10/4

LECTURE 8

new unit: contineos random variables

- **Readings:** Sections 3.1-3.3 ← did not read!

Lecture outline

leap in complexity
-calculus
-conceptual ↑ in difficulty

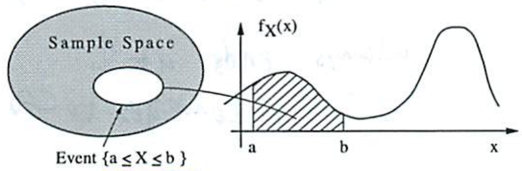
- Probability density functions
- Cumulative distribution functions
- Normal random variables

↑ most important one their is

Continuous r.v.'s and pdf's

- **A continuous r.v.** is described by a **probability density function** f_X

over the conthuem
-with discrete only had integers pmf = bar graph
- here can be any value pmf = integrate



$$\boxed{P(a \le X \le b) = \int_a^b f_X(x)\,dx}$$

density - rate of which probability accumulates
-integrate density over the interval

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$f_X(x) \ge 0$

$P(X=a) = \int_a^a = 0$

$$P(x \le X \le x+\boxed{\delta}) = \int_x^{x+\delta} f_X(s)\,ds \boxed{\approx f_X(x)\cdot\delta}$$

little slice $f_X(x)$ δ

similar to

$\boxed{\text{probability rate}}$

any individual pt has 0 probability

$$P(X \in B) = \int_B f_X(x)\,dx, \quad \text{for "nice" sets } B$$

- set over 2 intervals

just add

-disjoint!

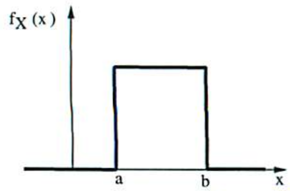
densities could be > 1
-~~each~~ not probabilites
-as long as area = 1

Means and variances

same as before
but sum replaced by integrals

- $\mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$
- $\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$
- $\mathrm{var}(X) = \sigma_X^2 = \int_{-\infty}^{\infty} (x - \mathrm{E}[X])^2 f_X(x)\, dx$

- **Continuous Uniform r.v.**



total area = 1

↓ so

height of function

every single little interval is equal prob.

- $f_X(x) = \frac{1}{b-a} \qquad a \le x \le b$
- $\mathrm{E}[X] = \int_a^b x \cdot \frac{1}{b-a}\, dx$ = still center of gravity = will be midpoint $\left(\frac{a+b}{2}\right)$
- $\sigma_X^2 = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\, dx = \frac{(b-a)^2}{12}$ same calculation need to integrate

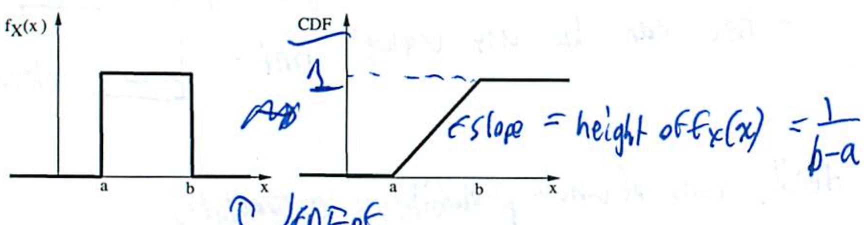
Cumulative distribution function (CDF)

probability to left of point

$$F_X(x) = \mathrm{P}(X \le x) = \int_{-\infty}^{x} f_X(t)\, dt$$

function of x

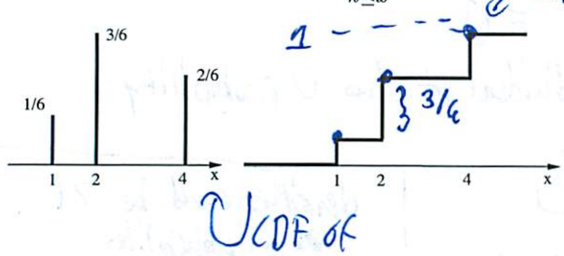
always starts at 0 asymptotically
always ends at 1
- as x converges to ∞



- Also for discrete r.v.'s:

$$F_X(x) = \mathrm{P}(X \le x) = \sum_{k \le x} p_X(k)$$

correct value higher at jump pts



Mixed distributions

- Schematic drawing of a combination of a PDF and a PMF

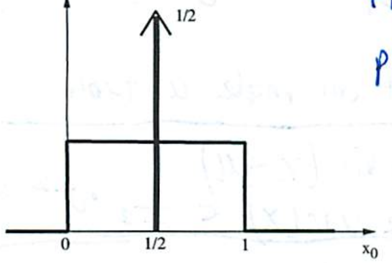
flip a coin

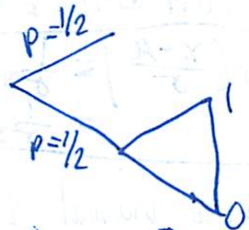
$p = \frac{1}{2} \to$ get .50 cents

-not contineas -since $P(x=a) \neq 0$ at $\frac{1}{2}$

-not discrete

-so say mixed



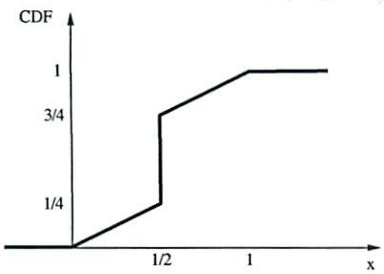


- The corresponding CDF:

$$F_X(x) = \mathrm{P}(X \leq x)$$

~~Uniform~~ Contineous

if you need to be formal about it, you can use this



mean variance

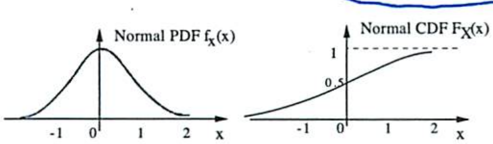
Gaussian (normal) PDF

most important probablity shows up in a lot of theoritical situations

- Standard normal $N(0,1)$: $\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$

Sum of many random variables is often approx normal distribution w/ the noise

-negitive exponential



Constant
↳ density must ∫ to = 1
So what constant do you need → that one

- $\mathrm{E}[X] = 0$ $\quad$ var$(X) = 1$

non trival calculus

- General normal $N(\mu, \sigma^2)$:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- It turns out that:
$\mathrm{E}[X] = \mu$ and Var$(X) = \sigma^2$.

σ small

so σ scales picture and var()

- Let $Y = aX + b$ linear function
  - Then: $\mathrm{E}[Y] = a\mu + b$ Var$(Y) = a^2\sigma^2$
  - Fact: $Y \sim N(a\mu + b, a^2\sigma^2)$

Calculating normal probabilities

- No closed form available for CDF
  - but there are tables (for standard normal)

$$P(X \le x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2}\, dx$$

Can't do it but can make a table

- If $X \sim N(\mu, \sigma^2)$, then $\dfrac{X-\mu}{\sigma} \sim N(0,\ )$

$$\text{var}\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}\text{var}(X-\mu) = \frac{1}{\sigma^2}\text{var}(X) = \frac{1}{\sigma^2}\cdot\sigma^2 = 1$$

standard normal distribution

- if not standard normal → normalize it!

- If $X \sim N(2, 16)$:

$$P(X \le 3) = P\left(\frac{X-2}{4} \le \frac{3-2}{4}\right) = \text{CDF}(0.25)$$

CDF

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

The constellation of concepts

$p_X(x)$ $\qquad$ $f_X(x)$

$F_X(x)$

$\mathbf{E}[X]$, var(X)

$p_{X,Y}(x,y)$ $\qquad$ $f_{X,Y}(x,y)$

$p_{X|Y}(x \mid y)$ $\qquad$ $f_{X|Y}(x \mid y)$

joint + conditional next time

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Recitation 8
October 5, 2010

1. Let Z be a continuous random variable with probability density function

$$f_z(z) = \begin{cases} \gamma(1+z^2), & \text{if } -2 < z < 1, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) For what value of γ is this possible?

   (b) Find the cumulative distribution function of Z.

2. Problem 3.9, pages 186–187 in the text.

   The taxi stand and the bus stop near Al's home are in the same location. Al goes there at a given time and if a taxi is waiting, (this happens with probability 2/3) he boards it. Otherwise he waits for a taxi or a bus to come, whichever comes first. The next taxi will arrive in a time that is uniformly distributed between 0 and 10 minutes, while the next bus will arrive in exactly 5 minutes. Find the CDF and the expected value of Al's waiting time.

3. Let λ be a positive number. The continuous random variable X is called **exponential** with parameter λ when its probability density function is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the cumulative distribution function (CDF) of X.

   (b) Find the mean of X.

   (c) Find the variance of X.

   (d) Suppose X_1, X_2, and X_3 are independent exponential random variables, each with parameter λ. Find the PDF of $Z = \max\{X_1, X_2, X_3\}$.

   (e) Find the PDF of $W = \min\{X_1, X_2\}$.

next Tue quiz review spring 10 quiz

Recitation 8

10/5

Contineos random values

- can have mixed (both)
- Must be all contineos to call contineos

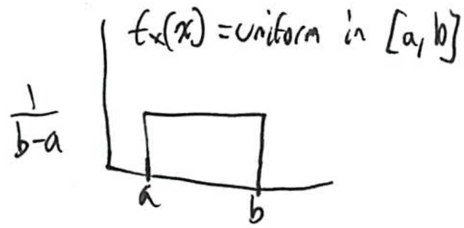
$$P(\{a \le X \le b\}) = \int_a^b f_X(x)\,dx$$

for all a, b

Normalization property

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dv$$

Uniform



Other

instead of weighted summation

- weighted integration

$Y = g(X)$ ← function

random variable

contineos?

- depends on what G is

(2)

Expected Value Rule

$Y = g(x)$

$$E[Y] = \int_{-\infty}^{\infty} g(x) f_x(x)\, dx$$

↑ can calc w/o know $p(Y)$

↑ weighted integration

$$var(x) = E[\underbrace{(x - E[x])^2}_{g(x)}]$$

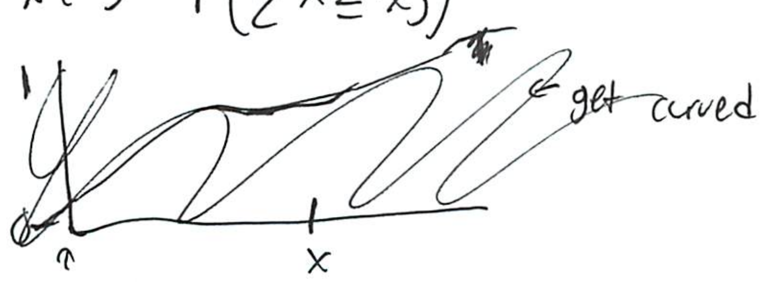
$$= E[x^2] - (E[x])^2$$ ← most conviented for calculations

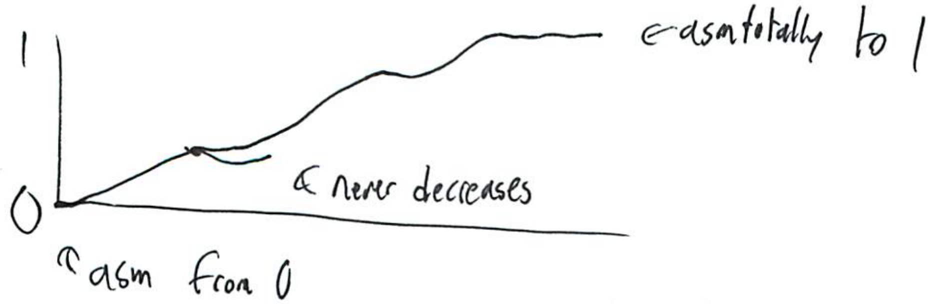
$$E[ax + b] = a E[x] + b$$

CDF

- works for contineous, discrete, mixed - anything

$$F_x(x) = P(\{X \leq x\})$$





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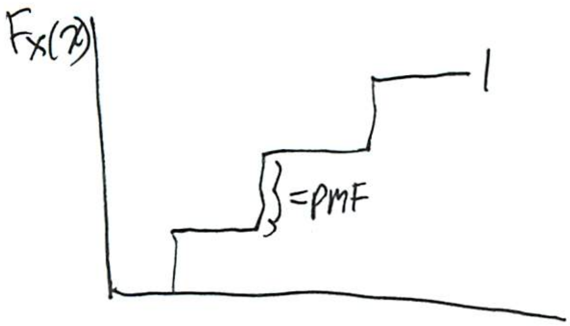
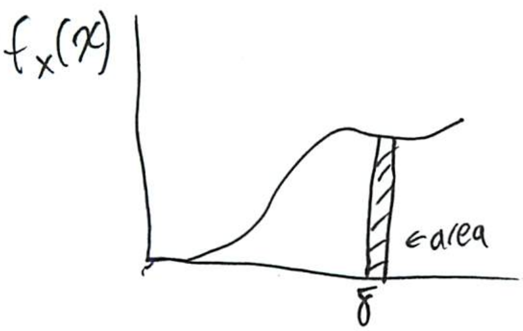


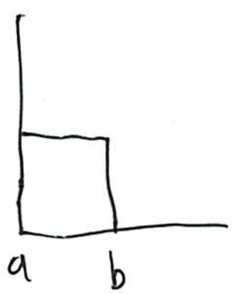
example
for discrete
-staircase

Continuous

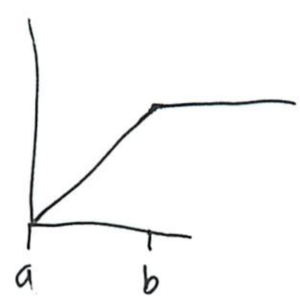


- any one value ≐ 0
- but interested in small δ

Normal function



CDF



$$PDF = \frac{d\,CDF}{d_x}$$

$$f_x(x) = F_x(x)$$

CDF capital letter

Skip lot problem

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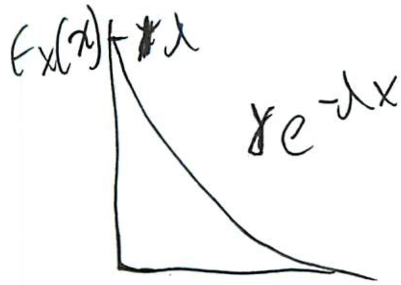
2. Quiz problem, more interesting
   1. Most important → normal / gaussian
   2. Second most important → exponential

(4)



how get γ

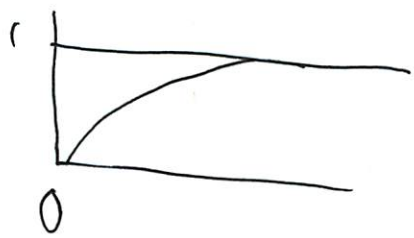
$$1 = \gamma \int_0^\infty e^{-\lambda x}\,dx$$

$$= \underbrace{\gamma \cdot \frac{1}{\lambda} e^{-\lambda x}}_{e^{-\lambda\infty} - e^{-\lambda 0}} \Big|_0^\infty$$

$$= \frac{\gamma}{\lambda}$$

confused - prof made mistake w/ γ and λ

$$CDF = F_X(x) = \int_0^x \lambda e^{-\lambda t}\,dt = 1 - e^{-\lambda x}$$



$$E[x] = \int_0^\infty x \cdot \lambda e^{-\lambda x}\,dx$$

- look at back of calculus table, but can't
- use ~~differentiation~~ integration by parts $\boxed{\int f g' = f \cdot g - \int f' \cdot g}$

(5)

$f(x) = x$
$g(x) = e^{-\lambda x}$

$$E[x] = x \cdot \lambda e^{-\lambda x}\Big|_0^\infty - \int_0^\infty \underbrace{1}_{f} \cdot \underbrace{\lambda e^{-\lambda x}}_{-g} dx$$

$$= 0 - \frac{e^{-\lambda x}}{\lambda}\Big|_0^\infty = \frac{1}{\lambda}$$ -Similar to geometric

-the discrete parts of geometric get really tight together + become contineous

---

$$Var(x) = \underbrace{\int_0^\infty x^2 \lambda e^{-\lambda x} dx}_{E[x^2]} - \underbrace{\left(\frac{1}{\lambda}\right)^2}_{(E[x])^2}$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\boxed{var(x) = \frac{1}{\lambda^2}}$$

(6)

Say have 3 random variables that are exponential (X_1, X_2, X_3)
- that are independant kinda

$$P(\{X_1 \le x_1, X_2 \le x_2, X_3 \le x_3\})$$
$$= P(\{X_1 \le x_1\}) P(\{X_2 \le x_2\}) P(\{X_3 \le x_3\})$$

roundabout way of defining probability

$$Z = \max\{X_1, X_2, X_3\}$$

- calc CDF
- differiate CDF → PDF

$$F_Z(z) = P(\{Z \le z\}) = P(\{X_1 \le z, X_2 \le z, X_3 \le z\})$$
$$= P(\{X_1 \le z\}) P(\{X_2 \le z\}) P(\{X_3 \le z\})$$
$$= (1 - e^{-\lambda z})^3$$

$(\text{CDF of each})^3$

$$W = \min\{X_1, X_2, X_3\}$$

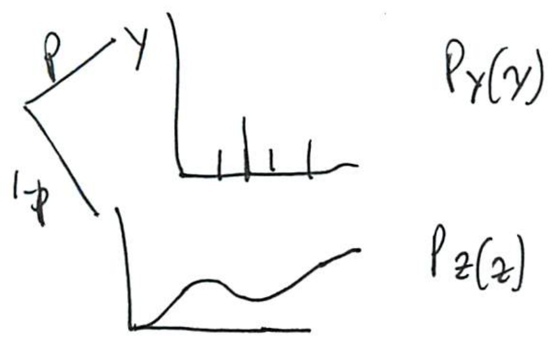
Exponential w/ parameter 3λ

$$F_W(w) = e^{-(3\lambda)w}$$

fast forwarding calculation

⑦

3. Mixed Random Variable



CDF = total prob. theorm

$$F_X(x) = \underbrace{p \cdot P(\{Y \le x\})}_{F_Y(y)} + (1-p) \underbrace{P(\{Z \le x\})}_{F_Z(z)}$$

$$F_X(x) = p\, F_Y(y) + (1-p)\, F_Z(z)$$

$$E[X] = p\, E[Y] + (1-p) E[Z]$$

In a hurry to get some where

← wait time

P(2/3) taxi is waiting $X = 0$

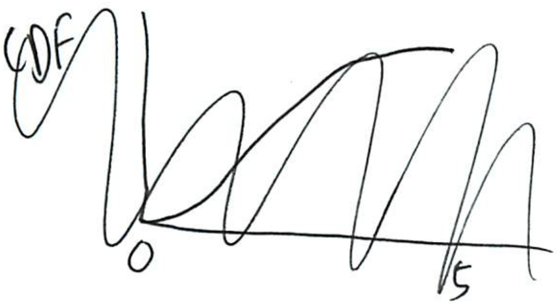
P(1/3) " not waiting

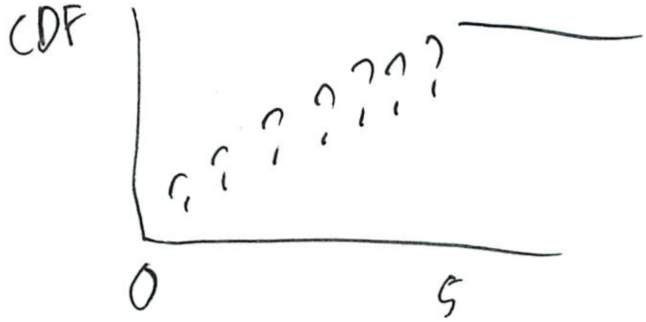
- call friend who takes 5 min

- $\frac{1}{2}$ chance taxi comes in those 5 min

8

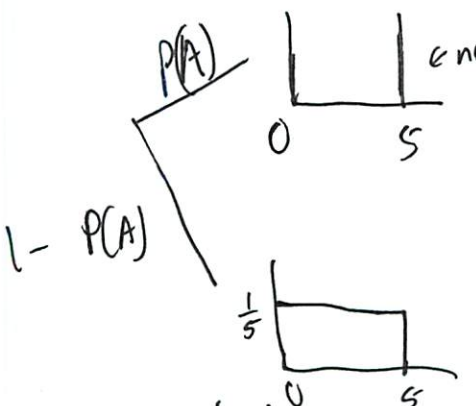
$X \in [0,5)$





- both discrete + mixture

A = finding taxi right away <u>or</u> picked up by friend



← need to calculate conditional that does not occur

taxi comes by, one is not there

- takes about a page of calculations
- do it on your own

10/5

LECTURE 9

material a little bit tricky

- **Readings:** Sections 3.4-3.5

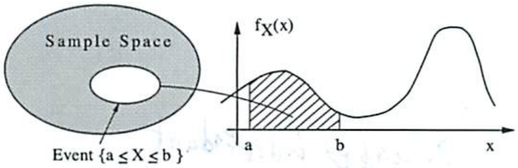
Outline

- PDF review
- Multiple random variables
  - conditioning
  - independence
- Examples

Summary of concepts

$p_X(x)$	CDF $F_X(x)$	$f_X(x)$
$\sum_x x p_X(x)$	$\mathbf{E}[X]$	$\int x f_X(x)\,dx$
	$\text{var}(X)$	
$p_{X,Y}(x,y)$		$f_{X,Y}(x,y)$
$p_{X\|A}(x)$		$f_{X\|A}(x)$
$p_{X\|Y}(x \mid y)$		$f_{X\|Y}(x \mid y)$
discrete	both	continuous

Continuous r.v.'s and pdf's



integrate mass over interval

densities - not necessarily probabilities!

