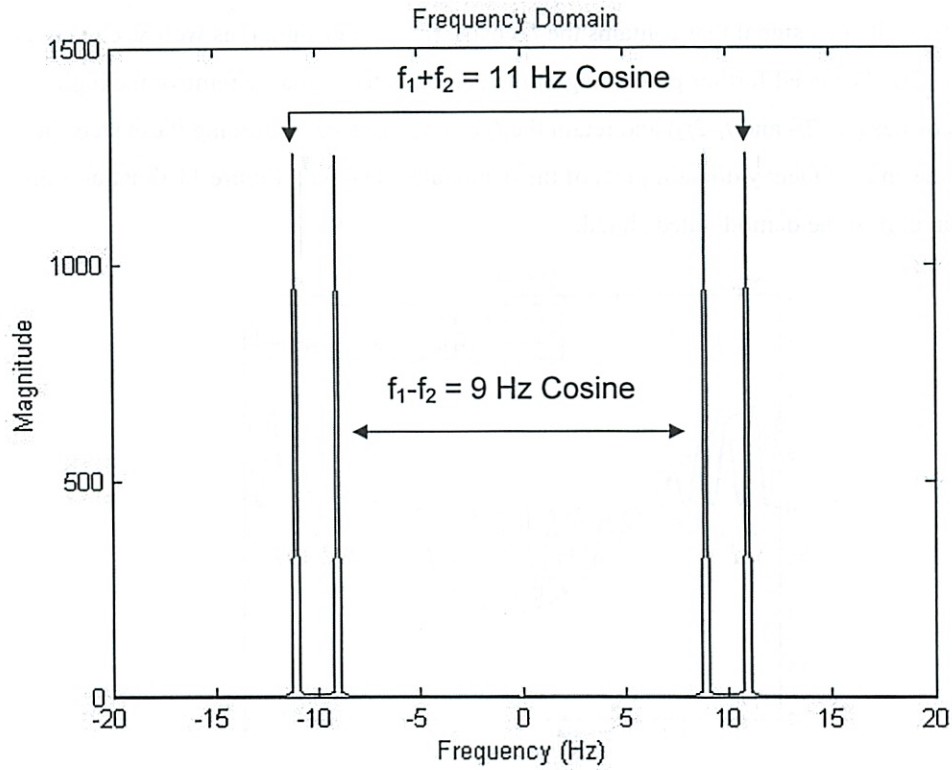


Figure 10

Demodulation

We know that demodulation involves shifting spectra, and that shifting spectra involves multiplication by cosines. Let's multiply the modulated signal from the previous section (the signal containing the f_1+f_2 and f_1-f_2 cosines) by another cosine with a frequency of f_2. The hope is that this operation will allow us to recover the f_1 cosine.

$$\frac{1}{2}\{\cos(2\pi(f_1+f_2)t)+\cos(2\pi(f_1-f_2)t)\}*\cos(2\pi f_2 t)$$

$$=\frac{1}{2}\{\cos(2\pi(f_1+f_2)t)*\cos(2\pi f_2 t)+\cos(2\pi(f_1-f_2)t)*\cos(2\pi f_2 t)\}$$

$$=\frac{1}{2}\{\frac{\cos(2\pi(f_1+2f_2)t)+\cos(2\pi f_1 t)}{2}+\frac{\cos(2\pi f_1 t)+\cos(2\pi(f_1-2f_2)t)}{2}\}$$

$$=\frac{1}{4}\cos(2\pi(f_1-2f_2)t)+\frac{1}{2}\cos(2\pi f_1 t)+\frac{1}{4}\cos(2\pi(f_1+2f_2)t)$$

From the above analysis, we see that demodulating by multipyling by a cosine with frequency f_2 results in a signal that contains the f_1 cosine (our target signal) as well as cosines at f_1+2f_2 and f_1-2f_2. We need further processing of the demodulated signal to remove the high frequency cosines (f_1+2f_2 and f_1-2f_2) and retain the f_1 cosine. Before discussing these steps, lets study the time and frequency domain plots of the demodulated signal. Figure 11 illustrates the time-domain plot of the demodulated signal.