$f(x) \geq 0$

$\int f(x)\,dx = 1$

$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)\,dx$$

- $\mathbf{P}(x \le X \le x+\delta) \approx f_X(x) \cdot \delta$
  height width
- $\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

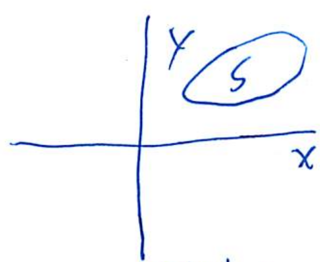
about
since need
second order, etc
like taylor series

1st new thing

Joint PDF $f_{X,Y}(x, y)$

if know densities of each
- don't know how the 2 are related

$$\boxed{\mathbf{P}((X,Y) \in S) = \int\int_S f_{X,Y}(x,y)\, dx\, dy}$$



$f_X(x)$
$f_Y(y)$

must be ≥ 0
$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$
that x and y lay anywhere

- Interpretation: definition: jointly continuous

$$\mathbf{P}(x \le X \le x+\delta,\ y \le Y \le y+\delta) \approx f_{X,Y}(x,y)\cdot\delta^2$$

* volume sitting on top of chart (3D) *

- Expectations:

$$\mathbf{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

discrete were $\Sigma\Sigma$

- From the joint to the marginal:

$$f_X(x)\cdot\delta \approx \mathbf{P}(x \le X \le x+\delta) = \int_{-\infty}^{\infty}\int_{x}^{x+\delta} f_{X,Y}(x,y)\,dx\,dy$$

^x is b/w x and x+δ

when calculating use little rectangles δ^2

$= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$

$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$

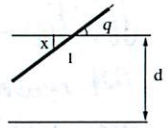
- X and Y are called independent if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

must be true for every pair

Buffon's needle

- Parallel lines at distance d
  Needle of length ℓ (assume $\ell < d$)
- Find $\mathbf{P}$(needle intersects one of the lines)



important: center point
θ

X and θ independent

$f_X(x) f_\Theta(\theta) = \frac{2}{d}\cdot\frac{2}{\pi}$

- $X \in [0, d/2]$: distance of needle midpoint to nearest line
- **Model:** X, Θ uniform, independent

$$f_{X,\Theta}(x,\theta) = \qquad 0 \le x \le d/2,\ 0 \le \theta \le \pi/2$$

~~angle of which needle fell~~

model 1st

- Intersect if $X \le \frac{\ell}{2}\sin\Theta$

then answer the question w/ model

$$\mathbf{P}\left(X \le \frac{\ell}{2}\sin\Theta\right) = \int\int_{x \le \frac{\ell}{2}\sin\theta} f_X(x) f_\Theta(\theta)\, dx\, d\theta$$

integrate over where it is true

$$= \frac{4}{\pi d}\int_0^{\pi/2}\int_0^{(\ell/2)\sin\theta} dx\, d\theta$$

$$= \frac{4}{\pi d}\int_0^{\pi/2}\frac{\ell}{2}\sin\theta\, d\theta = \boxed{\frac{2\ell}{\pi d}}$$

if $2L = D$ $\frac{1}{\pi}$

Conditioning

like ordinary
– but only applies in that conditional universe

- Recall

$$\mathbf{P}(x \le X \le x + \delta) \approx f_X(x) \cdot \delta$$

- By analogy, would like:

$$\mathbf{P}(x \le X \le x + \delta \mid Y \approx y) \approx f_{X|Y}(x \mid y) \cdot \delta$$

- This leads us to the **definition**:

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad \text{if } f_Y(y) > 0$$

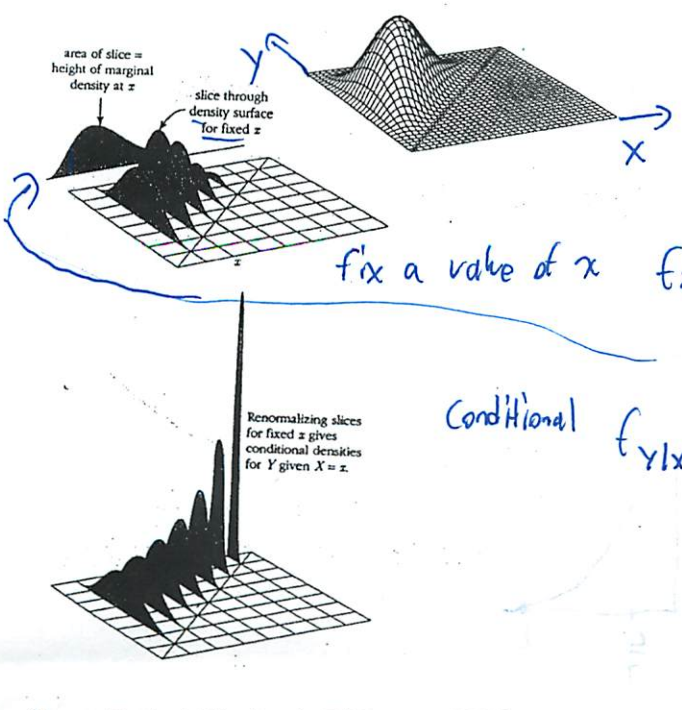
argue by analogy to discrete

- For given y, conditional PDF is a (normalized) "section" of the joint PDF

- If independent, $f_{X,Y} = f_X f_Y$, we obtain

$$f_{X|Y}(x|y) = f_X(x)$$

FIGURE 1. Joint, marginal, and conditional densities.



fix a value of x $f_X(x)$ = integrate over all y keeping x fixed

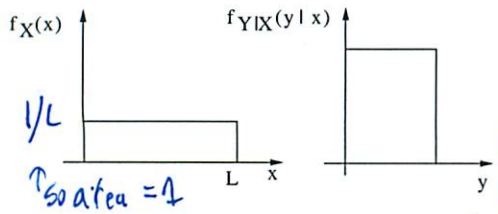
Conditional $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$ ← 1st pic ← slice

(from *Probability,* by J. Pittman, 1999)

Stick-breaking example

$0 \le Y \le X \le 1$

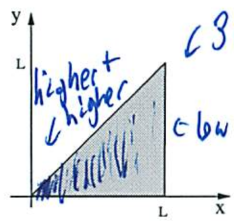
- Break a stick of length ℓ twice:
  break at X: uniform in $[0, 1]$;
  break again at Y, uniform in $[0, X]$



have indirect info → marginal ~~PDF~~ + conditional
So can find joint PDF

$$f_{X,Y}(x, y) = f_X(x) f_{Y|X}(y \mid x) = \frac{1}{\ell} \cdot \frac{1}{x} \quad \in x \in [0, L], \; y \in [0, x]$$

on the set:



← 3d

← low

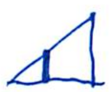
$0 \le y \le x \le \ell$

$f_{X,Y} = 0$, elsewhere

Set of possible X,L pairs

now compute stuff w/ it

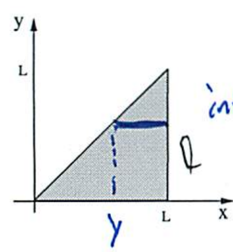
$$\mathrm{E}[Y \mid X = x] = \int y f_{Y|X}(y \mid X = x)\, dy =$$



$$= \int_0^x y \cdot \frac{1}{L} \cdot \frac{1}{x}\, dy = \frac{x}{2}$$

↑ uniform along that slice
— center of gravity is center
On average would break half into half again

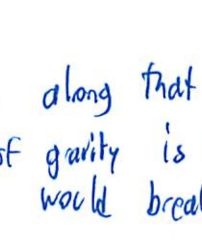
$$f_{X,Y}(x, y) = \frac{1}{\ell x}, \qquad 0 \le y \le x \le \ell$$



integrate joint over this interval

$$\begin{aligned} f_Y(y) &= \int f_{X,Y}(x, y)\, dx \\ &= \int_y^\ell \frac{1}{\ell x}\, dx \\ &= \boxed{\frac{1}{\ell} \log \frac{\ell}{y},} \qquad 0 \le y \le \ell \end{aligned}$$

$f_Y(y)$



$$\mathrm{E}[Y] = \int_0^\ell y f_Y(y)\, dy = \int_0^\ell y \frac{1}{\ell} \log \frac{\ell}{y}\, dy = \frac{\ell}{4}$$

if break half in half again
you are left w/ $\frac{1}{4}$

Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2010)

Recitation 9

October 7, 2010

1. Let X be an exponential random variable with parameter $\lambda > 0$. Calculate the probability that X belongs to one of the intervals $[n, n+1]$ with n odd.

2. (Example 3.13 of the text book, page 165) **Exponential Random Variable is Memoryless.** The time T until a new light bulb burns out is an exponential random variable with parameter λ. Ariadne turns the light on, leaves the room, and when she returns, t time units later, finds that the bulb is still on, which corresponds to the event $A = \{T > t\}$. Let X be the additional time until the bulb burns out. What is the conditional CDF of X, given the event A?

3. Problem 3.23, page 191 in the text.
   Let the random variables X and Y have a joint PDF which is uniform over the triangle with vertices $(0,0)$, $(0,1)$, and $(1,0)$.
   (a) Find the joint PDF of X and Y.
   (b) Find the marginal PDF of Y.
   (c) Find the conditional PDF of X given Y.
   (d) Find $\mathbf{E}[X \mid Y = y]$, and use the total expectation theorem to find $\mathbf{E}[X]$ in terms of $\mathbf{E}[Y]$.
   (e) Use the symmetry of the problem to find the value of $\mathbf{E}[X]$.

4. We have a stick of unit length, and we break it into three pieces. We choose randomly and independently two points on the stick using a uniform PDF, and we break the stick at these points. What is the probability that the three pieces we are left with can form a triangle?

Recitation 4

10/7

Quiz 1 Review Session 7:30 - 9:30 tonight ~~34~~ 32-141

Continuous RV

PDF $f_X(x)$ $\quad P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$ — for continuous

CDF $F_X(x) = P(X \leq x)$ — for any: -discrete -mixed -continuous

$$= \int_{-\infty}^{x} f_X(t)\,dt$$

$$f_X(x) = \frac{d}{dx} F_X(x)$$ } for continuous

PDF from CDF

Conditional CDF

- given event A w/ $P(A) > 0$

$$F_{X|A}(x) = P(X \leq x \mid A) = \frac{P(\{X = x\} \cap A)}{P(A)}$$

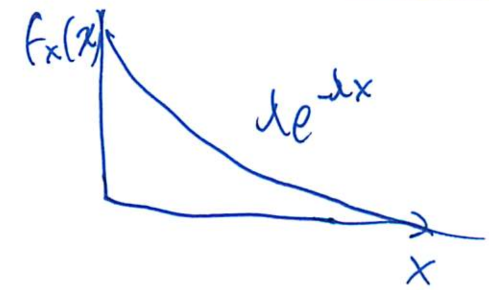
$A = \{X \in \underbrace{B}_{\text{interval of real line}}\}$

$$= \frac{\displaystyle\int_{\substack{X \leq x \\ x \in B}} f_X(x)\,dx}{P(A)}$$

$$f_{X|\{X \in B\}}(x) = \begin{cases} \displaystyle\int_{\substack{x \in B \\ X \leq x}} f_X(x)\,dx \Big/ P(A) & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}$$

②

Exponental Random Variables w/ Parameter λ

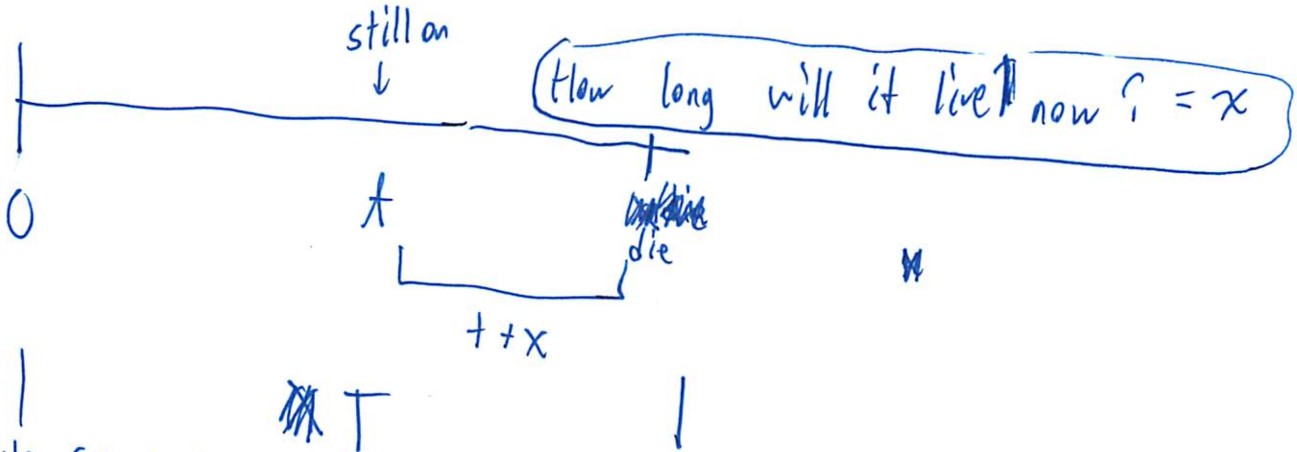


Memoryless

Exponental does not care what came before after certain value

ex

T = lifetime expontal w/ parameter λ



calculate complementry

$P(T \geq t+x \mid T > t)$ given $t+x \geq 0$

$$= \frac{P(T \geq t+e)}{P(T > t)} = \frac{1-CDF}{e^{-\lambda t}} = \frac{1-1-e^{-\lambda x}}{e^{-\lambda t}} = \frac{+e^{-\lambda(t+x)}}{e^{-\lambda t}}$$

③

Parts cancle

$= e^{-\lambda x}$

So now

$$P(T \le t + x \mid T > t) = 1 - e^{-\lambda x}$$

~~Complementary~~
Switch back

So distribution of remaining time to live, is same as checking any pt

But lightbulbs are not expontially - so bad example

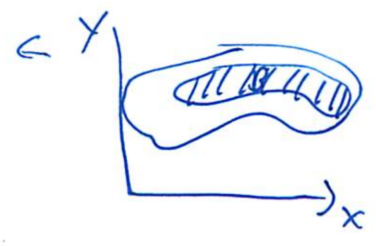
- don't talk about relative freq - 2 chap ahead

(Still unclear what he is trying to model)

Joint

$$P(a \le X \le b, c \le X \le d) = \int_a^b \int_c^d f_{X,Y}(x,y)\,dx\,dy$$

$$P((x,y) \in B) = \iint_B f_{X,Y}(x,y)\,dx\,dy$$



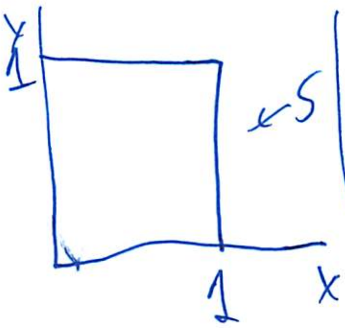
Noarmalization Property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

19.4. Stick of unit lenght - break at 2 random pts

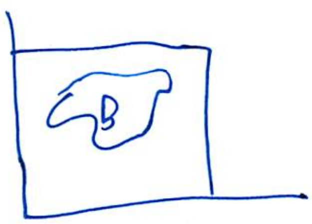
(not like last time where half + half again)

(4)



$$f_{X,Y}(x,y) = \begin{cases} C & \text{if } (x,y) \in S \\ 0 & \text{if } (x,y) \notin S \end{cases}$$

- $C = \dfrac{1}{\text{area } S}$ due to normalization property

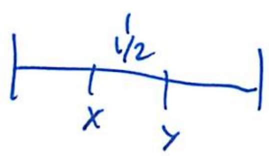


$$P(B) = \frac{\text{Area}(B)}{\text{area}(S)} = \frac{\text{Area}(B)}{1}$$

* Unlike lecture problem 1 break does not bias the second one

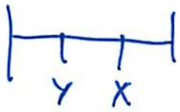
So how to form a triangle?

1. $x < y$

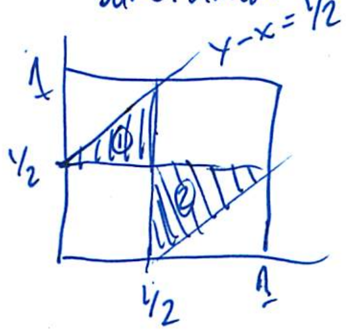


$x < \frac{1}{2}$, $y > \frac{1}{2}$, $(y-x) < 1/2$

2. $x > y$



Same/similar



$B = ① \cap ②$

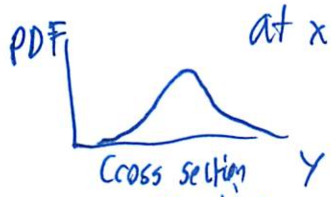
$P(B) = 1/4$

5. Marginal PDFs

messy :)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)$$

slice the joint ~~integrate~~



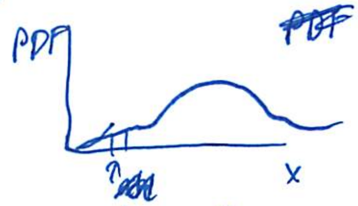
Cross section of slice

at x = __

← this is PDF of X

~~PDF~~

do for every X



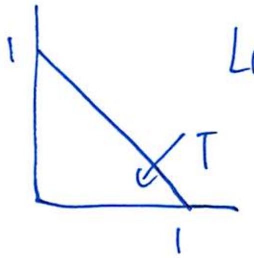
integrated val

~~[illegible]~~ each slice

normalize so $\int_{-\infty}^{\infty} = 1$

across

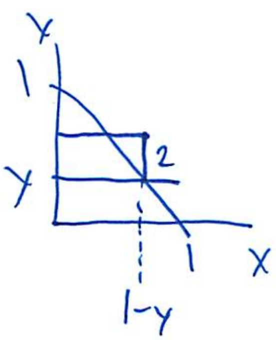
3.



Let $f_{X,Y}(x,y)$ uniform over triangle

$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } (x,y) \in T \\ 0 & \text{otherwise} \end{cases}$$

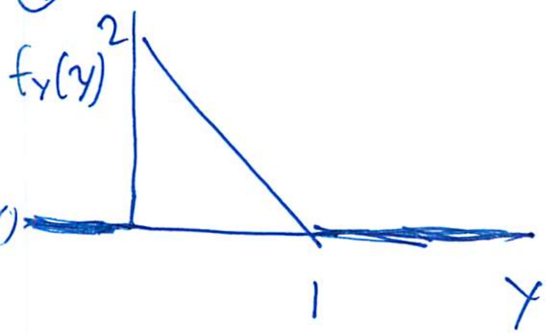
$c = 2$ ← (was wrong online)



$$f_Y(y) = \begin{cases} 0 & \text{if } y > 1 \text{ or } y < 0 \\ 2 \cdot (1-y) & 0 \le y \le 1 \end{cases}$$

See chart next pg

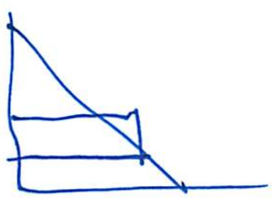
(6)



Conditional

$$f_{X|Y}(x|y) = \begin{cases} 0 & 0<y<1 \text{ and } \left.\begin{array}{l} x>1-y \\ x<0 \end{array}\right. \\ \frac{1}{1-y} & 0 \le x \le 1-y \\ \text{undefined} & y \ge 1 \text{ or } y<0 \end{cases}$$

slice + normalize



$$E[X \mid Y=y] = \frac{1-y}{2}$$

$0 \le y < 1$

$$E[X] = \int_{-\infty}^{\infty} f_Y(y)\, \underbrace{E[X \mid Y=y]}_{\frac{1-y}{2}}\, dy$$

Total expectation theorm
- divide + conqur (somewhat - contineous)

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Tutorial 4
October 7/8, 2010

1. Let X and Y be Gaussian random variables, with $X \sim N(0,1)$ and $Y \sim N(1,4)$.
   (a) Find $\mathbf{P}(X \leq 1.5)$ and $\mathbf{P}(X \leq -1)$.
   (b) What is the distribution of $\frac{Y-1}{2}$?
   (c) Find $\mathbf{P}(-1 \leq Y \leq 1)$.

2. Example 3.15, page 169 in text.

   Ben throws a dart at a circular target of radius r. We assume that he always hits the target, and that all points of impact (x, y) are equally likely. Compute the joint PDF $f_{X,Y}(x, y)$ of the random variables X and Y and compute the conditional PDF $f_{X|Y}(x|y)$.

3. Problem 3.20, page 191 in text.

   An absent-minded professor schedules two student appointments for the same time. The appointment durations are independent and exponentially distributed with mean thirty minutes. The first student arrives on time, but the second student arrives five minutes late. What is the expected value of the time between the arrival of the first student and the departure of the second student?

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

$= .86$ so $F_{X(x)}(z \le .86) = .8051$

The standard normal table. The entries in this table provide the numerical values of $\Phi(y) = \mathbf{P}(Y \le y)$, where Y is a standard normal random variable, for y between 0 and 3.49. For example, to find $\Phi(1.71)$, we look at the row corresponding to 1.7 and the column corresponding to 0.01, so that $\Phi(1.71) = .9564$. When y is negative, the value of $\Phi(y)$ can be found using the formula $\Phi(y) = 1 - \Phi(-y)$.

Tutorial 4

10/8

- HW due next Mon

1. X, Y Guassian Random variables

$X \sim N(0,1)$ ← x is random variable ~~Bot~~ gausian / normal distribution mean = 0, Var = 1

$Y \sim N(1,4)$

a) $P(X \leq 1.5)$ ← CDF

- look at table for (0, 1) Guassian (the ~~normal~~ "Standard")

aka $\Phi(1.5)$

$= .09332$

$P(X \leq -1)$

- tables is only (+)

↑ Symetric about the mean

So same as $P(X \geq 1)$

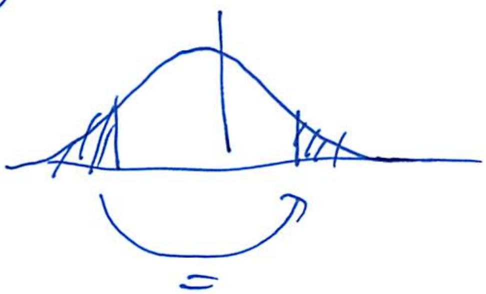
↑ but we have CDF table (only works for $\leq$)

- but if we flip

$= 1 - P(X \leq 1)$

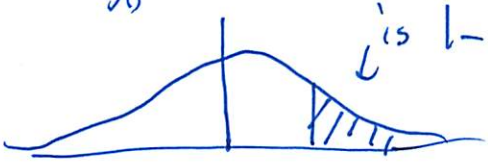
$= 1 - \Phi(1) = 1 - .8413 = .1587$

2



whole thing $= 1$ we know

so



is 1 − the rest
which is on the table

---

b) $Z = \frac{Y-1}{2}$

linear operation on normal is still normal

$E[Y] = 0$

$E[Y-1] = E[Y] - 1 = 0$

~~var~~ $E\left[\frac{Y-1}{2}\right] = \frac{1}{2} E[Y-1]$

$\cancel{= \frac{1}{2} \cdot 0 = 0}$

$\mathrm{var}(Y) = 4$

$\mathrm{var}(Y-1) = 4$

$\left(\frac{1}{2}\right)^2 \mathrm{var}(Y-1) = 1$

---

c) $P(-1 \le Y \le 1)$

now $Y \sim N(1, 4)$

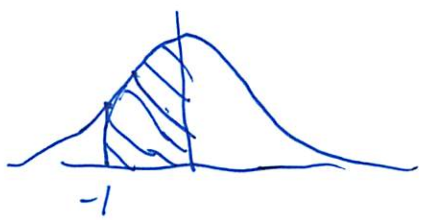
we say in b how to standardize

③

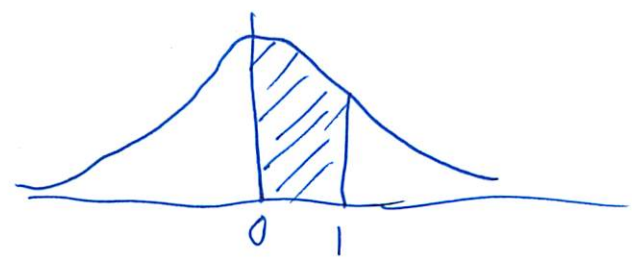
$$P\left(\frac{-1-1}{2} \le \frac{Y-1}{2} \le \frac{1-1}{2}\right)$$

$$\frac{Y-E[X]}{\sqrt{Var(X)}} = \frac{Y-E[\;]}{\sigma_Y}$$

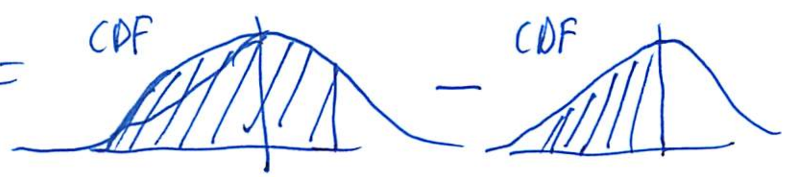
$$= P(-1 \le Z \le 0)$$



is same as



=



$$= P(Z \le 1) - P(Z < 0)$$

$$= \Phi(1) - \Phi(0)$$

$$= .8413 - .5$$

$$= .3413$$

(4)

3. Exponential Random Variables

-Only mentioned in recitation

$X, Y \sim \text{Exponential}(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$E[X] = 1/\lambda$

told are exponential

$E[X] = E[Y] = \frac{1}{\lambda} = 30$ *

So we know $\lambda = \frac{1}{30}$ in this problem

<u>CDF</u>

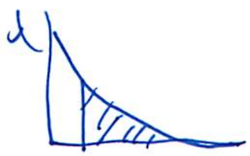
$P(X \leq x) = F_X(x)$

$$= \begin{cases} 1 - e^{-x/30} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

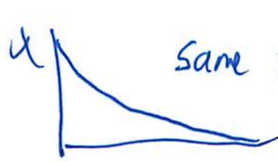
like geometric - memoryless property

-what happens does not depend on what happened

$E[X \mid X > 5] = E[X] + 5$



→ scale up to λ



same function

so $\int_{-\infty}^{\infty} = 1$

(5)

$$P(X \le x) = F_X(x) = \begin{cases} 1 - e^{-x/30} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

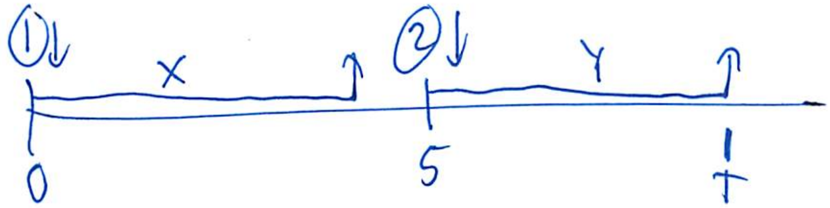
$$\int_0^x f_X(x')\,dx' = \int_0^{30} \frac{1}{30} e^{-x'/30} = -e^{-x/30}\Big|_0^x = 1 - e^{-x/30}$$

T = {time of student 2's departure − time of student 1's arrival}

X = duration of #1's ~~stay~~ in office
Y = " " #2's ~~stay~~ in office (wait time <u>not</u> included)

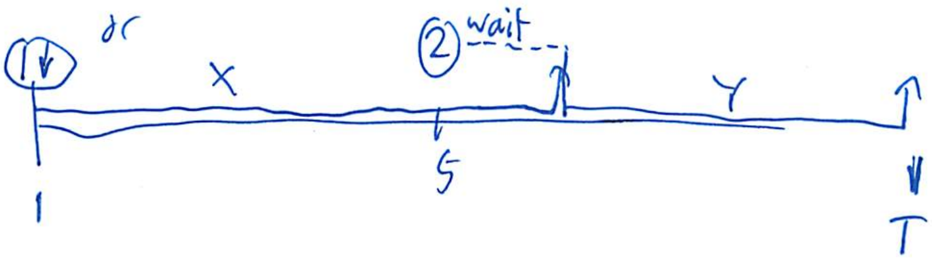
$E[T] = ?$

(A)
$(X \le 5)$



or

(B)



$$T = \begin{cases} 5 + Y & \text{if } A \\ X + Y & \text{if } B \end{cases}$$

(6)

Total expectation theorm

$$E[T] = P(A)\,E[T|A] + P(B)\,E[T|B]$$

$\boxed{P(A) + P(B) = 1}$

$P(A) = ?$

CDF taken at 5

$= 1 - e^{-5/30}$

$P(B) = e^{-5/30}$

$E[T|A] = \cancel{E[5+Y|A]}$

$= E[5+Y|A]$ ← ind. does not matter

$= E[5+Y]$

$= E[Y] + 5$

$= 35$

$E[T|B] = E[X+Y|B]$ ← does depend - B depends on x

dependant

$= E[X|B] + E[Y|B]$

$= E[X|B] + E[Y]$ ← is independent

$= E[X|X>5] + E[Y]$

$= E[X] + 5 + E[Y]$

$= 30 + 5 + 30 = 65$

(7)

Have everything we need to plug in

$$E[T] = (1 - e^{-5/30})\,35 + (e^{-5/30})\,65$$

$$= 35 + 30e^{-5/30}$$

- trickier problem

Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2010)

Recitation 10

October 12, 2010

- **Question 1.** The two parts of this question are about identities for a probabilistic model with sample space Ω, events A and B, and discrete random variable X. Any time conditioning on an event is indicated, the event has positive probability. An identity is *true* when it holds without any additional restrictions; it is *false* when there is any counterexample.

  1.1. Which **one** of the following statements is **true**?

  (a) $\mathbf{P}(A \cap B)$ may be larger than $\mathbf{P}(A)$.

  (b) The variance of X may be larger than the variance of $2X$.

  (c) If $A^c \cap B^c = \emptyset$, then $\mathbf{P}(A \cup B) = 1$.

  (d) If $A^c \cap B^c = \emptyset$, then $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$.

  (e) If $\mathbf{P}(A) > 1/2$ and $\mathbf{P}(B) > 1/2$, then $\mathbf{P}(A \cup B) = 1$.

  1.2. Which **one** of the following statements is **true**?

  (a) If $\mathbf{E}[X] = 0$, then $\mathbf{P}(X > 0) = \mathbf{P}(X < 0)$.

  (b) $\mathbf{P}(A) = \mathbf{P}(A \mid B) + \mathbf{P}(A \mid B^c)$

  (c) $\mathbf{P}(B \mid A) + \mathbf{P}(B \mid A^c) = 1$

  (d) $\mathbf{P}(B \mid A) + \mathbf{P}(B^c \mid A^c) = 1$

  (e) $\mathbf{P}(B \mid A) + \mathbf{P}(B^c \mid A) = 1$

- **Question 2.** Provide **clear reasoning**; partial credit is possible

Heather and Taylor play a game using independent tosses of an unfair coin. A head comes up on any toss with probability p, where $0 < p < 1$. The coin is tossed repeatedly until either the second time head comes up, in which case Heather wins; or the second time tail comes up, in which case Taylor wins. Note that a full game involves 2 or 3 tosses.

2.1. Consider a probabilistic model for the game in which the outcomes are the sequences of heads and tails in a full game. Provide a list of the outcomes and their probabilities of occurring.

2.2. What is the probability that Heather wins the game?

2.3. What is the conditional probability that Heather wins the game given that head comes up on the first toss?

2.4. What is the conditional probability that head comes up on the first toss given that Heather wins the game?

- **Question 3.** Provide **clear reasoning**; partial credit is possible

A casino game using a **fair** 4-sided die (with labels 1, 2, 3, and 4) is offered in which a **basic game** has 1 or 2 die rolls:

- If the first roll is a 1, 2, or 3, the player wins the amount of the die roll, in dollars, and the game is over.
- If the first roll is a 4, the player wins \$2 and the amount of a second ("bonus") die roll in dollars.

Let X be the payoff in dollars of the basic game.

3.1. Find the PMF of X, $p_X(x)$.

3.2. Find $\mathbf{E}[X]$.

3.3. Find the conditional PMF of the result of the first die roll given that $X = 3$. (Use a reasonable notation that you define explicitly.)

3.4. Now consider an **extended game** that can have any number of bonus rolls. Specifically:

* Any roll of a 1, 2, or 3 results in the player winning the amount of the die roll, in dollars, and the termination of the game.
* Any roll of a 4 results in the player winning \$2 and continuation of the game.

Let Y denote the payoff in dollars of the extended game. Find $\mathbf{E}[Y]$.

Recitation 10

10/12

Test Review

1.1 c is correct. We will prove that c is correct

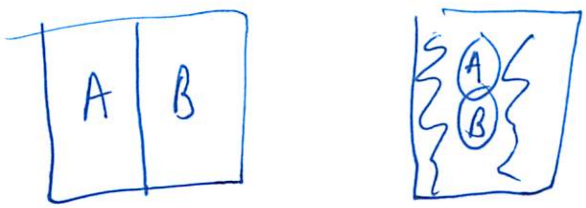
If $A^c \cap B^c = \emptyset$ then $P(A \cup B) = 1$

$\underbrace{(A^c \cap B^c)}_{\emptyset} = (A \cup B)^c$ ← make sure on formula sheet

or $(A^c \cap B^c)^c = A \cup B$

$\emptyset^c = \Omega$

Visually

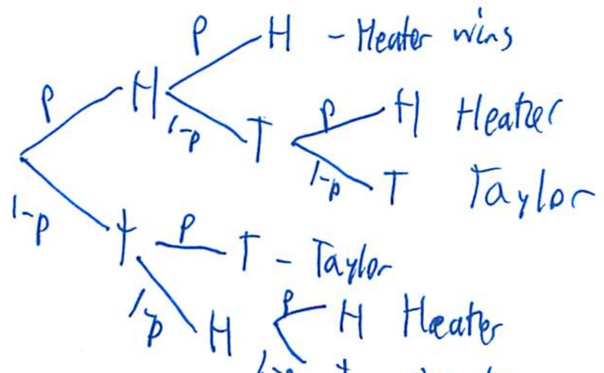


1.2 e is correct

$P(B|A) + P(B^c|A) = 1$

- since A is our universe

2.1



Need to rewrite			and the prob.
HH	=	$p \cdot p$	p^2
HTH		$p(1-p)p$	$p^2(1-p)$
HTT		$p(1-p)(1-p)$	$p(1-p)^2$
TT		$(1-p)(1-p)$	$(1-p)^2$
THH		$(1-p)pp$	$p^2(1-p)$
THT		$(1-p)p(1-p)$	$p(1-p)^2$

(2)

2.2. $P(\text{Heather wins}) = p^2 + p^2(1-p) + p^2(1-p)$

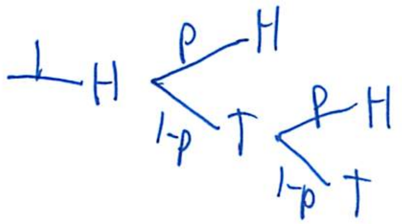
$= p^2 + 2p^2(1-p)$

Hopefully the quiz will be this simple!

2.3 $P(\text{Heather wins} \mid \text{1st toss is heads})$

now only look at top of tree

↑ "Shortcut method"



$= 1 \cdot p + 1 \cdot (1-p) \cdot p$

$= p + p(1-p)$

$= p(1+1-p) = p(2-p)$

<u>or</u>

$$\frac{P(\{\text{Heather wins}\} \cap \{\text{1st toss is heads}\})}{P(\{\text{1st toss is heads}\})}$$

top → $\{HH\} \cup \{HTH\}$

$$= \frac{p^2 + p^2(1-p)}{p}$$

$= p(2-p)$

Same
- but my "shortcut" method screws me up often

3

2.4 $P(\text{Heads 1st} \mid \text{Heater wins})$

- Could use Bayes's rule
- or formal way

$$= \frac{p^2 + p^2(1-p)}{P(\text{Heater wins})}$$ ← same as before

"short cut" would not work here

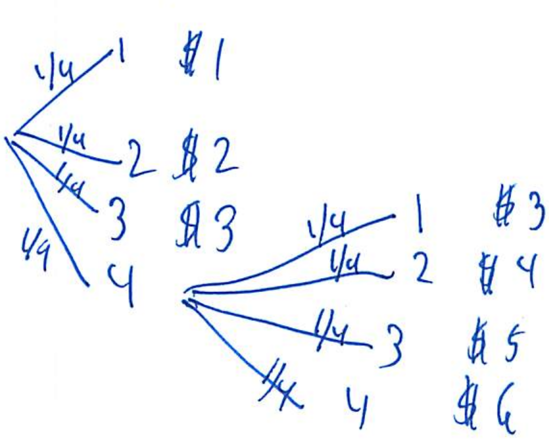
$$= \frac{p^2 + p^2(1-p)}{p^2 + 2p^2(1-p)}$$

$$= \frac{2-p}{3-2p}$$

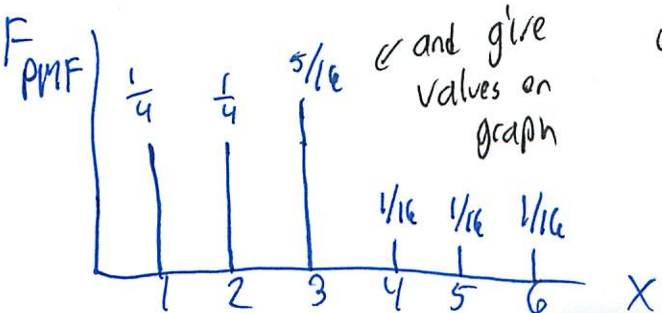
---

3. Casino game

X = amt we get



3.1 Find PMF



← and give values on graph

← Make sure know about this!

$$E[X] = \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{5}{16}\cdot 3 + \frac{1}{16}\cdot 4 \circ \frac{1}{16}\cdot 5 + \frac{1}{16}\cdot 6$$

↑ Make sure know definition

leave here

If had to calculate → could do total expectation theorm

$A = \{ \text{1st roll is } 4 \}$

$$E[X] = P(A)\,E[X|A] + P(A^c)\,E[X|A^c]$$

$$\frac{1}{4}\cdot\frac{2*(1+2+3+4)}{2} + \frac{3}{4}\cdot\left(\frac{1+2+3}{2}\right)$$

$$= \frac{21}{8}$$

3.3. Now find based on condition $X = 3$

$B = \{X = 3\}$

formal definition

$$P_{X|B}(z) = \frac{P(\{Z = z \cap B\})}{P(B)}$$

Z = Result of 1st roll

note 2 ways to get \$3

- do not try shortcut

$\frac{5}{16} - \frac{1}{4} = \frac{1}{16}$

- since must rescale/normalize
- but just do formal way

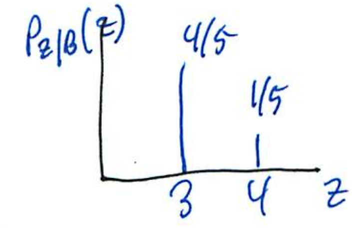
$P(B) = \frac{5}{16}$

For $z \neq 3, 4$ $\quad P_{Z|B}(z) = 0$

For $z = 3$ $\quad P_{Z|B}(z) = \frac{1/4}{5/16} = \frac{4}{5}$

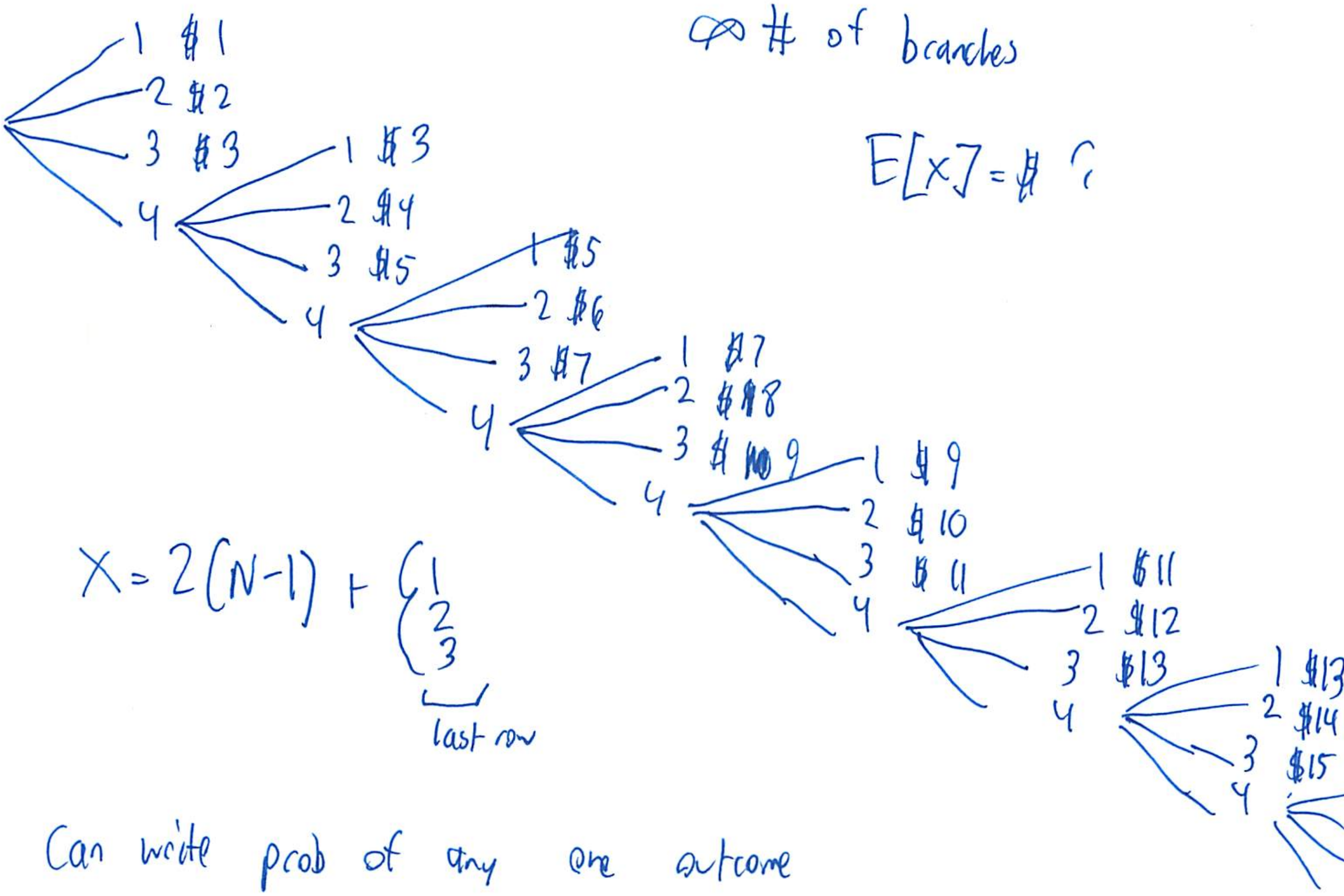
For $z = 4$ $\quad P_{Z|B}(z) = \frac{1/16}{5/16} = \frac{1}{5}$

We treated Z like a random variable



5

3.4. Always a single trick question!



∞ # of branches

$E[X] = \$\ ?$

$$X = 2(N-1) + \underbrace{\begin{cases}1\\2\\3\end{cases}}_{\text{last row}}$$

Can write prob of any one outcome

Adding series is <u>not</u> the best way

Write $X = \underset{\substack{\uparrow\\ \text{amt up to one pt}}}{W} + \underset{\substack{\uparrow\\ \text{last roll}}}{L}$

$$X = 2(N-1) + \begin{cases}1\\2\\3\end{cases}$$

$E[X] = E[W] + E[L]$ ← also distributes over summation

$= 2E[N] - 2 + \left(\frac{1+2+3}{2}\right)$

(6)

But now what is $E[N] = ?$

- geometric random variable
- $p = P(\text{Head}) = \frac{3}{4}$

$$E[N] = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

So together

$$E[X] = 2 \cdot \frac{4}{3} - 2 + 2$$

$$= \frac{8}{3}$$

10/13

LECTURE 10

2 big topics

Continuous Bayes rule; Inference
Derived distributions

- **Readings:**
  Section 3.6; start Section 4.1

Review

Chap 2 | Chap 3

$$p_X(x) \qquad f_X(x)$$

$$p_{X,Y}(x,y) \qquad f_{X,Y}(x,y)$$

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \qquad \boxed{f_{X|Y}(x \mid y)} = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Conditional PDF – ordinary PDF

denominator normalizes
↳ but picture has same shape

$$p_X(x) = \sum_y p_{X,Y}(x,y) \qquad f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$F_X(x) = \mathbf{P}(X \le x)$$

$$\mathbf{E}[X], \quad \text{var}(X)$$

The Bayes variations

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} = \frac{p_X(x)p_{Y|X}(y \mid x)}{p_Y(y)}$$

$$p_Y(y) = \sum_x p_X(x)p_{Y|X}(y \mid x)$$

Example:

- $X = 1, 0$: airplane present/not present
- $Y = 1, 0$: something did/did not register on radar

Inference

X → Noise → Y   Y related somehow to X

X: unknown, want to know

discrete $P_X(x)$ | $P_{Y|X}(y|x)$ | Want $P_{X|Y}(x|y)$
continuous $f_X(x)$ | $f_{Y|X}(y|x)$ | $f_{X|Y}(x|y)$

? tell me prob of diff values of X based on each value of Y

convert to continuous

Continuous counterpart

$$\underline{f_{X|Y}(x \mid y)} = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \underline{\frac{f_X(x)f_{Y|X}(y \mid x)}{f_Y(y)}}$$

$$f_Y(y) = \int_x f_X(x)f_{Y|X}(y \mid x)\,dx$$

need denominator so $\Omega = 1 = \int_{-\infty}^{\infty}$

Example: X: some signal; "prior" $f_X(x)$
Y: noisy version of X
$f_{Y|X}(y \mid x)$: model of the noise

Analog to Baye's rule

Discrete X, Continuous Y ← more confusing

$$p_{X|Y}(x \mid y) = \boxed{\frac{p_X(x) f_{Y|X}(y \mid x)}{f_Y(y)}}$$

$$f_Y(y) = \sum_x p_X(x) f_{Y|X}(y \mid x)$$

$P(X=x, Y=y) = P(X=x, y \le Y \le y+\delta)$

but y continuous so would be 0 not interesting

$= P(y \le Y \le y+\delta) P(X=x \mid y \le Y \le y+\delta)$

$\cong \delta f_Y(y) P(X=x \mid y \le Y \le y+\delta)$

$= \delta f_Y(y) p_{X|Y}(x|y)$

set = to cancle δ

$= P(X=x) P(y \le Y \le y+\delta \mid X=x)$ approx, see above, new universe

$= p_X(x) f_{Y|X}(y|x) \delta$

reverse things

Example:

- X: a discrete signal; "prior" $p_X(x)$
- Y: noisy version of X
- $f_{Y|X}(y \mid x)$: continuous noise model

Continuous X, Discrete Y

$$f_{X|Y}(x \mid y) = \frac{f_X(x) p_{Y|X}(y \mid x)}{p_Y(y)}$$

$$p_Y(y) = \int_x f_X(x) p_{Y|X}(y \mid x)\, dx$$

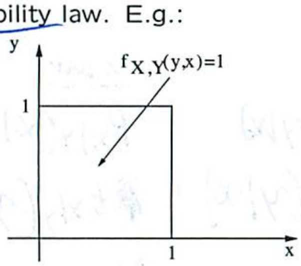
Example:

- X: a continuous signal; "prior" $f_X(x)$ (e.g., intensity of light beam);
- Y: discrete r.v. affected by X (e.g., photon count)
- $p_{Y|X}(y \mid x)$: model of the discrete r.v.