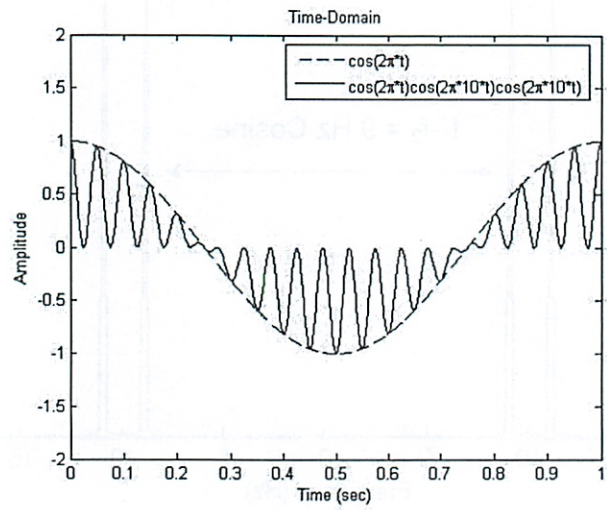


Figure 11

Figure 12 illustrates the spectrum of the demodulated signal; note how the spectrum has six peaks. The two spectral peaks at +/- 21 Hz correspond to the f_1+2f_2 cosine; the two peaks at +/- 19 Hz correspond to the f_1-2f_2 cosine; and the two spectral peaks at +/- 1 Hz correspond to the f_1 cosine.

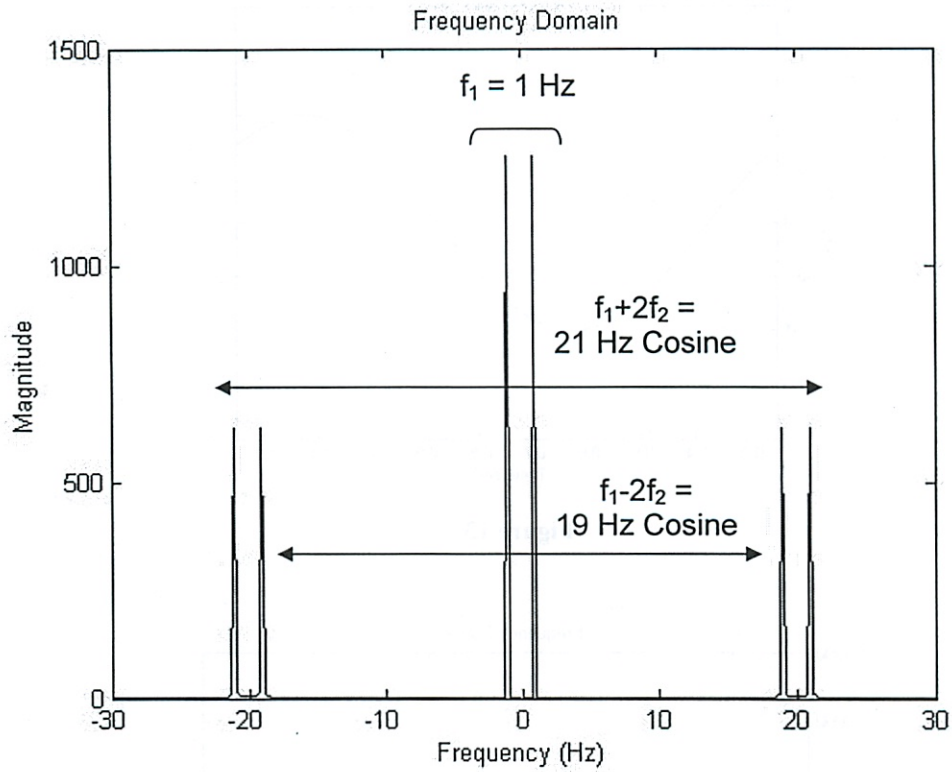


Figure 12

To remove the higher frequency cosines (f_1+2f_2 and f_1-2f_2) and only retain the f_1 cosine we will apply a low frequency filter with cutoff frequency $f_c = 10$ Hz. We will learn about filtering in the next lecture. Figure 13 and 14 illustrate the time and frequency plots of the demodulated signal after the low pass filter; note all that is left is the f_1 cosine (target signal).