Switch gears

What is a derived distribution

- It is a PMF or PDF of a function of one or more random variables with known probability law. E.g.:



  - Obtaining the PDF for
  $$g(X, Y) = Y/X$$
  involves deriving a distribution.
  Note: $g(X, Y)$ is a random variable

When not to find them

- Don't need PDF for $g(X, Y)$ if only want to compute expected value:

$$\mathrm{E}[g(X,Y)] = \int\int g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

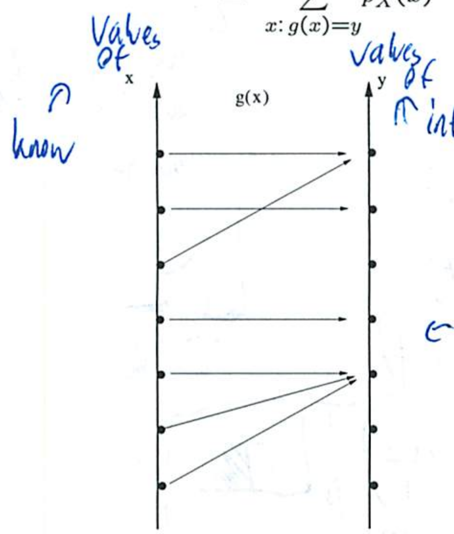
How to find them

- **Discrete case** : easy case
  - Obtain probability mass for each possible value of $Y = g(X)$

← for all

$\forall y$

$$p_Y(y) = \mathrm{P}(g(X) = y) = \sum_{x:\, g(x)=y} p_X(x)$$



$Y = 1$ if $x = 1$ or $x = 3$

$P(Y=1) = P(x=1) + P(x=3)$

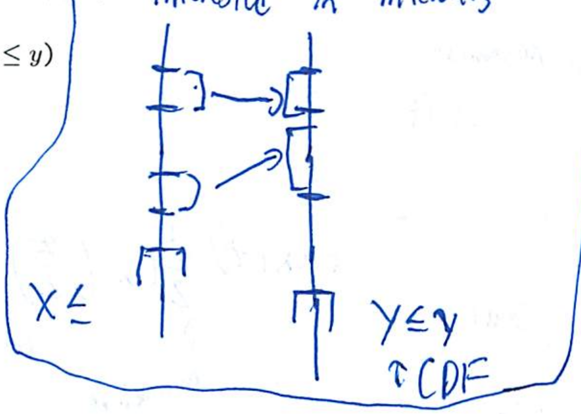
$P(y=4) = P(x=4)$

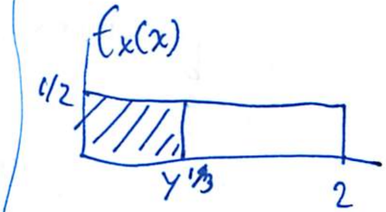
---

cookbook procedure

The continuous case

remember each point $= 0$
so interested in intervals





- **Two-step procedure:**
  - Get CDF of Y: $F_Y(y) = \mathrm{P}(Y \le y)$
  - Differentiate to get

$$f_Y(y) = \frac{dF_Y}{dy}(y)$$

to get density

Example

- X: uniform on [0,2]
- Find PDF of $Y = X^3$
- **Solution:**

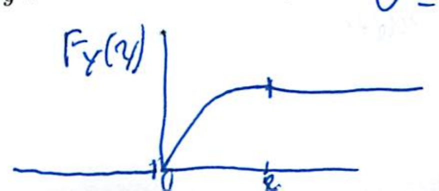
$$F_Y(y) = \mathrm{P}(Y \le y) = \mathrm{P}(X^3 \le y) = \mathrm{P}(X \le y^{1/3}) = \frac{1}{2}y^{1/3}$$

Monotonic if

$$f_Y(y) = \frac{dF_Y}{dy}(y) = \frac{1}{6y^{2/3}}$$

note picture only counts for
$0 \le y^{1/3} \le 2$
$0 \le y \le 8$

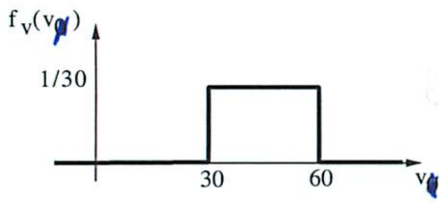
CDF
so $F_Y(y) = 0$ $y < 0$
$F_Y(y) = 1$ $y \ge 8$



$\frac{d\,F_Y(y)}{dy}$

Example

- Joan is driving from Boston to New York. Her speed is uniformly distributed between 30 and 60 mph. What is the distribution of the duration of the trip?
- Let $T(V) = \dfrac{200}{V}$.
- Find $f_T(t)$



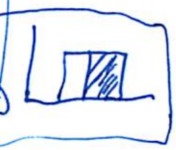
Set cruise control to random variables

T is contineas

- since V is contineas

Find CDF to get PDF

$$F_T(t) = P(T \le t)$$
$$= P\left(\frac{200}{V} \le t\right)$$
$$= P\left(V \ge \frac{200}{t}\right)$$
$$= \underbrace{\frac{1}{30}}_{\text{height}} \underbrace{\left(60 - \frac{200}{t}\right)}_{\text{base}}$$



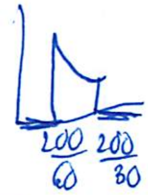
derivitive

$$f_T(t) = \frac{1}{30} \cdot \frac{200}{t^2}$$

for

$$\frac{200}{60} \le t \le \frac{200}{30}$$

0 elsewhere



Correct for

$$30 \le \frac{200}{t} \le 60$$

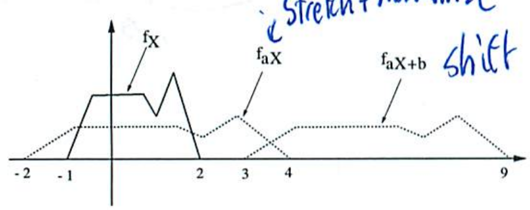
The pdf of Y=aX+b

easy functions → the linear functions

$Y = \overset{a}{2}X + \overset{b}{5}$:

stretch + normalize

shift



$$f_{aX}(z) = \underset{\text{normalize}}{\frac{1}{2}} f_X\underset{\text{stretch}}{\left(\frac{z}{2}\right)}$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y - \overset{\text{shift}}{b}}{a}\right)$$

← densities can't be negitive

- Use this to check that if X is normal, then $Y = aX + b$ is also normal.

if normal,

then aX+b also normal

leave as exercise for reader

→

Verify proof

$\underline{a > 0}$ ⊕

$$F_Y(y) = P(Y \le y) = P(aX + b \le y)$$
$$= P\left(X \le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

take derivative

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\frac{1}{a}$$

$\underline{a < 0}$ ⊖

direction of inequality changes

get complement → 1 − CDF

so have − sign

$$= -f_X\left(\frac{y-b}{a}\right)\frac{1}{a}$$

a is −

so evens out

$$= f_X\left(\frac{y-b}{a}\right)\frac{1}{a}$$

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Recitation 11
October 14, 2010

1. Let X be a discrete random variable that takes the values 1 with probability p and -1 with probability $1-p$. Let Y be a continuous random variable independent of X with the Laplacian (two-sided exponential) distribution

$$f_Y(y) = \tfrac{1}{2}\lambda e^{-\lambda|y|},$$

and let $Z = X + Y$. Find $\mathbf{P}(X = 1 \mid Z = z)$. Check that the expression obtained makes sense for $p \to 0^+$, $p \to 1^-$, $\lambda \to 0^+$, and $\lambda \to \infty$.

2. Let Q be a continuous random variable with PDF

$$f_Q(q) = \begin{cases} 6q(1-q), & \text{if } 0 \le q \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

This Q represents the probability of success of a Bernoulli random variable X, i.e.,

$$\mathbf{P}(X = 1 \mid Q = q) = q.$$

Find $f_{Q|X}(q|x)$ for $x \in \{0, 1\}$ and all q.

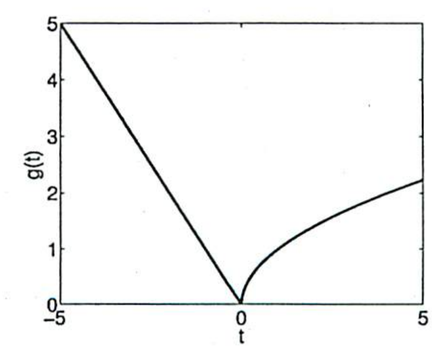
3. Let X have the normal distribution with mean 0 and variance 1, i.e.,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Also, let $Y = g(X)$ where

$$g(t) = \begin{cases} -t, & \text{for } t \le 0; \\ \sqrt{t}, & \text{for } t > 0, \end{cases}$$

as shown to the right.



Find the probability density function of Y.

Recitation 4

10/14

Bayer's Rule / Inference
Derifed Distribution

Class Avg 76 on Quiz 1

Bayer's Rule

- looks complex
- but just conditional probability → intersection of 2 shapes

$$P(A \cap B) = P(A)\, P(B|A) = P(B)\, P(A|B)$$

- and divide out
- just reversing conditioning

Continous Random Variables X + Y

$$f_{X,Y} = f_Y f_{X|Y}$$
$$= f_X f_{Y|X}$$

$$f_{X|Y}(x,y) = \frac{f_X(x)\, f_{Y|X}(y|x)}{f_Y(y)}$$

← marginal
- may know
- if just know Conditional
- express in integration

equivilently
$$\int_{-\infty}^{\infty} f_X(t)\, f_{Y|X}(y|t)\, dt$$

(2)

Discrete N + Contineous Y

$$f_Y(y) \cdot p(N=n \mid Y=y) = p_N(n) f_{Y|N}(y|n)$$

Bayey's Rule

$$P(N=n \mid Y=y) = \frac{p_N(n) f_{Y|N}(y|n)}{f_Y(y)}$$

or $\sum_i p_N(i) f_{Y|N}(y|i)$

$$f_{Y|N}(y|n) = \frac{f_Y(y) P(N=n \mid Y=y)}{\underbrace{p_N(n)}_{\text{or } \int_{-\infty}^{\infty} f_Y(t) P(N=n \mid Y=t)\,dt}}$$

1. Signal Detection

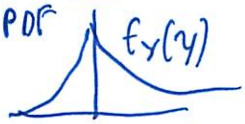
$$\frac{X}{\varepsilon\{-1, 1\}} \longrightarrow \boxed{\text{Noise Channel}} \longrightarrow Z = X + Y$$

$\downarrow Y$ (into the channel)

-1: $1-p$, 1: p

digital signal

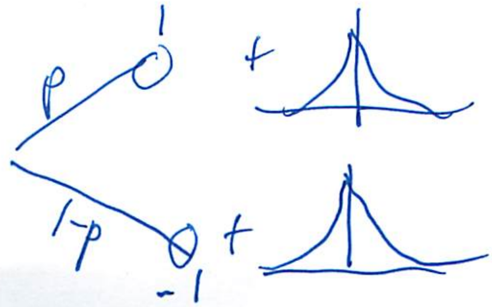
PDF $f_Y(y)$



exponential

$\frac{1}{2}\lambda e^{-\lambda|y|}$

*normalize

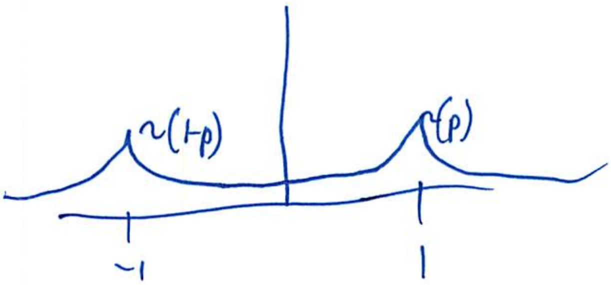


(3)

Z is mixture

– so contineous

$$P(X = 1 \mid Z = z)$$



know contineous given discrete

(what is that? p that what we got was a 1?

$$P(X=1 \mid Z=z) = \frac{P_X(1)\, f_{Z|X}(z|1)}{P_X(1)\, f_{Z|X}(z|1) + P_X(-1)\, f_{Z|X}(z|-1)}$$

$$= \frac{p \cdot \frac{1}{2}\lambda\, e^{-\lambda|z-1|}}{p \cdot \frac{1}{2}\lambda\, e^{-\lambda|z-1|} + (1-p)\frac{1}{2}\lambda\, e^{-\lambda|z+1|}}$$

$$= \frac{p\, e^{-\lambda|z-1|}}{p\, e^{-\lambda|z-1|} + (1-p)\, e^{-\lambda|z+1|}}$$

Use for decoding

get smudged version of digital signal

what did you send me?

infrence

– no 1 way to do it

(4)

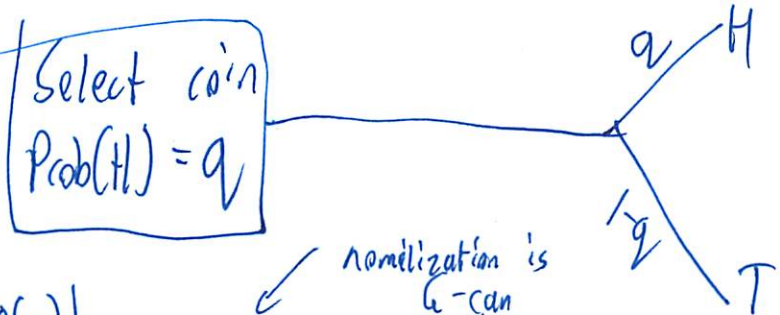
Many use MAP rule

- Choose $x = 1$ if $P(X=1 | z) > P(X=-1 | z)$
  - So in our situation if z is $+ \to 1$, $- \to -1$

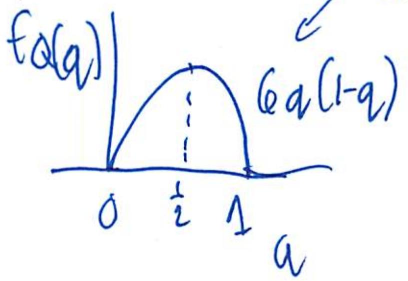
But depends on how likely each signal is

- a priori
- in this case does not make a difference
- noisy channel effects both values the same

2.



$X = \begin{cases} 1 & \text{w/ prob } q \\ 0 & \text{w/ prob } 1-q \end{cases}$



normalization is 6 - can verify got from $\int_0^1$

$f_{Q|X}(q|x)$ for $x = 0, 1$ — P of conditional given discrete

$$= \frac{f_Q(q) P(X=x | Q=q)}{p_X(x)}$$

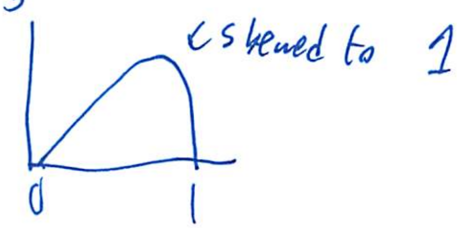
$\leftarrow$ $p_X(1) = 1$, $p_X(0) = 0$ since symetric

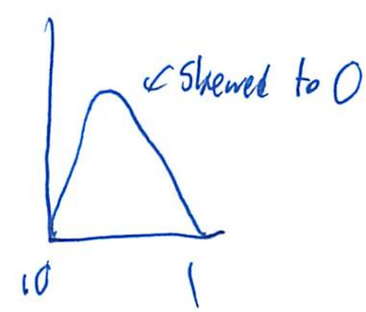
5

$$= \frac{6q(1-q)\cdot \begin{cases} q & \text{if } x=1 \\ 1-q & \text{if } x=0 \end{cases}}{1/2}$$

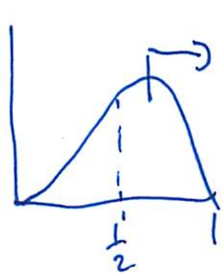
$$\left.\begin{aligned} f_{Q|X}(q|1) &= 12q^2(1-q) \\ f_{Q|X}(q|0) &= 12q(1-q)^2 \end{aligned}\right\} \text{ for } q \in [0,1]$$

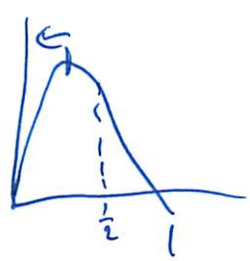
0 otherwise





Map rule. result?





(6) 2 tosses

q^2 HH

$2q(1-q)$ HT + TH

$(1-q)^2$ TT

$f_{Q|X}(q \mid HH)$

$\mid TT)$

$\mid HT + TH)$

1 million outcomes

- same method
- but 1,000,001 outcomes

posteriors have very specific form

$$\boxed{f_{Q|X}(q \mid x) = q^{a(x)}(1-q)^{b(x)}}$$

← is conditional PDF
-beta distribution

Why care about estimating unfair coins?

- are their any biased coin?
- is it just to torture students?
- binary data
- testing products
  - % of defective
    - you test if defective or not
    - estimate % of entire box

7 Derived Distribution

X = Random Variable ← PMF or PDF

$Y = g(X)$

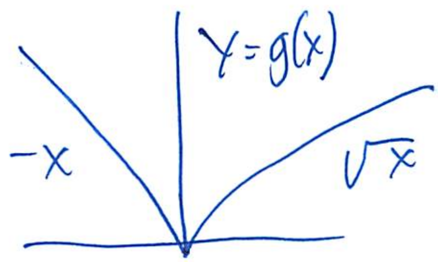
← PMF?
PDF?
neither?
} so work w/ CDF

CDF of X → CDF of Y

difference for PMF ↓ differentiate for CDF

example

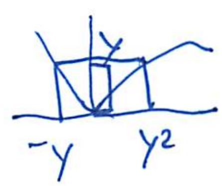
X = continuous



Find CDF of Y

$P(Y \leq y)$ = ?

↑ can only take positive values

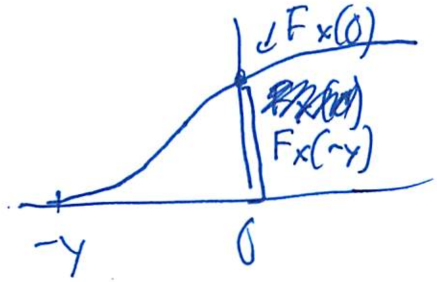


$= P(g(X) \leq y)$

$= P(X \in [-y, 0]) + P(X \in [\cancel{y} 0, y^2])$

broke down into 2 disjoint pieces

=



$$= \cancel{F_X(0)} - F_X(-y) + F_X(y^2) - \cancel{F_X(0)}$$

$$= + F_X(y^2) - F_X(-y)$$

will ending up wanting to differeniate

chain rule PDF(x)   chain rule

$$\underbrace{\frac{d}{dy} P(Y \le y)}_{f_Y(y)} = +f_X(-y) + 2y \cdot f_X(y^2)$$

- specific g not important
- important that we broke down calculations

Standard Normal

- Suppose X = standard normal

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

(9)

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}}\left(e^{-y^2/2} + 2y e^{-y^4/2}\right) & \text{if } y \geq 0 \\ 0 & \text{if } y \leq 0 \end{cases}$$

a lot of the process applies to more general situations

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Tutorial 5
October 14/15, 2010

1. Let Q be a random variable which is uniformly distributed between 0 and 1. On any given day, a particular machine is functional with probability Q. Furthermore, given the value of Q, the status of the machine on different days is independent.

   (a) Find the probability that the machine is functional on a particular day.

   (b) We are told that the machine was functional on m out of the last n days. Find the conditional PDF of Q. You may use the identity

$$\int_0^1 p^k(1-p)^{n-k}dp = \frac{k!(n-k)!}{(n+1)!}$$

2. Let X be a random variable with PDF f_X. Find the PDF of the random variable $Y = |X|$

   (a) when $f_X(x) = \begin{cases} 1/3, & \text{if } -2 < x \leq 1, \\ 0, & \text{otherwise;} \end{cases}$

   (b) when $f_X(x) = \begin{cases} 2e^{-2x}, \text{if } x > 0, \\ 0, \text{otherwise;} \end{cases}$

   (c) for general $f_X(x)$.

3. An ambulance travels back and forth, at a constant specific speed v, along a road of length ℓ. We may model the location of the ambulance at any moment in time to be uniformly distributed over the interval $(0, \ell)$. Also at any moment in time, an accident (not involving the ambulance itself) occurs at a point uniformly distributed on the road; that is, the accident's distance from one of the fixed ends of the road is also uniformly distributed over the interval $(0, \ell)$. Assume the location of the accident and the location of the ambulance are independent.

   Supposing the ambulance is capable of *immediate* U-turns, compute the CDF and PDF of the ambulance's travel time T to the location of the accident.

Tutorial 5

10/15

Contineous Bayer's Rule 3.7

$X \xrightarrow{\quad} Y$

hidden $P_{Y|X}$ observed

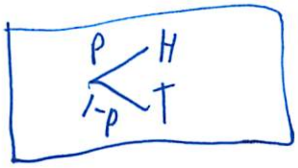
$f_{Y|X}$

know $P_X(x)$ but not values

Find $P_{X|Y}$ such that $f_{X|Y}$

Case 1 if x and y are both discrete

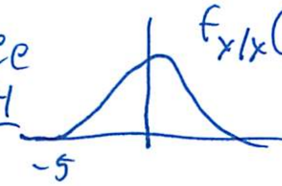
$P_{X|Y}(x|y) = ?$



← black box

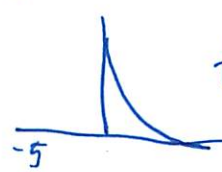
see

H



$f_{Y|X}(y|H)$

T



$f_{Y|X}(y|T)$

Now that seen Y - what is P of head

See $y = -5$ what is x

$P_{X|Y}(x|-5) = \begin{cases} 1 & x = 1 \\ 0 & x = 0 \end{cases}$ ← since if you have a -5 it can only be obtained from H

What if see $y = 1$

$P_{X|Y}(x|1) =$ not so easy
need Bayer's rule

(2)

Remember

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

Case 1 X + Y both continuous

$$P_{X|Y}(x|y) = \frac{P_{Y|X}(y|x) \cdot P_X(x)}{\sum P_{Y|X}(y|x) \cdot P_X(x)}$$

Case 2 if both were contineous

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) \cdot f_X(x)}{\int f_{Y|X}(y|x) \cdot f_X(x)\, dx}$$

many cases
So need to integrate

Case 3 Discrete X and contineous Y

$$P_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) \cdot P_X(x)}{\sum_x f_{Y|X}(y|x) \cdot P_X(x)}$$

conditioning is just new universe R.V. still discrete

Case 4 contineous X + discrete Y

$$f_{X|Y}(x|y) = \frac{P_{Y|X}(y|x) \cdot f_X(x)}{\int P_{Y|X}(y|x) \cdot f_X(x)\, dy}$$

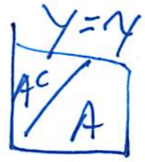
integral since X contineous

(3)

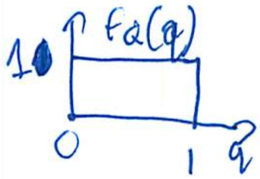
Case 5

Event A discrete Y

$$P[A \mid Y=y] = \frac{p_{Y|A}(y) \cdot P[A]}{p_{Y|A}(y) \cdot P[A] + p_{Y|A}(y) \cdot P[A^c]}$$



#1. Prob. Machine Functioning



$- - - - - - - - - -$) over days

If see machine not working for 10 days
So on 11th day likely to be not working

Observing days gives you an idea of q
 - which gives you idea of other days

* Each day is still independent and q does not change day by day *

Once you know q - looking at past days does not matter

(4)

a) A = machine working

$P(A) = ?$

$P(A \mid Q = q) = q$ ← define ??

Total Probability Theorem

$$P(A) = \int_0^1 P(A \mid Q = q) \cdot f_Q(q)\,dq$$

$$= \int_0^1 q \cdot \underbrace{1}_{\frac{1}{b-a} = \frac{1}{1-0} = 1} \,dq = \left[\frac{q^2}{2}\right]_0^1 = \frac{1}{2}$$

b)

B = machine is working m out of n days

What does tell us about PDF

$P_{Q|B}(q) = ?$

- have B
- want to find Q
- Bayes's Rule

$$P[B \mid Q = q] = \binom{n}{m} \cdot q^m (1-q)^{n-m} \qquad m \leq n$$

like case E → X cont. have event B

$$f_{Q|B}(q) = \frac{P[B \mid Q = q]\, f_Q(q)}{P[B \mid Q = q]\, f_Q(q)\, dq}$$

⑤

$$= \frac{\binom{n}{m} q^m (1-q)^{n-m} \cdot 1}{\int \binom{n}{m} q^m (1-q)^{n-m} \cdot 1 \, dq}$$

$$= \frac{q^m (1-q)^{n-m}}{\frac{m!(n-m)!}{(n+1)!}}$$

$\binom{n}{m}$ is constant

integral is given

Derived Distributions (4.1)

- read book
- lots of examples

- Given continuous RV X and the PDF(x) $\rightarrow f_X(x)$
- Find PDF of Y

Step 1

Calculate the CDF of Y.