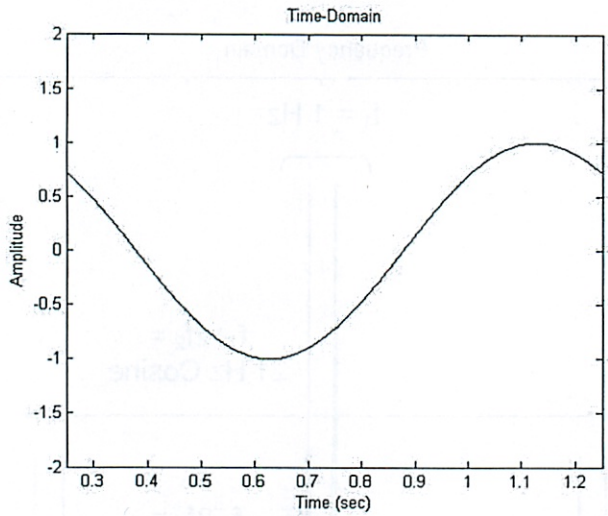


Figure 13

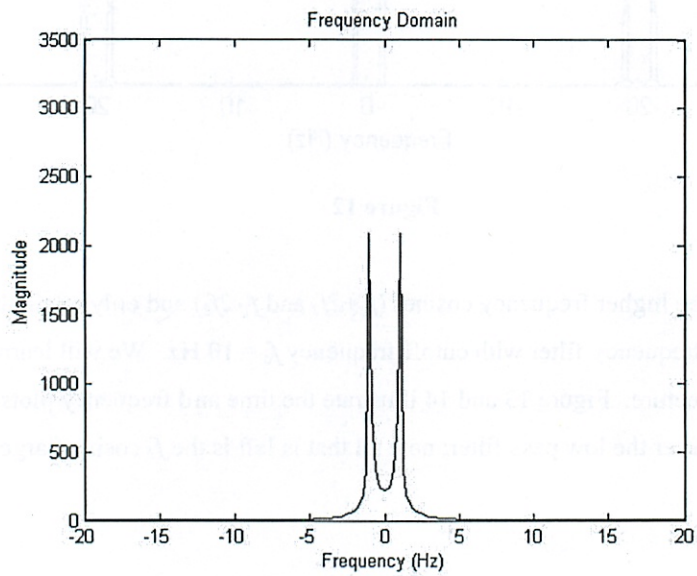


Figure 14

Modulating Signals With Nonzero Bandwidth

So far we have been dealing with the modulation of sinusoids; these signals have a single frequency component and zero bandwidth. Nonetheless signals with non-zero bandwidths, such as the voice signal whose time and frequency domain representations are shown in Figures 15 and 16, are also modulated by multiplying the signal by a cosine. The reason this works is that these

signals are made up of sine and cosines (recall Fourier series and transform), and modulation simply moves each of those sine/cosine components (and therefore the entire signal) up/down the spectrum.

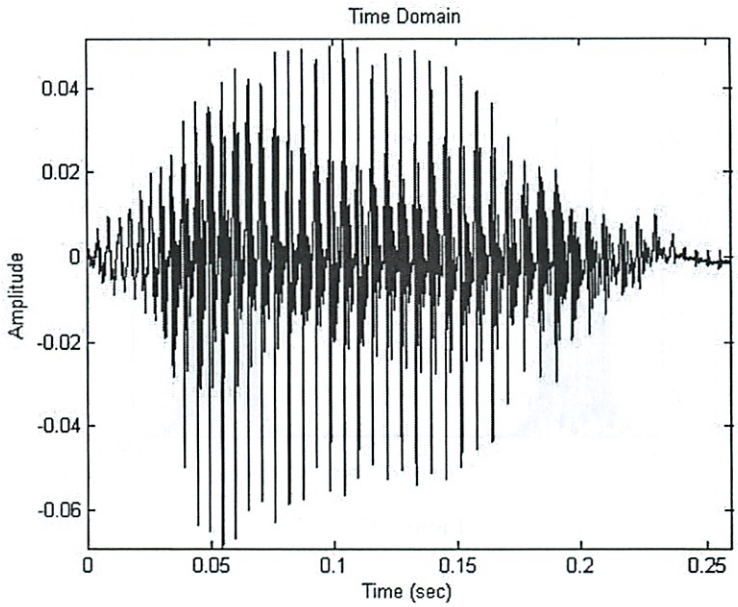


Figure 15

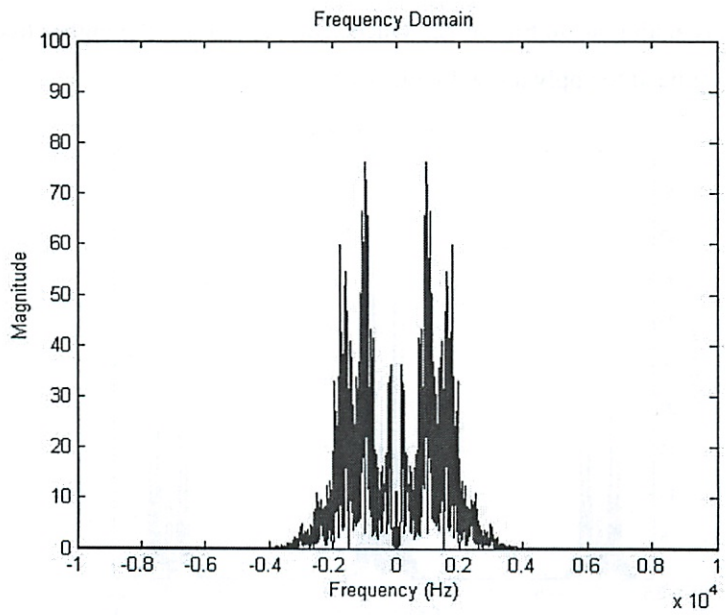


Figure 16

As an example, Figure 17 illustrates the spectrum of the voice signal in Figure 15 after it is modulated by a 10 kHz cosine. Note how the spectrum in Figure 16 is centered at +/- 10kHz; the plot is analogous to Figure 10.