$F_Y(y) = P[Y \le y] = P[g(x) \le y] = \ldots$

Step 2

Differentiate CDF to get PDF

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

6

Special Case

$$Y = aX + b$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$Y = aX + b$$

Discrete

$$p_Y(y) = P[Y = y] = P[aX + b = y] = P\left[X = \frac{y-b}{a}\right]$$
$$= p_X\left(\frac{y-b}{a}\right)$$

Contineous

$$f_Y(y) = \cancel{P[Y = y]}$$

Can't say, can't reason that way

$$F_Y(y) = P[Y \le y] \ldots$$

start w/ CDF

#2

$$X = f_X(x)$$

backwards

$$Y \doteq |X|$$

c) find $f_Y(y)$

$$F_Y(y) = P[Y \le y] = P[|X| \le y] = P[-y \le X \le y]$$

⑦

$= P[X \leq y] - P[X \leq -y]$ ← CDF

$= F_X(y) - F_X(-y)$

want PDF

$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[F_X(y) - F_X(-y)\right]$

$= f_X(y) - \left[f_X(-y)\right]$

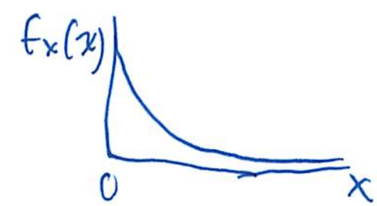
note

$\boxed{\frac{d}{dx} f(g(x)) = f(g(x)) \cdot g'(x) \quad = f'(-x) \cdot -1}$

$= f_X(y) + f_X(-y)$ ← answer to c
can't substitute
any further

---

b) $f_X(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{else} \end{cases}$

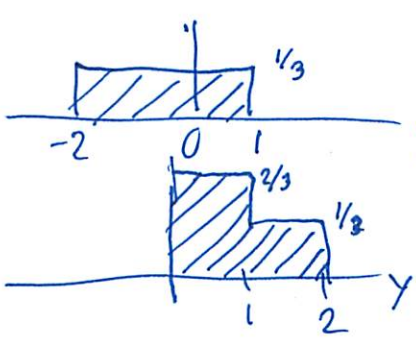


$Y = |X| = X$

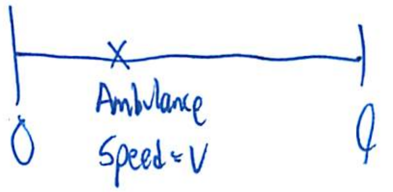
$f_Y(y) = \begin{cases} 2e^{-2y} & y > 0 \\ 0 & \text{else} \end{cases}$

---

a) $f_X(x) = \begin{cases} 1/3 & -2 \leq x \leq 1 \\ 0 & 0 \end{cases}$



3. Word Problem Derived Distributions



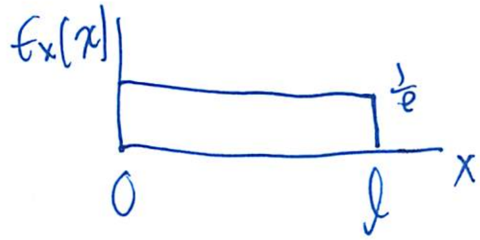
ambulance can turn around instantly

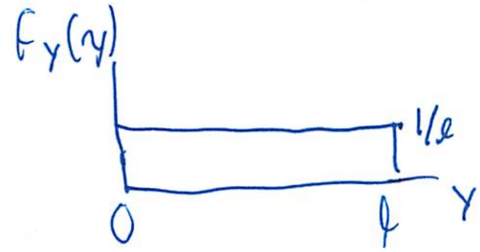
Some where an accident happens
- random

PDF and CDF of t = time for ambulance to discover accident

<u>assign symbols</u>

X = location of ambulance





Y = location of accident

$$T = \frac{|X - Y|}{V} = g(X, Y)$$

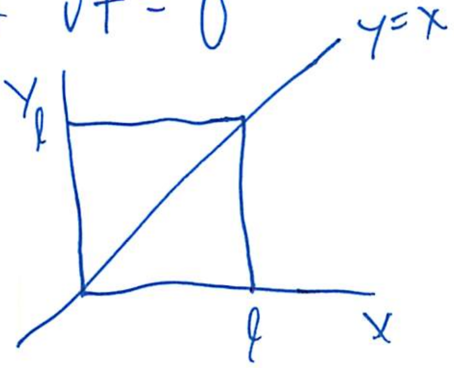
↑function 2 variables

$$F_T(t) = P[T \le t] = P\left[\frac{|X-Y|}{V} \le t\right] = P[-vt \le X - Y \le vt]$$

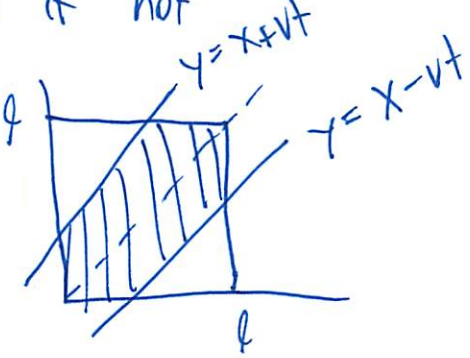
$$= P[Y - vt \le X \le vt + Y]$$

(9)

If $vt = 0$



But if not



? calculate area of shaded square

$$F_T(t) = \begin{cases} 0 & t=0 \\ \frac{2}{l^2}\cdot\text{area} & 0 \le t \le \frac{l}{v} \\ 1 & t \ge \frac{l}{v} \quad vt \ge l \end{cases}$$

to find area of shaded region

$= 2 \cdot$  $-$ 

$$= \frac{l^2}{2} - \frac{(l-vt)^2}{2}$$

$\longrightarrow$ So final CDF

$$\frac{2vt}{l} - \frac{(vt)^2}{l^2}$$

(10)

Now need PDF so take derivative

$\cancel{f_Y(y) = \frac{d}{dy} F_Y(y) =}$

$$f_T(t) = \frac{d}{dt} F_T(t) = \frac{d}{dt}\left[\frac{2vt}{l} - \frac{(vt)^2}{l^2}\right]$$

$$= \frac{2v}{\cancel{t}\,l} - \frac{2(vt)\cdot v}{l^2}$$

$$f_T(t) = \begin{cases} 0 & \text{else} \\ \frac{2v}{l} - \frac{2(vt)\cdot v}{l^2} & \text{if } 0 \le t \le \frac{l}{v} \end{cases}$$

Chap 3 General Random Variables

10/16

Continuous

- quite common
- velocity of vehicle
- finer grained
- allow use of calculus

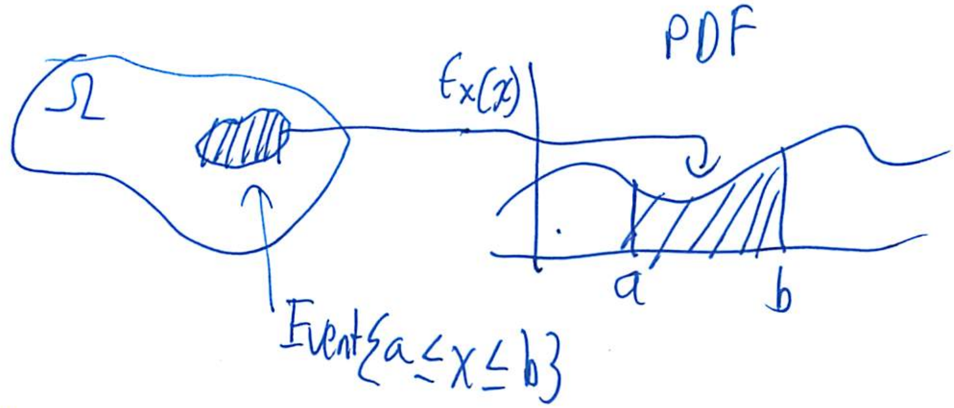
(doing after lectures)

$$P(X \in B) = \int_B f_X(x)\,dx$$

PDF

↳ for every subset B of the real line

$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$



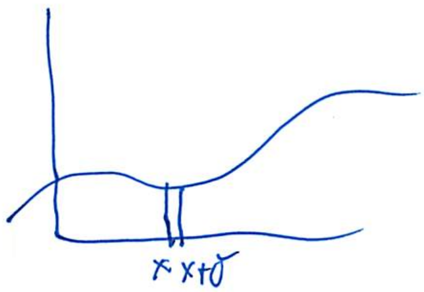
again
- non negitive

$$\int_{-\infty}^{\infty} f_X(x)\,dx = P(-\infty < X < \infty) = 1$$

(2) PDF at any 1 point

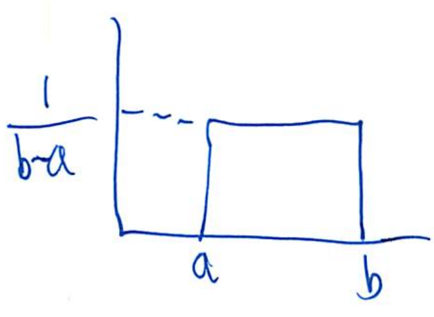
$$PDF(x, x+\delta) = \int_x^{x+\delta} f_X(t)\,dt \approx f_X(x) \cdot \delta$$

- like the normal integral defenition w/ rectangles
- since w/ contineous the $PDF(x) = 0$
- so must ~~...~~ add over, at least, a very small interval



If probablity evenly distributed b/w $[a, b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



~~can~~ note doing normalization property

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_a^b \frac{1}{b-a}\,dx$$

Can add up piece wise PDFs

$$1 = \int_{a_1}^{a_n} f_X(x)\,dx = \sum_{i=1}^{n-1} \int_{a_i}^{a_{i+1}} c_i\,dx$$

$$= \sum_{i=1}^{n-1} c_i(a_{i+1} - a_i)$$

and note it must be normalized

③
PDF can also take arbitrarily large #

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

becomes ∞ large as approaching 0
- but still valid PDF

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 \frac{1}{2\sqrt{x}}\,dx = \sqrt{x}\Big|_0^1 = 1$$

Expectation/mean

- Similar except integrating PDF <u>not</u> summing PMF
- still "center of gravity"

"integral is a limiting form of sum"

- also $Y = g(X)$ is still random variable
  - ↳ can turn out/result can be discrete
- either way mean satisfies expected value rule

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

  - analogous to discrete
- nth moment still $E[X^n]$
- Var is $(X - E[X])^2$

$$= E[X^2] - (E[X])^2$$

<u>still</u>
$E[Y] = aE[X] + b$
$var(Y) = a^2 var(X)$

(9)

Exponential Random Variable

defined as $f_X(x) = \begin{cases} \lambda e^{-\lambda x} dx & \text{if } x \geq 0 \\ \text{other} \end{cases}$

λ = positive parameter characterizing PDF

proof $\int_{-\infty}^{\infty} f_X(x) = \int_0^{\infty} \lambda e^{-\lambda x} dx = -e^{-\lambda x} \Big|_0^{\infty} = 1$

The probability that X exceeds a certain value decreases exponentially
(i like the light bulb qui)

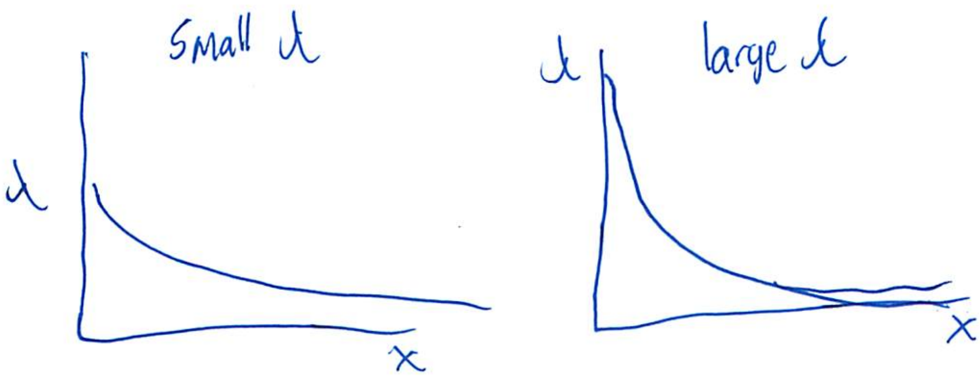
For any $a \geq 0$

$$P(X \geq a) = \int_a^{\infty} \lambda e^{-\lambda x} dx$$

$$= -e^{-\lambda x} \Big|_a^{\infty}$$

$$= e^{-\lambda a}$$

~~Thm~~ good model of how much time until an incident takes place



(3) Its closely related to <u>geometric RV</u> which is discrete time

$$E[X] = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2}$$

↳ verify $E[X] = \int_0^\infty x\lambda e^{-\lambda x}\,dx$

$$= \left(-x e^{-\lambda x}\right)\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\,dx$$

$$= 0 - \frac{e^{-\lambda x}}{\lambda}\Big|_0^\infty$$

$$= \frac{1}{\lambda}$$

verify $E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx$

$$= \left(-x^2 e^{-\lambda x}\right)\Big|_0^\infty + \int_0^\infty 2x e^{-\lambda x}\,dx$$

$$= 0 + \frac{2}{\lambda} E[X]$$

$$= \frac{2}{\lambda^2}$$

verify $var(X)$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

<u>example</u>

note $P(X \geq a) = P(X > a) = e^{-\lambda a}$

time until metior will land

Mean = 10 days ← so if this is given → $\lambda = \frac{1}{10}$

P(will land b/w 6AM + 6PM 1st day)

$$P\left(\tfrac{1}{4} \leq X \leq \tfrac{3}{4}\right) = P\left(X \geq \tfrac{1}{4}\right) - P\left(X > \tfrac{3}{4}\right) = e^{-1/40} - e^{-3/40}$$

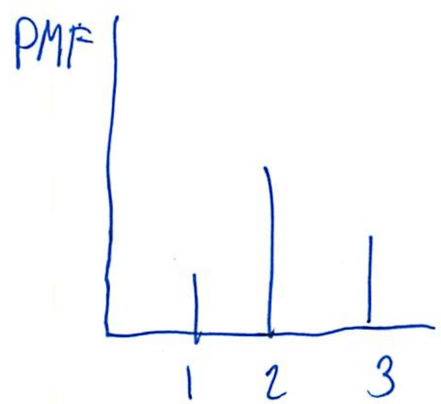
⑥

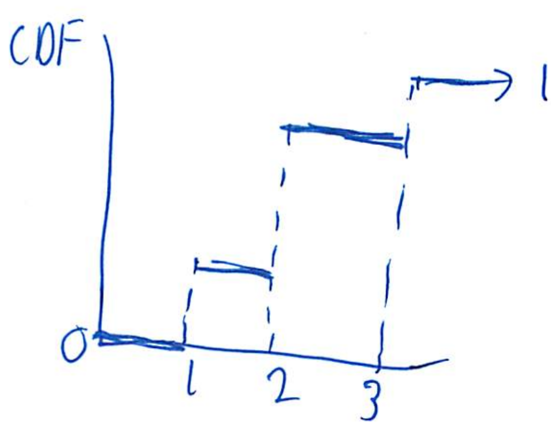
3.2 Cumulative Distribution Functions (CDF)

- single math concept to describe both discrete + random variables
- F_X
- Provides $P(X \leq x)$
- For every x

$$F_X(x) = P(X \leq x) = \begin{cases} \sum_{k \leq x} p_X(k) & \text{if } X = \text{discrete} \\ \int_{-\infty}^{x} f_X(t)\,dt & \text{if } X \Rightarrow \text{contineous} \end{cases}$$

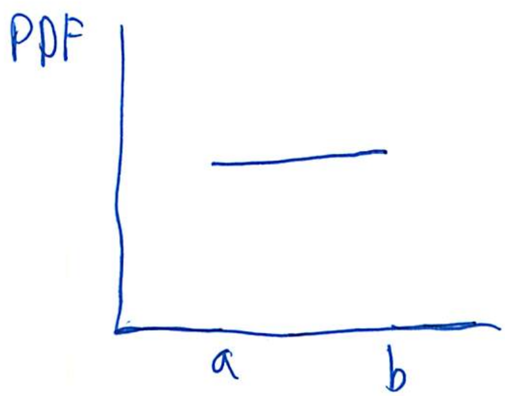
- it accumulates all the p to the left of desired x
- PDF, PMF, CMF } Probability law

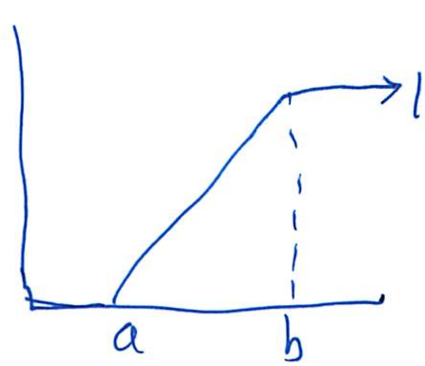




piecewise constant function

could sum $F_X(k) = \sum_{i=-\infty}^{k} p_X(i)$





Contineous function

(10) Max of Random Variables

$X = \max\{X_1; X_2, X_3\}$

$X_{1,2,3}$ = test scores $1 \to 10$ independent

PMF of score

$$p_X(k) = F_X(k) - F_X(k-1) \quad k = 1, 2, \dots 10$$

$$F_X(k) = P(X \le k)$$

$$= P(X_1 \le k, X_2 \le k, X_3 \le k)$$

$$= P(X_1 \le k)\, P(X_2 \le k)\, P(X_3 \le k)$$

$$= \left(\frac{k}{10}\right)^3$$

$$p_X(k) = \left(\frac{k}{10}\right)^3 - \left(\frac{k-1}{10}\right)^3$$

$$F_X(x) = F_{X_1}(x) \dots F_{X_n}(x)$$

Geometric + Exponential CDFs

- Since CDF for discrete + continuous + mixed used it to explore those

X = # of trials until 1st success

$$P(X = k) = p(1-p)^{k-1}$$

$$F_{geo}(n) = \sum_{k=1}^{n} p(1-p)^{k-1} = p\,\frac{1-(1-p)^n}{1-(1-p)} = 1-(1-p)^n$$

8

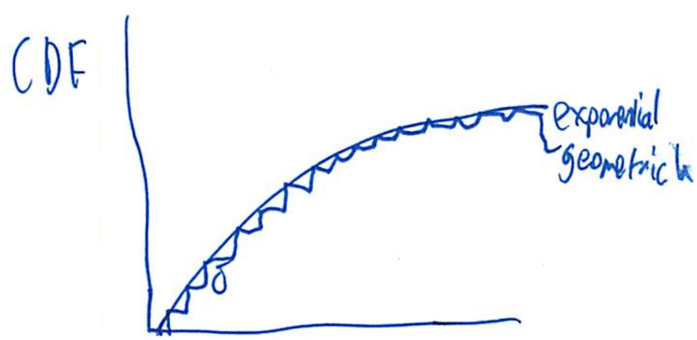
Now Exponential

$$F_{exp}(x) = P(X \leq x) = 0 \quad \text{for } x \leq 0$$

$$F_{exp}(x) = \int_0^x \lambda e^{-\lambda t}\, dt$$

$$= -e^{-\lambda t} \Big|_0^x$$

$$= 1 - e^{-\lambda x} \quad \text{for } x > 0$$



- So can see it matches as $\lim \delta \to 0$

- so it matches

$$\delta = -\frac{\ln(1-p)}{\lambda}$$

$$e^{-\lambda\delta} = 1-p$$

then $F_{exp}(n\delta) = F_{geo}(n)$ exponential + geometric CDFs are = when $x = n\delta$, $n = 1, 2, 3, \ldots$

Will see more in chap 6

(9)

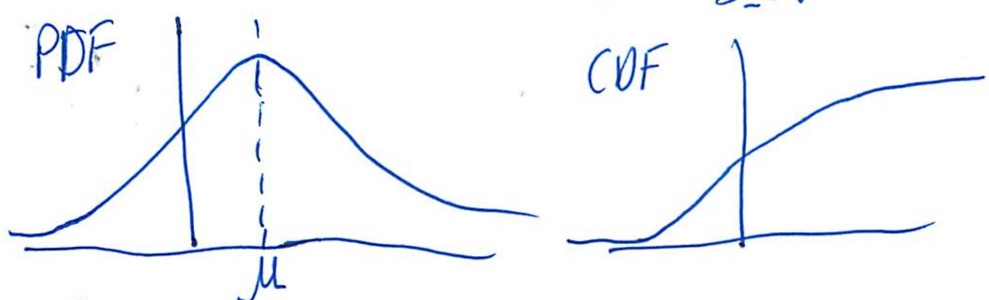
3.3 Normal Random Variables

a contineous RV X is said to be normal / Guassian if PDF has form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

μ, σ are 2 scalar parameters characterizing PDF
w/ σ assumed positive

Verify normalization property $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx = 1$



$E[X] = \mu$ $\quad$ $\mathrm{var}(X) = \sigma^2$

$Y = aX + b$ $\quad$ $E[Y] = a\mu + b$ $\quad$ $\mathrm{var}(Y) = a^2\sigma^2$

Standard Normal

$\mu = 0 \quad \sigma = 1$

CDF called Φ

$$\Phi(y) = P(Y \le y) = P(Y < y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-t^2/2}\,dt$$

See table in book

Note $\Phi(-y) = 1 - \Phi(y)$

(16)

Note on the table

ie $\Phi(1.77)$

$\downarrow$

1.7 $\longrightarrow$ .7

read ~~there~~ there

X = normal RV mean μ var σ^2

Can standardize X

$Y = \dfrac{X-\mu}{\sigma}$

Y is a linear function of X

$E[Y] = \dfrac{E[X]-\mu}{\sigma} = 0$ $Var(Y) = \dfrac{Var(X)}{\sigma^2} = 1$

So now Y is a standard normal random variable, use table

So is it?