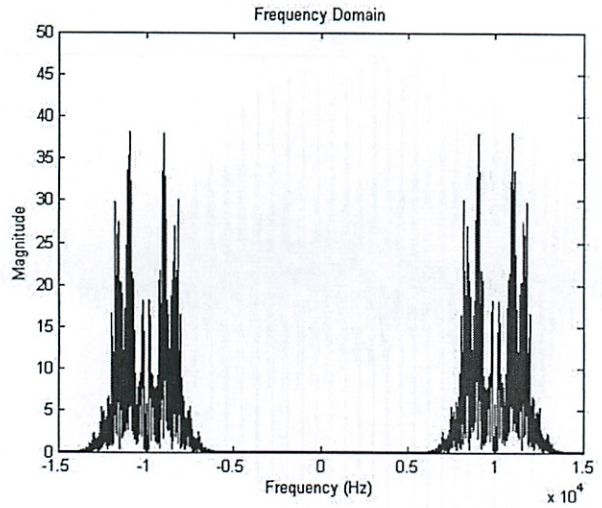


Figure 17

Now we will demodulate the voice signal back to DC (0 Hz). Figure 18 illustrates the spectrum of the voice signal after demodulation; multiplying the modulated signal (Figure 17) by a 10kHz cosine. Note that we have one copy of the voice signal spectrum at 0Hz and two copies at +/- 20kHz, which is analogous to Figure 12. Once again, to remove the higher frequency copies at +/-20kHz we need to apply a low frequency filter.

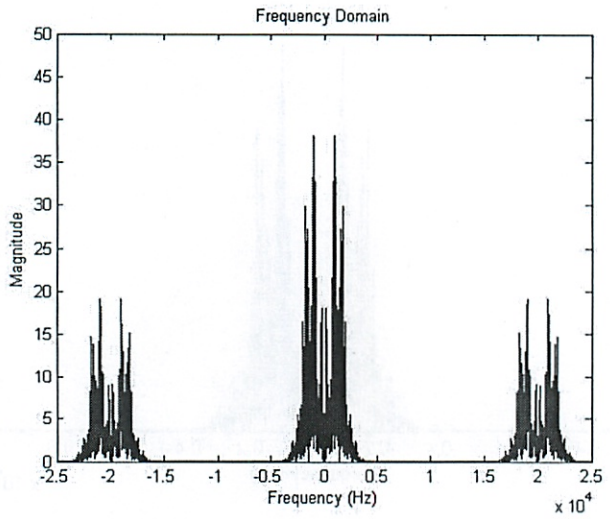


Figure 18

Modulation and Demodulation Summary

Figure 19 is a block diagram that summarizes the steps involved in modulating and demodulating a signal by a frequency f_2. Modulation involves multiplying the input signal x(t) by a cosine with a frequency f_2. Demodulation involves multiplying the modulated signal again by a cosine with a frequency f_2, and then applying a low pass filter with cutoff frequency f_2. The low pass filter removes the high frequency components centered about $2f_2$.