$1 - \phi\left(\dfrac{X-\mu}{\sigma}\right) =$ for $X > .5$

from table

$\phi\left(\dfrac{X-\mu}{\sigma}\right) =$ for $X < .5$

behind the scenes $P(X \leq x) = P\left(\dfrac{X-\mu}{\sigma} \leq \dfrac{x-\mu}{\sigma}\right) = P\left(Y \leq \dfrac{X-\mu}{\sigma}\right) = \phi\left(\dfrac{x-\mu}{\sigma}\right)$

(11)

normal variables model well the additive effects of many independent factors

- a sum of lots of independently distributed RV have an approx of normal CDF irregardless of the CDF of actual variables

---

3.4 Joint PDFs of Multiple Random Variables

2 continuous RV w/ same experiments are <u>jointly continuous</u> and can be described by joint PDF $f_{X,Y}$

If $f_{X,Y}$ is nonnegative and

$$P((X,Y)\in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$

for every subset of B on 2D plane

Integration carried over $B = \{(x,y) \mid a \le x \le b, c \le y \le d\}$

$$P(a \le X \le b, c \le Y \le d) = \int_c^d \int_a^b f_{X,Y}(x,y)\,dx\,dy$$

B is the entire 2D plane $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

So sum over the small rectangle

$$P(a \le X \le a+\delta, c \le Y \le c+\delta) = \int_c^{c+\delta}\int_a^{a+\delta} f_{X,Y}(x,y)\,dx\,dy$$

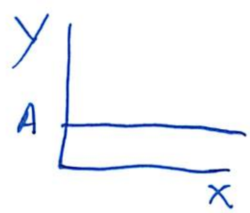
$$\approx f_{X,Y}(a,c)\,\delta^2$$

probability per unit area in vicinity of (a,c)

<u>Joint PDF</u> contains all of the relevant probablistic information on RV X, Y and their dependencies

~~[illegible]~~

So if wanted $\{X \in A\}$



$$P(X \in A) = P(X \in A \text{ and } Y \in (-\infty, \infty)) = \int_A \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx$$

X Y

$$= \int_A f_X(x)\,dx$$

So <u>Marginal PDFs</u>

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

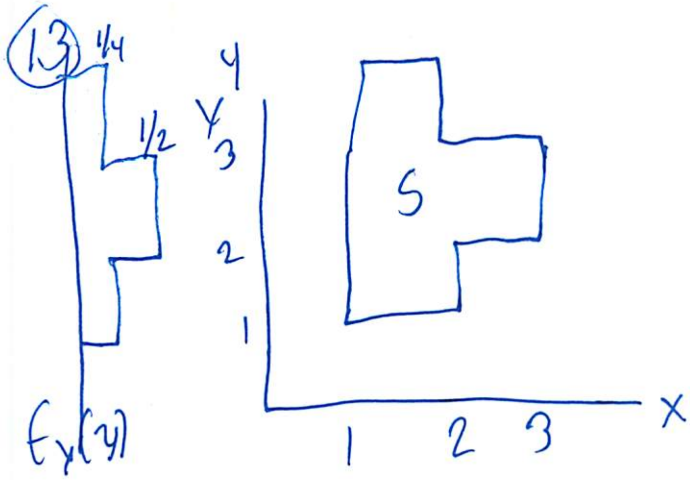
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

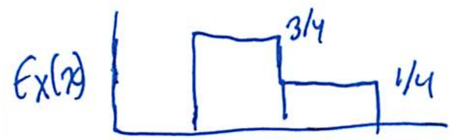
And you have a normizilation property so $c = 1$

<u>Uniform Joint PDF</u>

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\text{area } S} & \text{if } (x,y) \in S \\ 0 & \text{otherwise} \end{cases}$$

(13)





← integrate (w/ respect to y) over vert line corresponding to that x

So for any set $A \subset S$ prob (X, Y) lies in A is

$$P((X,Y) \in A) = \iint_{(x,y)\in A} f_{X,Y}(x,y)\,dx\,dy = \frac{1}{\text{area } S} \iint_{(x,y)\in A} dx\,dy = \frac{\text{area } A}{\text{area } S}$$

Joint CDF

- define joint CDF

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s,t)\,dt\,ds$$

- CDF is just as valuable in joint
- Can get PDF from CDF by differentiating

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x\,\partial y}(x,y)$$

Expectation

$$Z = g(X,Y)$$

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,f_{X,Y}(x,y)\,dx\,dy$$

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

(14)

More than 2 R.V. variables

$$P((X,Y,Z) \in B) = \iiint_{(x,y,z)\in B} f_{X,Y,Z}(x,y,z)\,dx\,dy\,dz$$

↑ for any set B

have relationships

$$f_{X,Y}(x,y) = \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dz$$

and

$$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dy\,dz$$

So $E[\cancel{X}]$

$$E[g(X,Y,Z)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y,z)\, f_{X,Y,Z}(x,y,z)\,dx\,dy\,dz$$

So if g is linear

$$E[aX+bY+cZ] = aE[X] + bE[Y] + cE[Z]$$

3.5 Conditioning

- Similar to discrete
- except if condition on event $\{Y=y\}$ which has 0 prob.

Conditional PDF

$$P(X \in B \mid A) = \int_B f_{X|A}(x)\,dx$$

↑ for any subset B of the real line

So normalize

$$\int_{-\infty}^{\infty} f_{X|A}(x)\,dx = 1$$

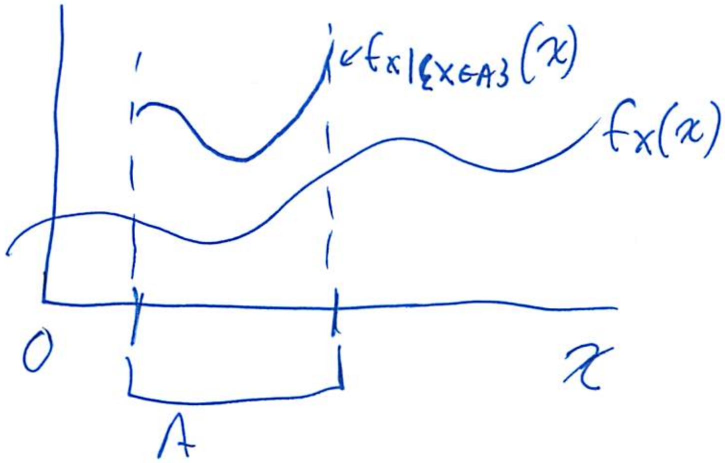
Then special case

$$\cancel{A}\ P(X\in B \mid X\in A) = \frac{P(X\in B,\ X\in A)}{P(X\in A)} = \frac{\int_{A\cap B} f_X(x)\,dx}{P(X\in A)}$$

So

$$f_{X\mid\{X\in A\}}(x) = \begin{cases} \dfrac{f_X(x)}{P(X\in A)} & \text{if } X\in A \\ 0 & \text{Otherwise} \end{cases}$$

- So 0 outside conditioning set
- inside same shape
- just normalized scaled by $\frac{1}{P(X\in A)}$
- now just ordinary PDF in new universe



W/ multiple similar notion of joint conditional PDF

$$f_{X,Y\mid C}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P(C)} & \text{if } (X,Y)\in A \\ 0 & \text{other} \end{cases}$$

$C=\{(X,Y)\in A\}$

Conditional PDF of X

$$f_{X\mid C}(x) = \int_{\infty}^{\infty} f_{X,Y\mid C}(x,y)\,dy$$

if partitions

$$f_X(x) = \sum_{i=1}^{n} P(A_i)\, f_{X\mid A_i}(x)$$

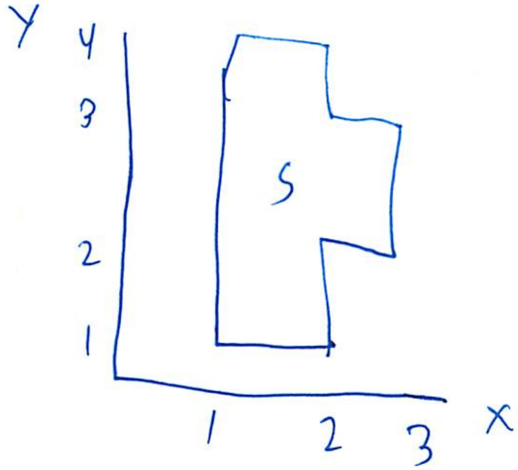
(16)

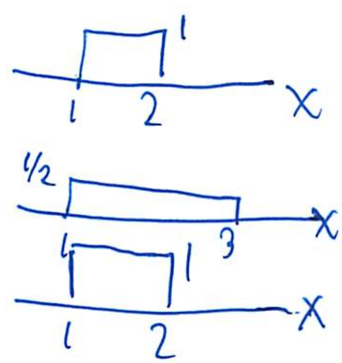
Conditioning 1 RV on another

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

↑ Conditional PDF of X given $Y = y$

- view Y as fixed #
- Consider $f_{X|Y}(x|y)$ as function of single variable x
- Same slope of joint PDF because denom $f_Y(y)$ does not depend on x └ $f_{X,Y}(x,y)$





(Skipping integration w/ σ - obvious)

Continuous Probability w/ zero prob event $\{Y = y\}$ can be defined now

You can calc joint PDF w/ conditional $f_{X|Y}$ and f_Y

(17)

Conditional Expectation

$E[X|A]$ - similar to unconditional case
and similar to discrete
(not going to copy)

Independence

if joint PDF is product of marginal PDFs

$f_{X,Y}(x,y) = f_X(x) f_Y(y)$ for all x, y

$f_{X|Y}(x|y) = f_X(x)$ for all y w/ $f_Y(y) > 0$ and all x

$f_{Y|X}(y|x) = f_Y(y)$ for all x w/ $f_X(x) > 0$ and all y

if X, Y independent events $\{X \in A\}$ and $\{Y \in B\}$ are ind.

$F_{X,Y}(x,y) = P(X \le x, Y \le y) = P(X \le x) P(Y \le y) = F_X(x) F_Y(y)$

$E[g(X) h(Y)] = E[g(X)] E[h(Y)]$

~~$var(g(X)h(Y)) = var(X) + var$ sum of variance)~~

$var(X+Y) = var(X) + var(Y)$

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3.6 Continuous Bayes' Rule

$$\xrightarrow[f_X(x)]{X} \boxed{\text{Measurement}} \xrightarrow[f_{Y|X}(y|x)]{Y} \boxed{\text{Inference}} \xrightarrow[f_{X|Y}(y|x)]{}$$

$$f_{X|Y}(x|y) = \frac{f_X(x) f_{Y|X}(y|x)}{f_Y(y)}$$

normalize so

$$f_{X|Y}(x|y) = \frac{f_X(x)\, f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(t) f_{Y|X}(y|t)\, dt}$$

Inference

- did in recitation

actually will skip rest - covered in recitation

Michael Plasmeier

If you staple over the numbers, I can't read them.

I show all work
No OH this time

25/10

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Problem Set 5
Due October 18, 2010

1. Random variables X and Y are distributed according to the joint PDF

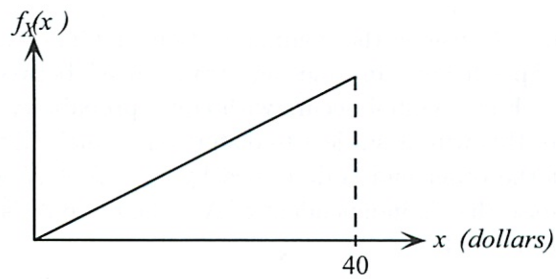
$$f_{X,Y}(x,y) = \begin{cases} ax, & \text{if } 1 \le x \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Evaluate the constant a.

   (b) Determine the marginal PDF $f_Y(y)$.

   (c) Determine the expected value of $\frac{1}{X}$, given that $Y = \frac{3}{2}$.

2. Paul is vacationing in Monte Carlo. The amount X (in dollars) he takes to the casino each evening is a random variable with the PDF shown in the figure. At the end of each night, the amount Y that he has on leaving the casino is uniformly distributed between zero and twice the amount he took in.

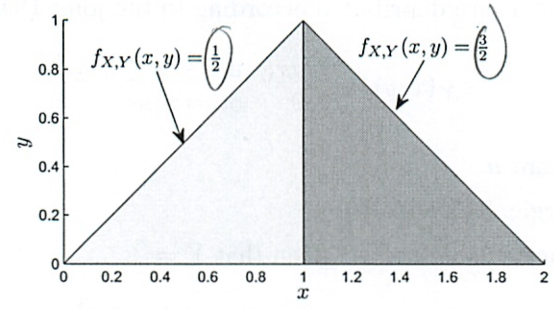


   (a) Determine the joint PDF $f_{X,Y}(x,y)$. Be sure to indicate what the sample space is.

   (b) What is the probability that on any given night Paul makes a positive profit at the casino? Justify your reasoning.

   (c) Find and sketch the probability density function of Paul's profit on any particular night, $Z = Y - X$. What is $\mathbf{E}[Z]$? Please label all axes on your sketch.

3. X and Y are continuous random variables. X takes on values between 0 and 2 while Y takes on values between 0 and 1. Their joint pdf is indicated below.



(a) Are X and Y independent? Present a convincing argument for your answer.
(b) Prepare neat, fully labelled plots for $f_X(x)$, $f_{Y|X}(y \mid 0.5)$, and $f_{X|Y}(x \mid 0.5)$.
(c) Let $R = XY$ and let A be the event $X < 0.5$. Evaluate $\mathbf{E}[R \mid A]$.
(d) Let $W = Y - X$ and determine the cumulative distribution function (CDF) of W.

4. **Signal Classification:** Consider the communication of binary-valued messages over some transmission medium. Specifically, any message transmitted between locations is one of two possible symbols, 0 or 1. Each symbol occurs with equal probability. It is also known that any numerical value sent over this wire is subject to distortion; namely, if the value X is transmitted, the value Y received at the other end is described by $Y = X + N$ where the random variable N represents additive noise that is independent of X. The noise N is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 4$.

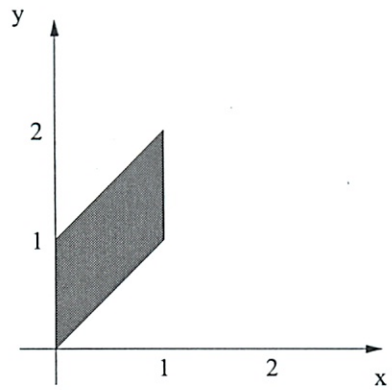
(a) Suppose the transmitter encodes the symbol 0 with the value $X = -2$ and the symbol 1 with the value $X = 2$. At the other end, the received message is decoded according to the following rules:
- If $Y \geq 0$, then conclude the symbol 1 was sent.
- If $Y < 0$. then conclude the symbol 0 was sent.

Determine the probability of error for this encoding/decoding scheme. Reduce your calculations to a single numerical value.

(b) In an effort to reduce the probability of error, the following modifications are made. The transmitter encodes the symbols with a repeated scheme. The symbol 0 is encoded with the vector $\overline{X} = [-2, -2, -2]^\intercal$ and the symbol 1 is encoded with the vector $\overline{X} = [2, 2, 2]^\intercal$. The vector $\overline{Y} = [Y_1, Y_2, Y_3]^\intercal$ received at the other end is described by $\overline{Y} = \overline{X} + \overline{N}$. The vector $\overline{N} = [N_1, N_2, N_3]^\intercal$ represents the noise vector where each N_i is a random variable assumed to be normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 4$. Assume each N_i is independent of each other and independent of the X_i's. Each component value of $\overline{Y}$ is decoded with the same rule as in part (a). The receiver then uses a majority rule to determine which symbol was sent. The receiver's decoding rules are:
- If 2 or more components of $\overline{Y}$ are greater than 0, then conclude the symbol 1 was sent.
- If 2 or more components of $\overline{Y}$ are less than 0, then conclude the symbol 0 was sent.

Determine the probability of error for this modified encoding/decoding scheme. Reduce your calculations to a single numerical value.

5. The random variables X and Y are described by a joint PDF which is constant within the unit area quadrilateral with vertices $(0, 0)$, $(0, 1)$, $(1, 2)$, and $(1, 1)$.



   (a) Are X and Y independent?
   (b) Find the marginal PDFs of X and Y.
   (c) Find the expected value of $X + Y$.
   (d) Find the variance of $X + Y$.

6. A defective coin minting machine produces coins whose probability of heads is a random variable P with PDF

$$f_P(p) = \begin{cases} 1 + \sin(2\pi p), & \text{if } p \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

   In essence, a specific coin produced by this machine will have a fixed probability $P = p$ of giving heads, but you do not know initially what that probability is. A coin produced by this machine is selected and tossed repeatedly, with successive tosses assumed independent.

   (a) Find the probability that the first coin toss results in heads.
   (b) Given that the first coin toss resulted in heads, find the conditional PDF of P.
   (c) Given that the first coin toss resulted in heads, find the conditional probability of heads on the second toss.

G1†. Let C be the circle $\{(x, y) \mid x^2 + y^2 \leq 1\}$. A point a is chosen randomly on the boundary of C and another point b is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x- and y-axes with diagonal ab. What is the probability that no point of R lies outside of C?

†Required for 6.431; optional for 6.041

Michael Plasmeier **P-Set 5** 10/18

1. Random variables distributed by joint PDF

$$f_{X,Y}(x,y) = \begin{cases} ax & \text{if } 1 \le x \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Evaluate constant a

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_X(x,y)\,dx\,dy = \int_1^2\int_1^2 \frac{1}{2-1}\cdot\frac{1}{2-1}\,dx\,dy$$

$$= \frac{1}{1}\cdot\frac{1}{1}\,\Big|_1^2\,\Big|_1^2$$

$$a = \underline{1}$$

(-0.5)

← makes sense like a 1×1 unit square
- well could be
- could have $X < Y$

b) Determine the marginal pdf $f_Y(y)$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$\cancel{\int_{-\infty}^{\infty} ax\,dx}$$

$$\cancel{\frac{ax^2}{2} + c\,\Big|_{-\infty}^{\infty}}$$

~~integral does not converge~~

} do?

Eg reasoning no examples

definitions correct but not here

- improper integral
  - endpoints approach a # or $\infty, -\infty$ 
- Unbounded → will converge if limit exists
  - $\int_{-\infty}^{\infty}$

(1b)

Oh is bounded

$$f_{y}(y) = \int_{1}^{2} a x \, dx$$

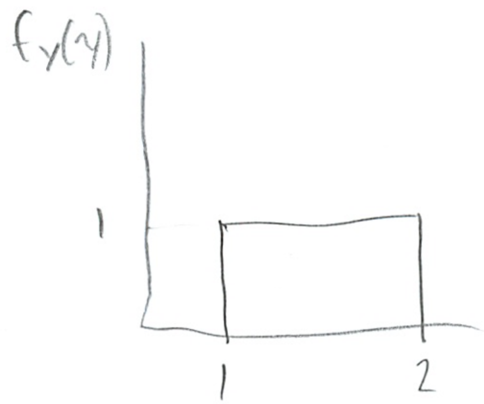
$$\frac{a x^2}{2} + c \Big|_{1}^{2}$$

$$\frac{a\,4}{2} - \frac{a}{2}$$

$$2a - a$$

a ← constant PDF

and $a = \underline{1}$



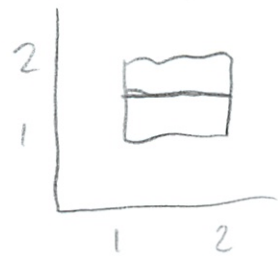
makes sense

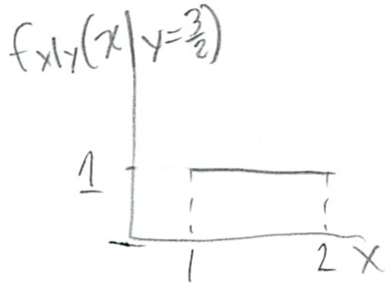
(−1)

(2)

c) Determine $E\left[\frac{1}{X} \middle| Y = \frac{3}{2}\right]$

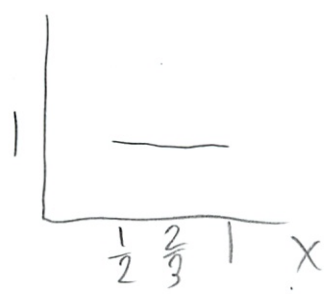
so fix $Y = \frac{3}{2}$





$$E[X|Y=y] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y) \, dx$$

$$E[X] = \frac{3}{2}$$



↳ what is middle? $\frac{2}{3}$ or $\frac{1}{4}$

I would say $\frac{1}{\frac{3}{2}} = \frac{2}{3}$

(-1)

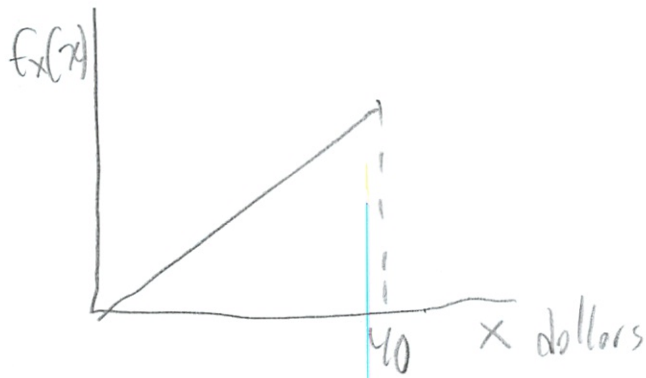
0/2.5

③

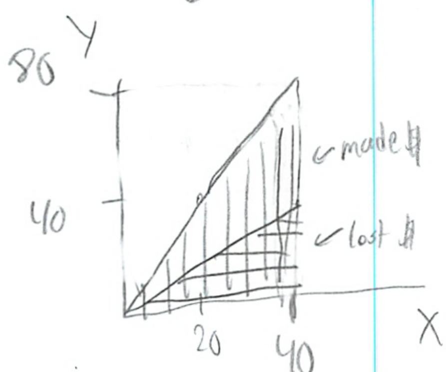
2. Paul is vacationing in Monte Carlo

X = \$ taken to casino

Y = between 0 and $2X$



a) Determine joint PDF, $f_{X,Y}(x,y)$



? how represent
· go back + reread notes

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\text{area}} & \text{if } 0 \le x \le 40,\ 0 \le y \le 80 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{area} = \frac{1}{2} \cdot 80 \cdot 40$$
$$= 1600$$

---

b) $P(\text{made \$}) = 1 - P(\text{lost \$})$

the ... not ...   the ...   the ...

(4) .

area of lost \$

$\frac{1}{2} \cdot 40 \cdot 40$

$800 = \int_0^{y}\int_0^{40} f_{X,Y}(x,y)\,dx\,dy$

Look at the 2 different areas

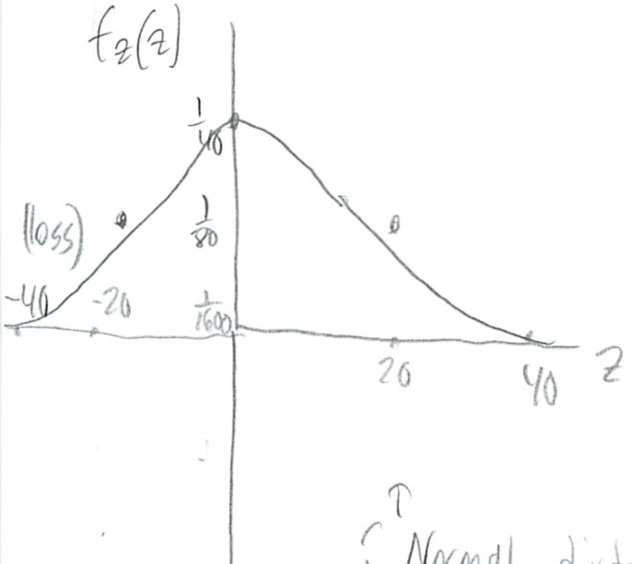
Can calculate P(loss) easily - so calculate that

$P(\text{made } \$) = \frac{1}{1600-800} = \frac{1}{2}$

c) Find and sketch the PDF of Pauls profit on a night

$Z = Y - X$

$E[Z] = ?$



20 can be 40-20
39-19
⋮
20-0
and continuous

0 can be 40-40
39-39

↑ but its continuous, this is discrete reasoning

↑ Normal distribution - would make sense

$E[Z] = 0$

↑ μ for normal distributions

Makes sense some nights you win, some you dont

or is it

but why would it be like that?

(5)

3. X, Y continuous variables

area $= \frac{1}{2} \cdot 1 \cdot 2 = 1$
so don't need to normalize
densities $\frac{1}{2} \cdot 1 \cdot 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 \cdot 1 \cdot \frac{3}{2}$
$\frac{1}{4} + \frac{3}{4} = 1$ evens out as well

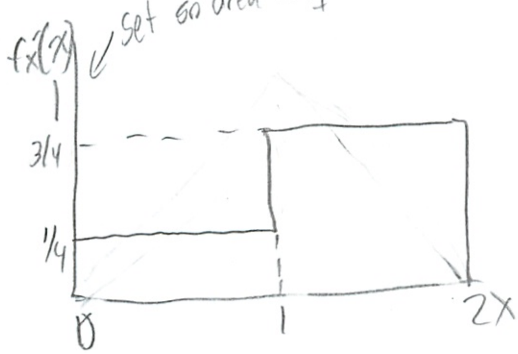
a) are they independent?