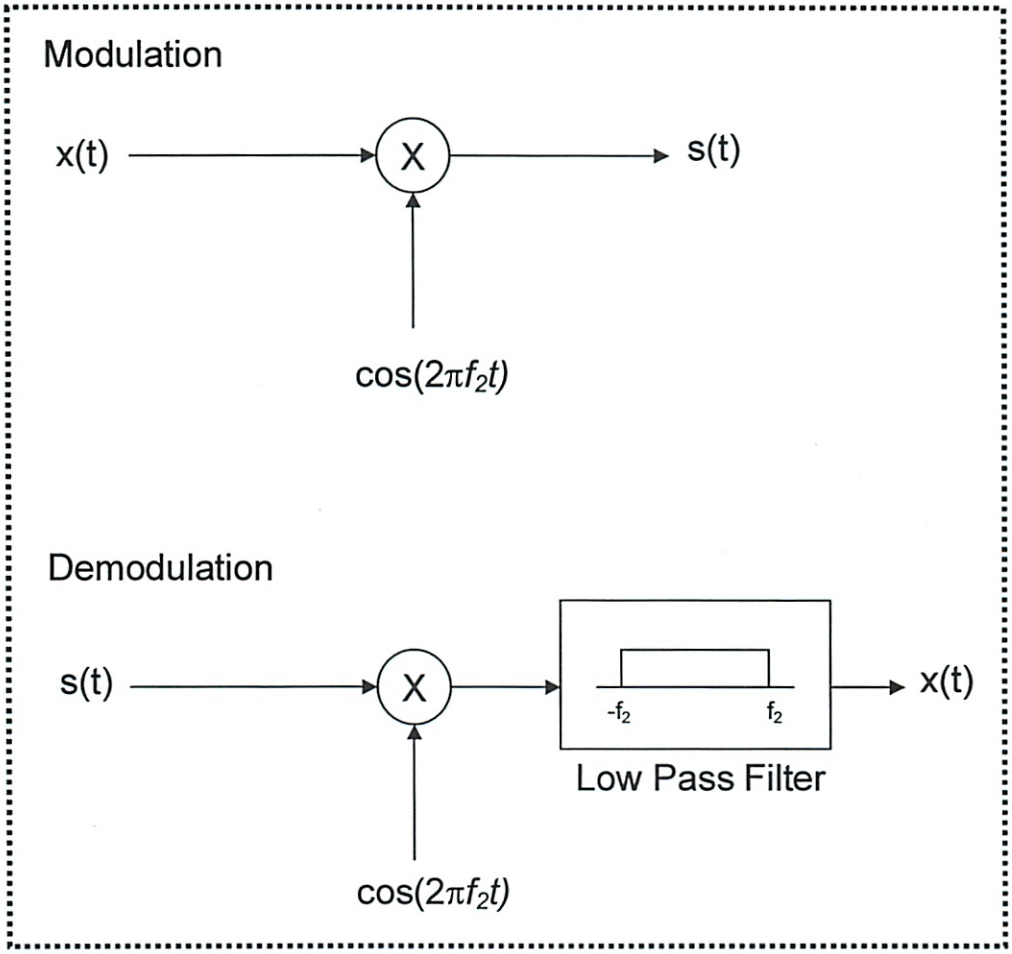


Figure 19

6.02 Recitation

Exam 2 Review

4/7

Lectures 6-15

1. BER and SNR
   - noise is added to LTI system
   - measuring noise
   - ratio $\frac{\text{signal}}{\text{noise}}$
2. Coding to remove errors
   - Hamming dist
   - Linear / Parities
   - Reed Soloman
   - Convolution / Viterbi / Trellis
3. Contention Resolution
   - TDMA
   - Aloha
     - stabalized
   - CSMA (carrier sense)
4. Frequency perspective of LTI systems
   - freq response
   - filters
   - fourier transforms

Adler checksums:

②

BER/SNR

$x[n]$ = signal

$0 \le n \le N-1$

$$\mu_x = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

$$\sigma^2 = var = \frac{1}{N} \sum_{n=0}^{N-1} (x[n] - \mu_x)^2$$

$$= \frac{1}{N} \left(\sum_{n=0}^{N-1} x[n]^2 \right) - \mu_x^2$$

$P_x = \sigma_x^2$

↑ how much you change is power for the signal

$$x[n] \longrightarrow \boxed{LTI} \longrightarrow y_{nf}[n] + noise[n] \longrightarrow y[n]$$

$$= x[n] * h[n]$$

↑ convolve

$$P_n = var(noise()) = \frac{1}{N} \sum_{n=0}^{N-1} (noise[n])^2 - \mu_{noise}^2$$

In order to know noise ~~the~~ must know what x was sent

$$SNR = \frac{P_x}{P_N}$$

No noise → $P_N = 0$ so $SNR = \infty$ $BER = 0$

(3)

BER decreases as SNR increases

Will model noise as Gaussian / Normal RV

$$\text{noise} \sim (\mu, \sigma^2)$$

$$= \sigma \underbrace{N}_{N(0,1)} + \mu$$

$$P(\text{noise} < t) = P(\sigma N + \mu < t)$$

$$= P\left(N < \frac{t-\mu}{\sigma}\right)$$

$$= \underbrace{\Phi}_{\text{CDF}}\left(\frac{t-\mu}{\sigma}\right)$$

Note $\Phi(x) + \Phi(-x) = 1$

What is sent

$0 \rightarrow 0V$

$1 \rightarrow 1V$

What is recieved:

$$x[\,] \rightarrow x[\,] + \underbrace{\text{noise}[\,]}_{N(0,\sigma^2)} = y[n]$$

Equal prob of sending 1 or 0

(4)

$BER = F(SNR)$

Compute var + μ

$\mu_x = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$

$\sigma_x^2 = \frac{1}{2} \cdot 0^2 + \frac{1}{2} \cdot 1^2 - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$SNR = \frac{P_x}{P_n} = \frac{1/4}{\sigma^2} = \left(\frac{1}{2\sigma}\right)^2$

st dev of noise

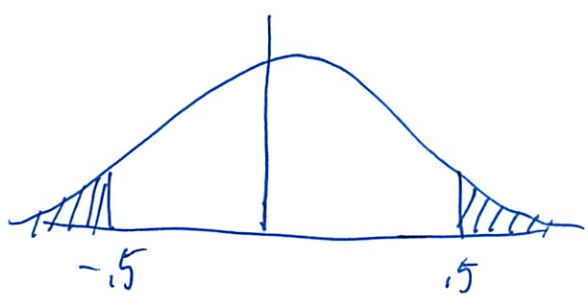
Cheating by using emperical mean - but ok w/ law of large #