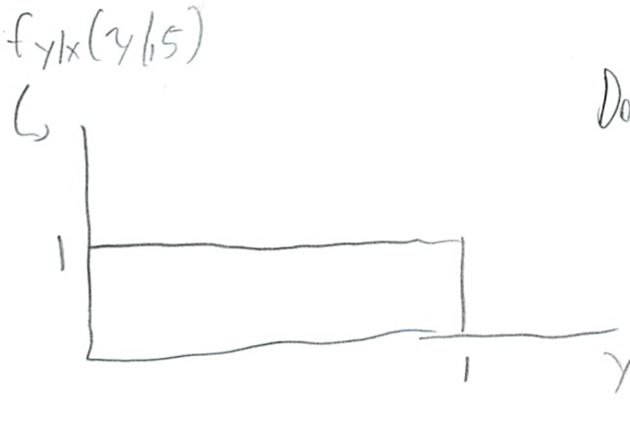
Does $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ for all x, y

No. If you know $f_X(.5)$ than that tells you by $f_Y(y)$ that $y \le .5$

b) Prepare neat, fully labeled plots $f_X(x)$, $f_{Y|X}(y|.5)$
$f_{X|Y}(x,.5)$

how likly is it to be that x value



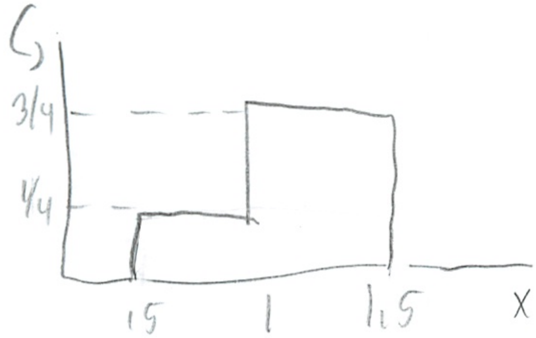


Don't need to know which side of line - just thats its even along line

this is slowly making sense

⑥

$f_{x|y}(x|1.5)$



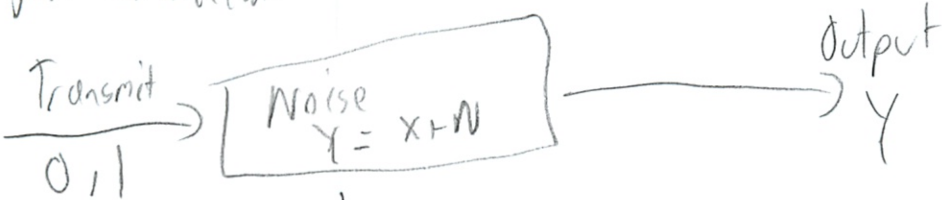
$\frac{1}{2} \cdot \frac{1}{2} + \frac{3}{2} \cdot \frac{1}{2}$

$\frac{1}{4} + \frac{3}{4} = 1$

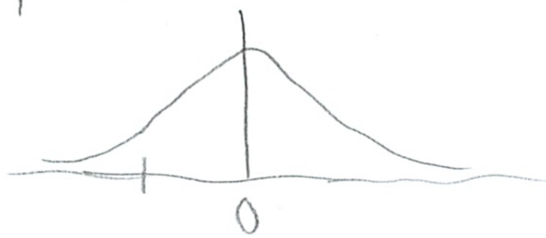
(7)

4. Signal Classification

↓ took a few pages to figure out



I'ile recitation 11 except normal dist.



$\sigma^2 = 4$

$\sigma = 2$ ← double standard normal

$\Phi\left(\frac{X-\mu}{\sigma}\right) = \Phi\left(\frac{Y}{2}\right)$

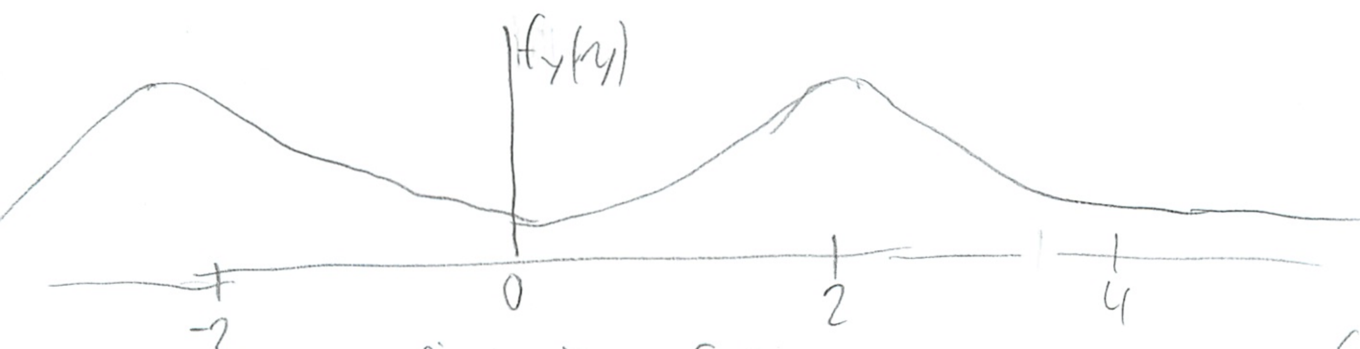
a) Suppose

a 0 is $-2 = X$ $\quad p = \frac{1}{2}$

a 1 is $2 = X$ $\quad p = \frac{1}{2}$

Output $\quad Y \geq 0$ then 1 was sent

$\quad Y < 0$ then 0 was sent



Discrete X + Continuous Y

? what order ?

$$P(X=2 \mid Y=y) = \frac{P_X(2)\, f_{Y|X}(y \mid 2)}{P_X(2)\, f_{Y|X}(y \mid 2) + P_X(-2)\, f_{Y|X}(y \mid -2)}$$

(8)

$$= \frac{\frac{1}{2} \cdot \phi\left(\frac{y}{2}\right)}{\frac{1}{2} \cdot \phi\left(\frac{y}{2}\right) + \frac{1}{2}\phi\left(\frac{y}{2}\right)}$$

$$= \frac{\phi\left(\frac{y}{2}\right)}{2\phi\left(\frac{y}{2}\right)} = \frac{1}{2}$$

???

[Textbook P(error) p157

What do we want?

$$P(Y < 0) = P\left(\frac{Y-\mu}{\sigma} < \frac{0-\mu}{\sigma}\right)$$

$$= \phi\left(\frac{0}{2}\right)$$

$$= \phi(0)$$

$$= .5$$

oh same as what I got before - cool

$$P(Y \geq 0) = P(Y > 0) = 1 - P(Y < 0) = 1 - P\left(\frac{Y-\mu}{\sigma} < \frac{0-\mu}{\sigma}\right)$$

$$= 1 - \phi\left(\frac{0}{2}\right)$$

$$= 1 - \phi(0)$$

$$= 1 - .5$$

$$= .5$$

(9) ~~So what is P either will error~~ → or

~~P(~~

No that is just P is each

— makes sense

I need

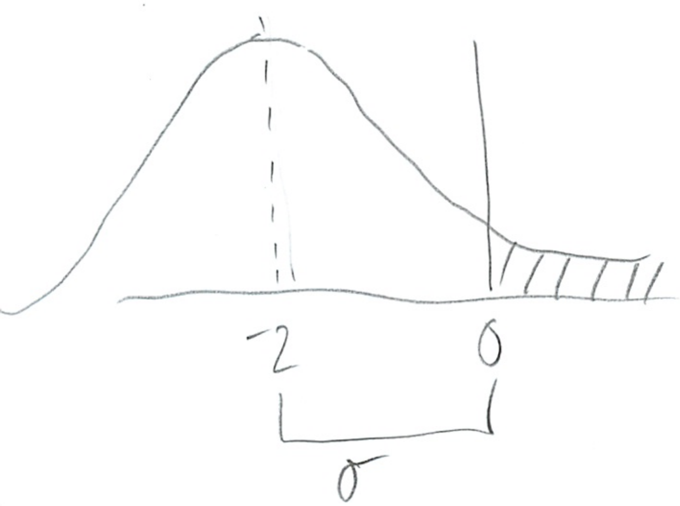
$$f_{Y|X}(Y<0 \mid X=-2) = \frac{f_Y(y)\, f_{X|Y}(x \mid y)}{f_X(x)}$$

know ↑ $\frac{1}{2}$ ↑ found above .5 know was .5

$$= \frac{.5\, f_{X|Y}(x \mid y)}{.5}$$ what is this?

Or do I need to do that — no

I messed above up



oh forgot to fill these in

$$P(Y \geq 0) = 1 - P(N<0) = 1 - P\left(\frac{N-(-2)}{2} < \frac{0-(-2)}{2}\right)$$

$$= 1 - \Phi(1)$$

$$= 1 - .8413$$

$$= .1587$$

yeah was same as top just adjusted

(10) And is the same for the other one by symattry

$P(Y<2) = P\left(\frac{Y--2}{2} < \frac{0--2}{2}\right)$

$\phi(1)$

.1587

Now need to combine

$P(\text{error} \mid X=-2) \cup P(\text{error} \mid x=2)$

$\frac{.1587 \cdot 1}{\frac{1}{2}} + \frac{.1587 \cdot 1}{\frac{1}{2}}$ ← notation?

.1587 since only 1 symbol can be transmitter each time

(11)

b) 0 is now $\bar{X} = [-2,-2,-2]^T$; so basically everything is sent 3 times

1 is now $\bar{X} = [2,2,2]^T$

$\bar{N} = [N_1, N_2, N_3]^T$

each random $\mu = 0$, $\sigma = 4$

$\bar{Y} = \bar{N} + \bar{X}$

If 2 or more components of $\bar{Y} > 0 \rightarrow$ think 1 sent

If 2 " " " " $\bar{Y} < 0 \rightarrow 0$ sent

Probability of Y_i still same

But can be

C C C ← correct

C C I

C I C

I C C

$P(I) = .1587$

$P(C) = .8413$

should represent in a better way too complicate though

$$= \frac{.8413^3 + .8413^2 \, .1587 \binom{3}{1}}{4}$$

← only failure can be cubed?

← must be Ω

$$= \frac{.59546 + .3369}{4}$$

$= .466$ ← should be much smaller

(12) $\Omega = .8413^3\binom{3}{3} + \binom{3}{1}.8413^2 \cdot .1587 + \binom{3}{2}.8413 \cdot .1587^2$

$+\binom{3}{3}\ .1587^3$

$\binom{3}{1} = 3$

$\binom{3}{2} = \frac{3!}{2!1!} = \frac{6}{2} = 3$

$\binom{3}{3} = 1$

$\Omega = 1$ ok confirmed function, but does not help

ah, right

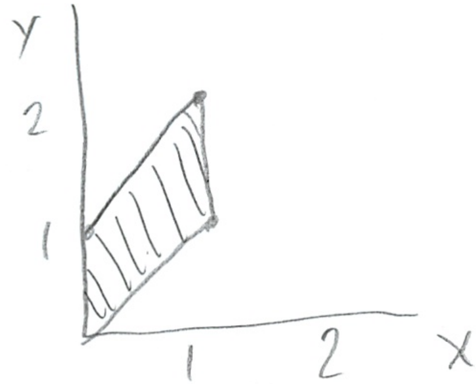
.93236 is P(signal correct)

P(error) = 1 − P(correct)

= .06764 much better

(13)

5. RV, X, Y are described by joint PDF

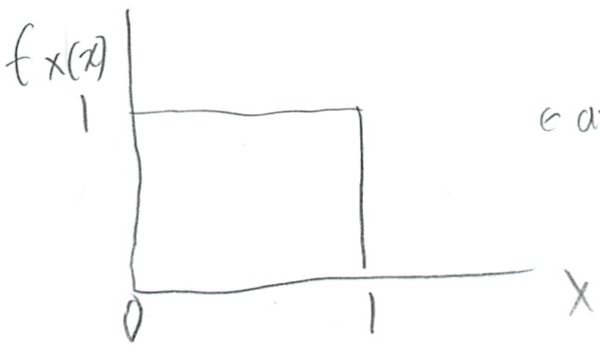


a) Are X, Y independent?   +1

No, If I knew $X = .5$, I would know $.5 < Y < 1.5$

b) Find the marginal PDFs of X

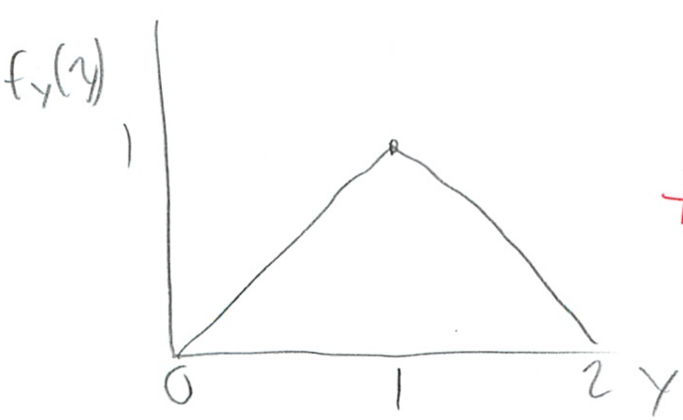
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$$



← at each point on X - how high the box

+0.5

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$$



+0.5

Slowly learning the stuff doing the p-set
What you should do as you do the p-set

(14)

c) Find $E[X]$ and $E[Y]$

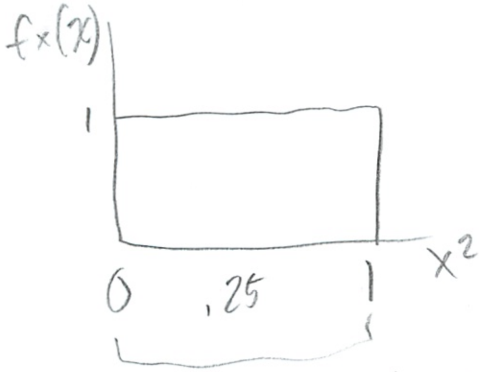
- graphically center of gravity

$E[x] = .5$ +0.5

$E[y] = 1$ $E[X+Y]=?$ (-0.5)

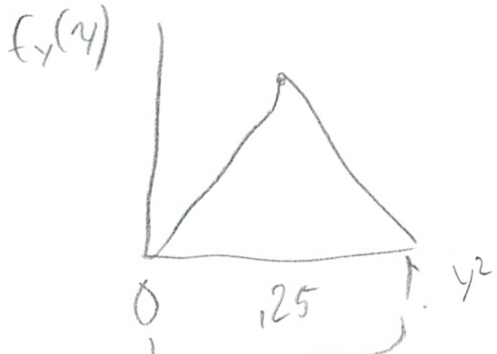
d) Find $Var(x)$ and $var(y)$

$E[x^2] = .5$



log -center still .5

$E[Y^2] = .25$



log

2.5/4.5

$var(x) = .5 - .5 = 0$

$var(y) = .25 - 1 = .75$

Not what was asked (-1.5)

(15)

6. A defective machine makes an unfair coin

$$f_X(x) = \begin{cases} 1 + \sin(2\pi x) & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Heads

- So coin slanted
- but don't know by how much
- This sounds similar to a tutorial problem

a) Find p that 1st coin toss results in heads

$$\underset{\text{hidden}}{X} \xrightarrow{f_{Y|X}} \underset{\substack{\text{Observed} \\ \text{head}}}{Y}$$

Find $P_{X|Y}$

Contineous X, discrete Y

need - what is it??? really confused on what this could be

$$f_{X|Y}(x|y) = \frac{P_{Y|X}(y|x) \cdot f_X(x)}{\int_0^1 P_{Y|X}(y|x) \cdot f_X(x)\,dx}$$

have

have

$P_{Y|X}(y|x = .5)$

$= ?\ 1 + \sin(2\pi x)$ but that is in other place

Recitation 11 #2 may be better model

(16).

← But what in all world should this be?

$$= \frac{1+\sin(2\pi x) \cdot \begin{cases} 1 & \text{if } y=1 \\ 0 & \text{if } y=0 \end{cases}}{\frac{1}{2}\int_0^1 1+\sin(2\pi x) \cdot \; dx}$$

← but it is discrete output
continuous input
reverse of problem

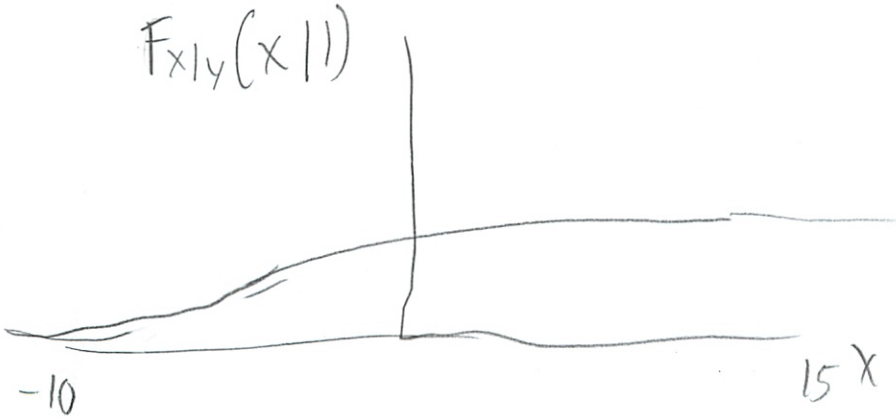
ask
~~$f_{X|Y}(x|1)$~~ ? integral TI 89

$$= \frac{1+\sin(2\pi x) \cdot \begin{cases} 1 & \text{if } y=1 \\ 0 & \text{if } y=0 \end{cases}}{.527388}$$

ask
$$f_{X|Y}(x|1) = \frac{1+\sin(2\pi x)}{.527388} \quad \text{for } x \in [0,1]$$

graph

$F_{X|Y}(x|1)$



looks like CDF, but not
- or is it since
mix continuous/discrete?

P(heads) depends what x was

(17)

b) Given that 1st coin toss resulted in heads, find conditional PDF of P

- like the machine problem in the tutorial

Wait didn't I answer that already?

a) again) and then the answer to a was just that

$$p_P(p) = \begin{cases} 1 + \sin(2\pi p) & \text{if } p \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Since every toss is the same

(-1)

c) $P(\overset{\text{2nd day}}{\text{Heads}} \mid \underset{\text{day}}{\text{1st}} \text{ heads}) =$

$$= \frac{P(\text{2nd heads} \cap \text{1st heads})}{P(\text{1st heads})}$$

Tosses are independent

But they provide info about p

this is like machine problem in tutorial

$$P\left(\begin{matrix}\text{2nd day} \\ \text{heads} \\ = C\end{matrix} \middle| P = p\right) = \begin{cases} 1 + \sin(2\pi p) & \text{if } p \in [0,1] \\ 0 \end{cases}$$

(-1)

$$p_{P|C}(p) = \frac{P(C \mid P=p)\, f_P(p)}{\int P(C \mid P=p)\, f_P(p)\, dp}$$

(18)

$$P(C \mid 1) = P(x \mid 1) = \frac{1+\sin(2\pi x)}{.527388}$$

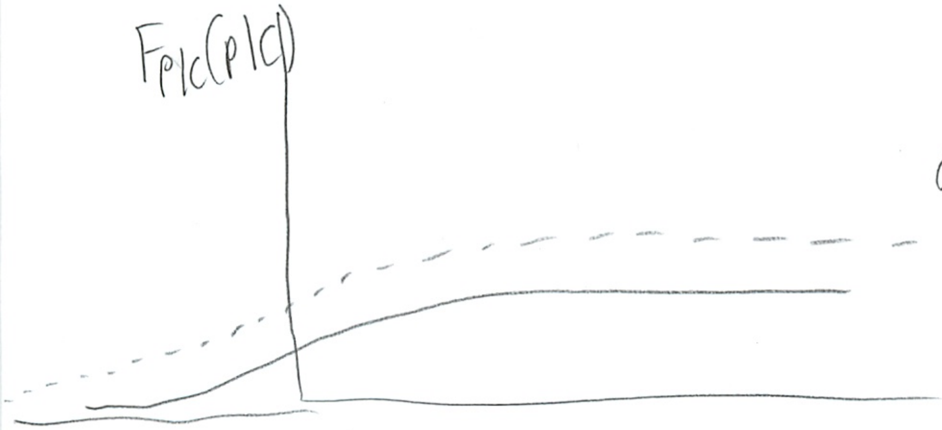
$$F_{p|c}(p|C) = \frac{\frac{1+\sin(2\pi x)}{.527388} \cdot 1+\sin(2\pi x)}{\int_0^1 \frac{1+\sin(2\pi x)}{.527388} + 1+\sin(2\pi x)\,dx}$$

$$= \frac{\frac{1+2\sin(2\pi x)+\sin^2(2\pi x)}{.527388}}{3.05}$$

$$= .6207\left(1+2\sin(2\pi x)+\sin^2(2\pi x)\right)$$

Likely super wrong

(-1)



part b

old one dotted

for comparison

0/3

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

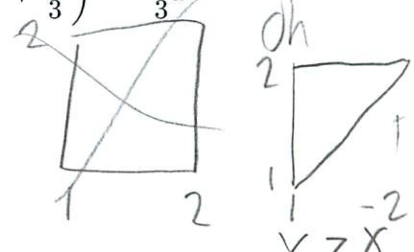
Problem Set 5: Solutions

Review w/o paper back

1. (a) Because of the required normalization property of any joint PDF,

$$1 = \int_{x=1}^{2}\left(\int_{y=x}^{2} ax\,dy\right)dx \;=\; \int_{x=1}^{2} ax(2-x)\,dx \;=\; a\left(2^2 - 1^2 - \frac{2^3}{3} + \frac{1^3}{3}\right) \;=\; \frac{2}{3}a$$

so $a = 3/2$. a and prob is not even



(b) For $1 \le y \le 2$,

$$f_Y(y) = \int_1^y ax\,dx \;=\; \frac{a}{2}(y^2-1) \;=\; \frac{3}{4}(y^2-1),$$

and $f_Y(y) = 0$ otherwise.

(c) First notice that for $1 \le x \le 3/2$,

$$f_{X|Y}(x \mid 3/2) = \frac{f_{X,Y}(x,3/2)}{f_Y(3/2)} = \frac{(3/2)x}{\frac{3}{4}\left(\left(\frac{3}{2}\right)^2 - 1^2\right)} = \frac{8x}{5}.$$

Opps - why did I ignore $Y \ge X$?

Therefore,

? I did not do it that formally - ? should have | just had square on brain

$$\mathbf{E}[1/X \mid Y = 3/2] \;=\; \int_1^{3/2} \frac{1}{x}\frac{8x}{5}\,dx \;=\; 4/5.$$

2. (a) By definition $f_{X,Y}(x,y) = f_X(x)f_{Y|X}(y \mid x)$. $f_X(x) = ax$ as shown in the graph. We have that

again pay attention to formal def

$$1 \;=\; \int_0^{40} ax\,dx \;=\; 800a.$$

So $f_X(x) = x/800$. From the problem statement $f_{Y|X}(y \mid x) = \frac{1}{2x}$ for $y \in [0, 2x]$. Therefore,

$$f_{X,Y}(x,y) = \begin{cases} 1/1600, & \text{if } 0 \le x \le 4 \text{ and } 0 < y < 2x, \\ 0, & \text{otherwise.} \end{cases}$$ did I write this

(b) Paul makes a positive profit if $Y > X$. This occurs with probability

$$\mathbf{P}(Y > X) \;=\; \int\int_{y>x} f_{X,Y}(x,y)\,dy\,dx \;=\; \int_0^{40}\int_x^{2x} \frac{1}{1600}\,dy\,dx \;=\; \frac{1}{2}.$$

We could have also arrived at this answer by realizing that for each possible value of X, there is a 1/2 probability that $Y > X$.