$\sigma = \frac{1}{2\sqrt{SNR}}$

recieved sent

$BER = P(1|0)\,P(0) + P(0|1)\,P(1)$

$= P(\text{noise} > .5) \cdot \frac{1}{2} + P(\text{noise} < -.5)\frac{1}{2}$



$= P(\text{noise} < -.5)$

$= \Phi\left(\frac{-.5-0}{\sigma}\right)$

plug in from above

BER is fn of SNR

$= \Phi\left(-\frac{1}{2\sigma}\right)$

$$= \Phi\left(-\sqrt{SNR}\right)$$

- prob becomes $\downarrow$ as SNR $\uparrow$

Freq Domain

$x[n]$ $\in$ real values
in time domain
periodic N

$\llcorner$ $x[t] = x[t+N] = x[t-N] = x[t+2N] \ldots$ etc $\forall t$

Fourier Transform

for each k a complex #

$$x[n] = \sum_{k \in [N]} a_k \cdot e^{j \cdot \frac{2\pi}{N} \cdot k \cdot n}$$

$\in$ ensamble

$k+N$: $j \cdot \frac{2\pi}{N} \cdot kn + \underbrace{j \frac{2\pi}{N} \cdot N \cdot n}_{2\pi N}$ $= \exp\left(j \frac{2\pi}{N} \cdot kn\right) \cdot \exp(j 2\pi n)$

indexed to $\left\{-\frac{N}{2}, -\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}-1\right.$

$0, 1, 2, \ldots, N-1$

$1, 2, 3, \ldots, N$

the k values

Inverse Fourier Transform

$$a_k = \frac{1}{N} \sum_{n \in [N]} x[n] \cdot \exp\left(-j \cdot \frac{2\pi}{N} \cdot k \cdot n\right)$$

(6)

Now write in terms of cos and sin

FT of $\cos\left(\frac{2\pi}{N} \cdot r \cdot n\right) = x[n]$

$0 \le r \le N-1$

Can apply formula or use tricks

$$= \frac{1}{2}\left(\exp\left(j \cdot \frac{2\pi}{N} \cdot r \cdot n\right) + \exp\left(-j \frac{2\pi}{N} \cdot r \cdot n\right)\right)$$

Can read off coefficients of complex exponentials

$$= \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} a_k \cdot \exp\left(j \cdot \frac{2\pi}{N} \cdot k \cdot n\right)$$

Only thing left:

$$a_k = \begin{cases} \frac{1}{2} & \text{if } k = \pm r \\ 0 & \text{otherwise} \end{cases}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

repeat above

(7)

Suppose have signal

$$\alpha \cdot \cos\left(\frac{2\pi}{N} \cdot r \cdot n\right) + \beta \sin\left(\frac{2\pi}{N} \cdot q \cdot n\right)$$

Do FT

Use linearity to find FT for each component then add

That was signals

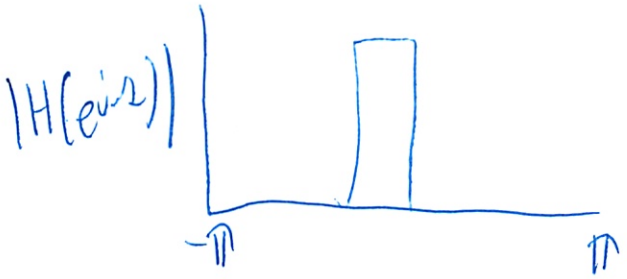
Now <u>channel respones</u>

$$x[\,] \rightarrow \boxed{\text{LTI}} \xrightarrow{y[\,]} = x[\,] \underset{n}{*} h[\,]$$

Convolve

$$H(e^{j\Omega}) = \sum_m h(m) \cdot e^{-j\Omega m}$$

Fitering



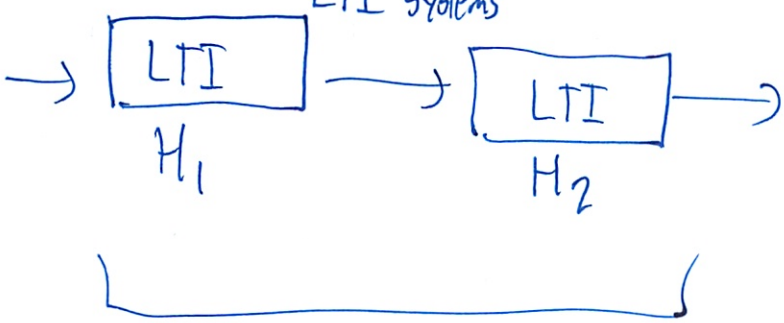
$\Omega \in [-\pi, \pi]$

$$x[n] = \sum_k a_k e^{j\Omega_k n} \xrightarrow[\text{LTI}]{} \sum a_k \cdot e^{j\Omega_k n} \cdot H(e^{j\Omega_k n})$$

8

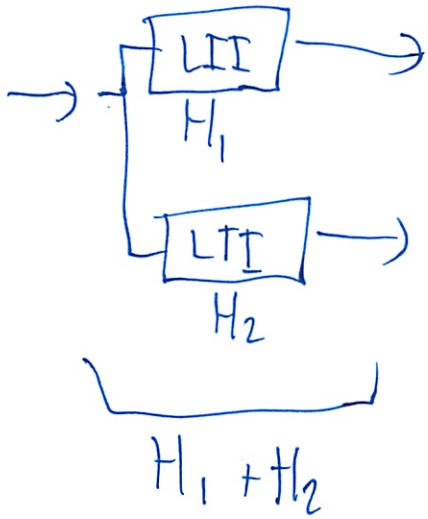
Now <u>building filters</u>

- putting LTI systems in series



$$H = H_1 \cdot H_2$$

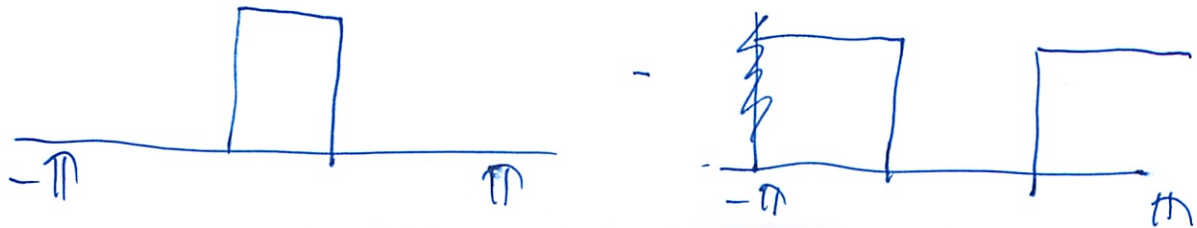
↑ multip in freq



$$H_1 + H_2$$

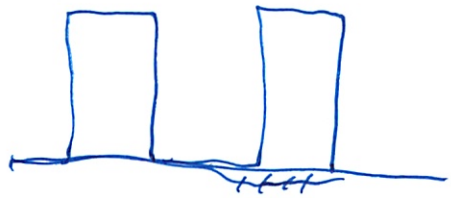
Suppose have low pass filter
Want high pass filter

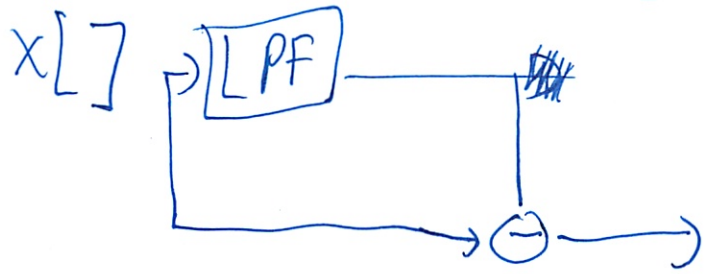
low



9

Also have band pass filter





is a HPF

For band pass multiply 2 HPF

- put in series

6.02 Checkoff

4/8

- he asked if I was busy this week
- 4C

a_p fine

also need a_{-p}

a_k only valid b/w

But $\cos(p)$ could be diff

- But periodic, so only in certain range

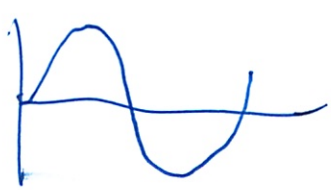
$p = 11 = 0$

a_k - Take modulus 11

Code 1

Why $-\frac{N}{2}$? convention

- Periodic could start anywhere



Want to change freq indicies

list containing freq - -

e)

So change r

k matches a_k

So make sure # matches

6.02 Recitation

Quiz 2 Review 2

9/12

(7 min late)

Today: Codes - convolution + viterbi

TDMA, Aloha, CSMA

Convolution

(I finally learned this on pset)

samples/bit = 2

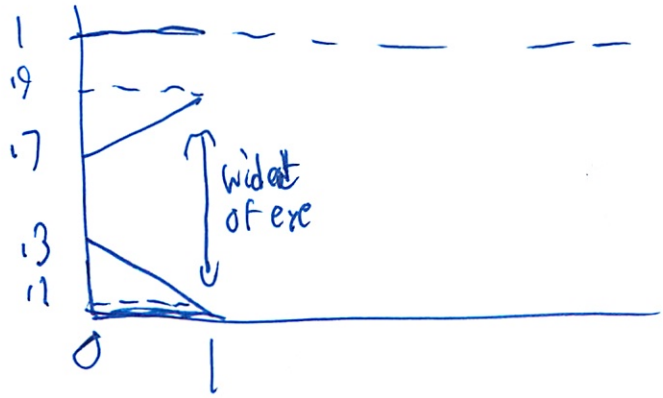
$y[n] = x[n]h_0 + x[n-1]h_1 + x[n-2]h_2$

$h_0 = 0.7$

$h_1 = 0.2$

$h_2 = 0.1$

$h_k = 0$ otherwise



Making an eyediagram

- what are all the possible lines?
- Convolve to find the .7 and .9

(2)

Now noise is added

$y_{nf}[n] \rightarrow y[\,] = y_{nf}[\cdot] + \text{noise}[\,]$

$\sim \text{Normal}(0, \sigma^2)$

To compute BER

- 2 options

$P(\text{detected } 1 \mid \text{trans. } 0)\, P(0) + P(0 \mid 1)\, P(1)$

- but errors different
- So actually 4 options/cases (what I thought)

$P(\text{prev}=0, \text{cur}=0)\; P(\text{detect } 1 \mid \text{prev}=0, \text{cur}=0) +$ ← This $(0 + \text{noise}) > .5$

$P(1, 0)\; P(1 \mid 1, 0) + (1 + \text{noise}) > .5$ ← at end

$P(0, 1)\; P(0 \mid 0, 1) +$

$P(1, 1)\; P(0 \mid 1, 1)$

These all $\frac{1}{4}$ since = prob

$P(\text{noise} > .5)$

Saying $\text{var} = \sigma^2 = 1$

(3)

$$= P(\sigma N > .5)$$

$$= P\left(N > \frac{.5}{\sigma}\right)$$

$$= P\left(N < \frac{-.5}{\sigma}\right)$$

$$= \Phi\left(-\frac{.5}{\sigma}\right)$$

$$= 1 - \Phi\left(\frac{.5}{\sigma}\right)$$

More freq

last week - periodic, a_k
- channel response
- filters

Finding a_k

Given $x[\ ]$ that is periodic w/ N

$$f = \frac{2\pi}{N}$$

If signal has that period - what freq are possible

how often you repeat

9

When multiply freq by N - get some multiple of 2π

So freq must be $\frac{2\pi}{N} \cdot k$ - all freq that make up signal

$k = 0, \ldots, N-1$

$-\frac{N}{2}, \ldots, \frac{N}{2}$ Choose by Convention

$$x[n] = \sum_{k \in \langle N \rangle} a_k \cdot \exp\left(j \cdot \frac{2\pi}{N} \cdot k \cdot n\right)$$

$$a_k = \frac{1}{N} \sum_{n \in \langle N \rangle} x[n] \cdot \exp\left(-j \frac{2\pi}{N} \cdot k \cdot n\right)$$

easier w/ sin, cos

$\cos(\Omega n) \rightarrow$

$\frac{2\pi k}{N}$

$$a_f = \begin{cases} \frac{1}{2} & \text{if } f = -\Omega \text{ or } \Omega \\ 0 & \text{otherwise} \end{cases}$$

$-\frac{2\pi k}{N}$ $\frac{2\pi k}{N}$

k is just convient notation for a certain freq

$$x[n] = 7 \sin\left(\frac{7\pi}{13} \cdot n\right)$$

$= \frac{7\pi}{13} = \text{freq}$

$$\text{period} = \frac{2\pi}{\Omega} = \frac{2\pi \cdot 13}{7\pi} = \frac{26}{7}$$

(5) So it repeats in N

So can say

$$\sin\left(7 \cdot \frac{2\pi}{26} \cdot n\right)$$

So this is N

You want freq to look $\underbrace{\frac{2\pi}{N} \cdot k}$ in formula

So $x(n) = 7 \sin\left(7 \cdot \frac{2\pi}{26} \cdot n\right)$

So k we have $-13, \ldots, 12$

$a_{-7} = \frac{7j}{2}$

$a_7 = -\frac{7j}{2}$

$a_{\text{otherwise}} = 0$

---

linear codes → rectangular codes

Single error can correct 1 error

lots of them

- prove property of code

(6)

(n, k, d)

n = total # bits

k = # of message bits

d = hamming distance of code

- parity, so linear
- so d is same as b/w 0 and any non 0 code

0000 0000

Change by 1 bit

1000 1010

So d is 3 or less

Keep trying for less

See for working codes

2^n want to correct $n+1$ errors

Have $n-k$ parity bits

Must capture errors

$2^{n-k} \geq n+1$