(c) The joint density function satisfies $f_{X,Z}(x,z) = f_X(x)\,f_{Z|X}(z|x)$. Since Z is conditionally uniformly distributed given X, $f_{Z|X}(z \mid x) = \frac{1}{2x}$ for $-x \le z \le x$. Therefore, $f_{X,Z}(x,z) = 1/1600$ for $0 \le x \le 40$ and $-x \le z \le x$. The marginal density $f_z(z)$ is calculated as

$$f_Z(z) \;=\; \int_x f_{X,Z}(x,z)\,dx \;=\; \int_{x=|z|}^{40} \frac{1}{1600}\,dx \;=\; \begin{cases} \frac{40-|z|}{1600}, & \text{if } |z| < 40, \\ 0, & \text{otherwise.} \end{cases}$$

? forgot it I got this?

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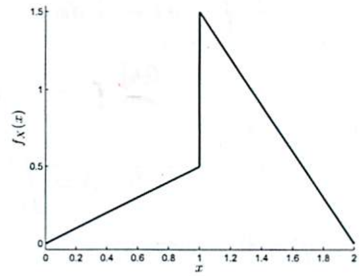
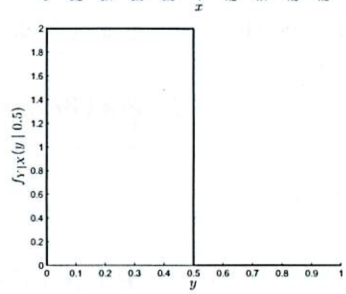
3. (a) In order for X and Y to be independent, any observation of X should not give any information on Y. If X is observed to be equal to 0, then Y must be 0.
In other words, $f_{Y|\{X=0\}}(y \mid 0) \neq f_Y(y)$. Therefore, X and Y are not independent.

(b) $$f_X(x) = \begin{cases} x/2, & \text{if } 0 \le x \le 1, \\ -3x/2 + 3, & \text{if } 1 < x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

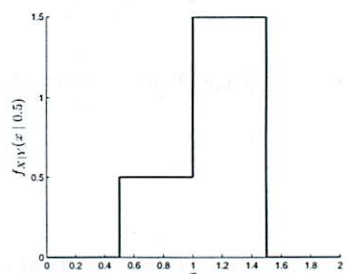


← did not get anything like these pictures I think — or maybe I did but lower

$$f_{Y|X}(y \mid 0.5) = \begin{cases} 2, & \text{if } 0 \le y \le 1/2, \\ 0, & \text{otherwise.} \end{cases}$$



$$f_{X|Y}(x \mid 0.5) = \begin{cases} 1/2, & \text{if } 1/2 \le x \le 1, \\ 3/2, & \text{if } 1 < x \le 3/2, \\ 0, & \text{otherwise.} \end{cases}$$

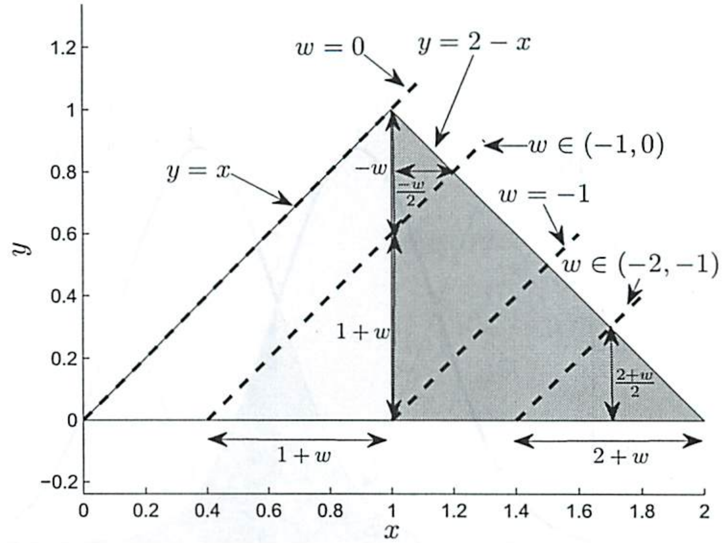


(c) The event A leaves us with a right triangle with a constant height. The conditional PDF is then 1/area = 8. The conditional expectation yields: ? did I figure that out ?

$$\begin{aligned} \mathbf{E}[R \mid A] &= \mathbf{E}[XY \mid A] \\ &= \int_0^{0.5} \int_y^{0.5} 8xy \, dx \, dy \\ &= 1/16. \end{aligned}$$

(d) The CDF of W is $F_W(w) = \mathbf{P}(W \le w) = \mathbf{P}(Y - X \le w) = \mathbf{P}(Y \le X + w)$. $\mathbf{P}(Y \le X + w)$ can be computed by integrating the area below the line $Y = X + w$ for all possible values of w. The lines $Y = X + w$ are shown below for $w = 0$, $w = -1/2$, $w = -1$ and $w = -3/2$. The probabilities of interest can be calculated by taking advantage of the uniform PDF over the two triangles. Remember to multiply the areas by the appropriate joint density $f_{X,Y}(x, y)$! Take note that there are 4 regions of interest: $w < -2$, $-2 \le w \le -1$, $-1 < w \le 0$ and $w > 0$.

did I even do any CDF?



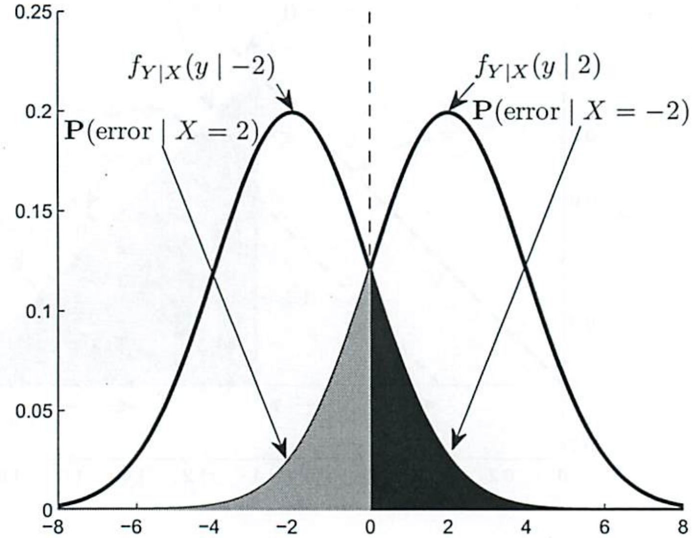
The CDF of W is

$$F_W(w) = \begin{cases} 0, & \text{if } w < -2, \\ 3/2 \cdot 1/2(2+w)^2/2, & \text{if } -2 \leq w \leq -1, \\ 1/2 \cdot 1/2(1+w)^2 + 3/2 \cdot (1/2 \cdot 1 \cdot 1 - 1/2(-w/2 \cdot -w)), & \text{if } -1 < w \leq 0, \\ 1, & \text{if } w > 0 \end{cases}$$

$$= \begin{cases} 0, & \text{if } w < -2, \\ 3/8 \cdot (2+w)^2, & \text{if } -2 \leq w \leq -1, \\ 1/8 \cdot (-w^2 + 4w + 8), & \text{if } -1 < w \leq 0, \\ 1, & \text{if } w > 0. \end{cases}$$

As a sanity check, $F_W(-\infty) = 0$ and $F_W(+\infty) = 1$. Also, $F_W(w)$ is continuous at $w = -2$ and at $w = -1$.

4. (a) If the transmitter sends the 0 symbol, the received signal is a normal random variable with a mean of -2 and a variance of 4. In other words, $f_{Y|X}(y \mid -2) = \mathcal{N}(-2, 4)$. Also, $f_{Y|X}(y \mid 2) = \mathcal{N}(2, 4)$ These conditional pdfs are shown in the graph below.



think got

The probability of error can be found using the total probability theorem.

$$
\begin{aligned}
\mathbf{P}(\text{error}) &= \mathbf{P}(\text{error} \mid X = -2)\mathbf{P}(X = -2) + \mathbf{P}(\text{error} \mid X = 2)\mathbf{P}(X = 2) \\
&= \frac{1}{2}(\mathbf{P}(Y \geq 0 \mid X = -2) + \mathbf{P}(Y < 0 \mid X = 2)) \\
&= \frac{1}{2}(\mathbf{P}(N \geq 2 \mid X = -2) + \mathbf{P}(N < -2 \mid X = 2)) \\
&= \frac{1}{2}(\mathbf{P}(N \geq 2) + \mathbf{P}(N < -2)) \\
&= \frac{1}{2}(\mathbf{P}(\frac{N-0}{2} \geq \frac{2-0}{2}) + \mathbf{P}(\frac{N-0}{2} < \frac{-2-0}{2})) \\
&= \frac{1}{2}((1 - \Phi(1)) + (1 - \Phi(1))) \\
&= 0.1587.
\end{aligned}
$$

(b) With 3 components, the probability of error given an obervation of X is the probability of decoding 2 or 3 of the components incorrectly. For each component, the probability of error is 0.1587. Therefore,

$$
\begin{aligned}
\mathbf{P}(\text{error} \mid \text{sent } 0) &= \binom{3}{2}(0.1587)^2(1 - 0.1587) + (0.1587)^3 \\
&= 0.0676.
\end{aligned}
$$

✓ yeah

By symmetry, $\mathbf{P}(\text{error} \mid \text{sent } 1) = \mathbf{P}(\text{error} \mid \text{sent } 0)$.
Therefore, $\mathbf{P}(\text{error}) = \mathbf{P}(\text{error} \mid \text{sent } 0)\mathbf{P}(\text{sent } 0) + \mathbf{P}(\text{error} \mid \text{sent } 1)\mathbf{P}(\text{sent } 1) = 0.0676$.

5. (a) There are many ways to show that X and Y are not independent. One of the most intuitive arguments is that knowing the value of X limits the range of Y, and vice versa. For instance, if it is known in a particular trial that $X \geq 1/2$, the value of Y in that trial cannot be smaller

can do it when real world solution

than 1/2. Another way to prove that the two are not independent is to calculate the product of their expectations, and show that this is not equal to $\mathbf{E}[XY]$.

(b) Applying the definition of a marginal PDF,

for $0 \le x \le 1$,

$$\begin{aligned} f_X(x) &= \int_y f_{X,Y}(x,y)\,dy \\ &= \int_x^{x+1} 1\,dy \\ &= 1; \end{aligned}$$

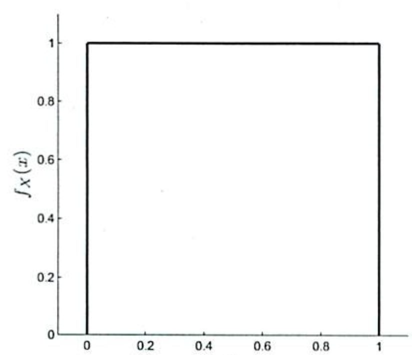
for $0 \le y \le 1$,

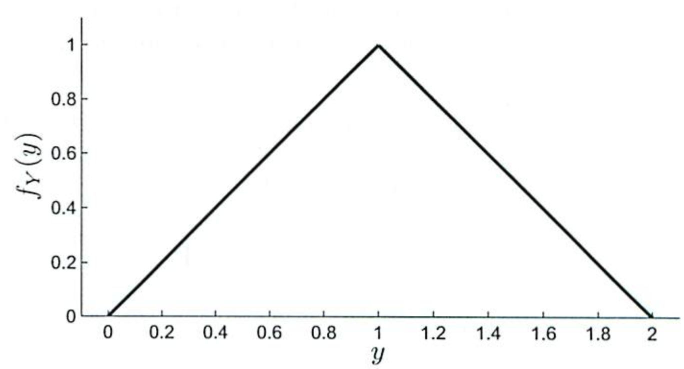
$$\begin{aligned} f_Y(y) &= \int_x f_{X,Y}(x,y)\,dx \\ &= \int_0^{y} 1\,dx \\ &= y; \end{aligned}$$

and for $1 \le y \le 2$,

$$\begin{aligned} f_Y(y) &= \int_x f_{X,Y}(x,y)\,dx \\ &= \int_{y-1}^{1} 1\,dx \\ &= 2-y. \end{aligned}$$

think got this





(c) By linearity of expectation, the expected value of a sum is the sum of the expected values. By inspection, $\mathbf{E}[X] = 1/2$ and $\mathbf{E}[Y] = 1$.
Thus, $\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y] = 3/2$.

(d) The variance of $X + Y$ is

$$\mathbf{E}[(X+Y)^2] - \mathbf{E}[X+Y]^2 = \mathbf{E}[X^2] + 2\mathbf{E}[XY] + \mathbf{E}[Y^2] - (\mathbf{E}[X+Y])^2. \quad (1)$$

In part (c), $\mathbf{E}[X+Y]$ was computed, so only the other three expressions need to be calculated. First, the expected value of X^2:

$$\mathbf{E}[X^2] = \int_0^1 x^2 \int_x^{x+1} 1\, dy\, dx = \int_0^1 x^2\, dx = 1/3.$$

Also, the expected value of Y^2 is

$$\mathbf{E}[Y^2] = \int_0^1 \int_x^{x+1} y^2\, dy\, dx = \int_0^1 (3x^2 + 3x + 1)/3\, dx = 7/6.$$

Finally, the expected value of XY is

$$\begin{aligned} \mathbf{E}[XY] &= \int_0^1 x \int_x^{x+1} y\, dy\, dx \\ &= \int_0^1 (2x^2 + x)/2\, dx\, dy = 7/12. \end{aligned}$$

Substituting these into (1), we get $\text{var}(X+Y) = 1/3 + 7/6 + 7/6 - 9/4 = 5/12$.

Alternative (shortcut) solution to parts (c) and (d)

Given any value of X (in ([0,1]), we observe that $Y - X$ takes values between 0 and 1, and is uniformly distributed. Since the conditional distribution of $Y - X$ is the same for every value of X in [0,1], we see that $Y - X$ independent of X. Thus: (a) X is uniform, and (b) $Y = X + U$, where U is also uniform and independent of X. It follows that $\mathbf{E}[X+Y] = \mathbf{E}[2X+U] = 3/2$. Furthermore, $\text{var}(X+Y) = 4\,\text{var}(X) + \text{var}(U) = 5/12$.

6. (a) Let A be the event that the first coin toss resulted in heads. To calculate the probability $\mathbf{P}(A)$, we use the continuous version of the total probability theorem:

$$\mathbf{P}(A) = \int_0^1 \mathbf{P}(A \mid P = p) f_P(p)\, dp = \int_0^1 p(1 + \sin(2\pi p))\, dp,$$

which after some calculation yields

$$\mathbf{P}(A) = \frac{\pi - 1}{2\pi}.$$

did I calculate that?

(b) Using Bayes rule,

Use formal rule

$$\begin{aligned} f_{P|A}(p) &= \frac{\mathbf{P}(A \mid P = p) f_P(p)}{\mathbf{P}(A)} \\ &= \begin{cases} \dfrac{2\pi p(1 + \sin(2\pi p))}{\pi - 1}, & \text{if } 0 \le p \le 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(c) Let B be the event that the second toss resulted in heads. We have

$$\begin{aligned}
\mathbf{P}(B \mid A) &= \int_0^1 \mathbf{P}(B \mid P = p, A) f_{P|A}(p)\, dp \\
&= \int_0^1 \mathbf{P}(B \mid P = p) f_{P|A}(p)\, dp \\
&= \frac{2\pi}{\pi - 1} \int_0^1 p^2(1 + \sin(2\pi p))\, dp.
\end{aligned}$$

After some calculation, this yields

$$\mathbf{P}(B \mid A) = \frac{2\pi}{\pi - 1} \cdot \frac{2\pi - 3}{6\pi} = \frac{2\pi - 3}{3\pi - 3} \approx 0.5110.$$

G1†. Let $a = (\cos\theta, \sin\theta)$ and $b = (b_x, b_y)$. We will show that no point of R lies outside C if and only if

$$|b| \leq |\sin\theta|, \quad \text{and} \quad |a| \leq |\cos\theta| \tag{2}$$

The other two vertices of R are $(\cos\theta, b_y)$ and $(b_x, \sin\theta)$. If $|b_x| \leq |\cos\theta|$ and $|b_y| \leq |\sin\theta|$, then each vertex (x, y) of R satisfies $x^2 + y^2 \leq \cos^2\theta + \sin^2\theta = 1$ and no points of R can lie outside of C. Conversely if no points of R lie outside C, then applying this to the two vertices other than a and b, we find

$$\cos^2\theta + b^2 \leq 1, \quad \text{and} \quad a^2 + \sin^2\theta \leq 1.$$

which is equivalent to 2.

These conditions imply that (b_x, b_y) lies inside or on C, so for any given θ, the probability that the random point $b = (b_x, b_y)$ satisfies (2) is

$$\frac{2|\cos\theta| \cdot 2|\sin\theta|}{\pi} = \frac{2}{\pi}|\sin(2\theta)|$$

and the overall probability is

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{2}{\pi}|\sin(2\theta)| d\theta = \frac{4}{\pi^2} \int_0^{\pi/2} \sin(2\theta) d\theta = \frac{4}{\pi^2}$$

.

†Required for 6.431; optional for 6.041

$W = X + Y$

derived distributions

harder + abstract

10/18

Quiz 2 up to Wed lecture on Nov 2

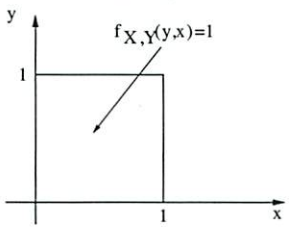
LECTURE 11

Derived distributions; convolution; covariance and correlation

- **Readings:**
  Finish Section 4.1;
  Section 4.2

$Y = g(X)$
- CDF + differentiate

Example



Find the PDF of $Z = g(X,Y) = \boxed{Y/X}$

integrate density
will be just area

would need to find new formula since can't calc area of that

$F_Z(z) = P\left(\frac{Y}{X} \le z\right) = \frac{z}{2}$ $\quad z \le 1$

are their z which this if false? $0 \le z \le 1$

$F_Z(z) = 1 - \frac{1}{2} \cdot 1 \cdot \frac{1}{z}$ $\quad z \ge 1$

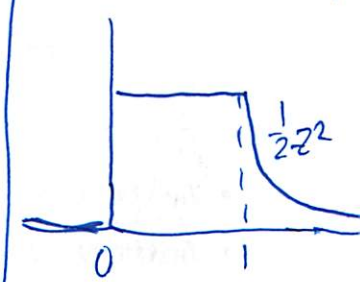
(slope z') here

So total area − unshaded triangle
$1 - \frac{1}{2} \cdot 1 \cdot \frac{1}{z}$

$F_Z(z) = 0 \qquad z < 0$

now differentiate for PDF

$$f_Z(z) = \begin{cases} 0 & z < 0 \\ \frac{1}{2} & 0 \le z \le 1 \\ \frac{1}{2} \cdot \frac{1}{z^2} & z > 1 \end{cases}$$

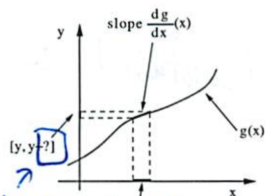


(on hw I did not use CDF − was that correct?)

A general formula

- Let $Y = g(X)$ function 1 → 1 x → y
  g strictly monotonic.

density = prob per unit length/area



depends on ~~slope~~ slope of g function

- Event $x \le X \le x + \delta$ is the same as
  $g(x) \le Y \le g(x + \delta)$
  or (approximately)
  $g(x) \le Y \le g(x) + \delta |(dg/dx)(x)|$

- Hence,

$$\delta f_X(x) = \delta f_Y(y) \left| \frac{dg}{dx}(x) \right|$$

$\boxed{\text{where } y = g(x)}$

slope of line at that location

delta small so cancle

$$f_X(x) = f_Y(y) \left| \frac{dg}{dx}(x) \right|$$

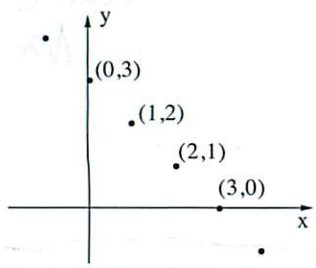
$y = g(x)$
$x = g^{-1}(y)$ ← so get expression w/ only ys

See seperate sheet

Start w/ discrete - simpler

The distribution of $X+Y$

- $W = X + Y$; X, Y independent



$w=3$

all x,y pairs where $3 = x+y$

Want $P(\text{points}) = \sum P(\text{point})$

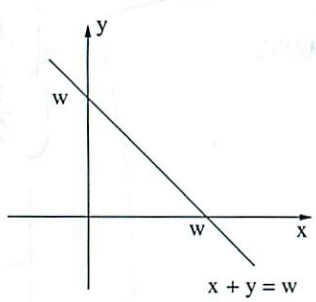
See seperate sheet

$$
\begin{aligned}
p_W(w) &= \mathrm{P}(X+Y=w)\\
&= \sum_x \mathrm{P}(X=x)\mathrm{P}(Y=w-x)\\
&= \sum_x p_X(x)p_Y(w-x)
\end{aligned}
$$

- Mechanics:
  - Put the pmf's on top of each other
  - Flip the pmf of Y
  - Shift the flipped pmf by w (to the right if $w > 0$)
  - Cross-multiply and add

The continuous case

- $W = X + Y$; X, Y independent



W same as Y, but add X (a constant) to it

if you add constant

just shift density of Y

- $f_{W|X}(w \mid x) = f_Y(w - x)$

Now can find joint density

~~from~~ marginal · conditional

- $f_{W,X}(w, x) = f_X(x) f_{W|X}(w \mid x)$

$\qquad = f_X(x) f_Y(w - x)$

Then find marginal

- integrate away variable don't care about

- $\boxed{f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x)\,dx}$

get rid of joint

) cross multiply + integrate

Switching gears

Two independent normal r.v.s

- $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$, independent

$$
\begin{aligned}
f_{X,Y}(x,y) &= f_X(x) f_Y(y) \\
&= \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{ -\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2} \right\}
\end{aligned}
$$

- PDF is constant on the ellipse where

$$
\frac{(x-\mu_x)^2}{2\sigma_x^2} + \frac{(y-\mu_y)^2}{2\sigma_y^2}
$$

is constant

get contours

Spead like a bell in 3D
X more likely to be big
~ broader range in X than Y

- Ellipse is a circle when $\sigma_x = \sigma_y$

if circle

When not the same, need to squeeze one access (to get to standard normal?)
Scaling axis in + out

The sum of independent normal r.v.'s

- $X \sim N(0, \sigma_x^2)$, $Y \sim N(0, \sigma_y^2)$, independent

- Let $W = X + Y$

$$
\begin{aligned}
f_W(w) &= \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\, dx \\
&= \frac{1}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{\infty} e^{-x^2/2\sigma_x^2} e^{-(w-x)^2/2\sigma_y^2}\, dx \\
\text{after (algebra)} &= ce^{-\gamma w^2}
\end{aligned}
$$

density of X, Y

to the argument

end result

- Conclusion: W is normal

$$\frac{1}{\sqrt{2\pi}\sigma_w} e^{-\frac{1}{2}\frac{w^2}{\sigma_w^2}}$$

  - mean=0, variance=$\sigma_x^2 + \sigma_y^2 = \sigma_w^2$
  - same argument for nonzero mean case

End result is back to normal
nice to live in normal world
(not standard normal)

how do 2 random variables depend on each other

Covariance

- $\text{cov}(X,Y) = \mathbf{E}\Big[(X - \mathbf{E}[X]) \cdot (Y - \mathbf{E}[Y])\Big]$ ← how large is product?

- Zero-mean case: $\text{cov}(X,Y) = \mathbf{E}[XY]$

$E[XY] > 0$ $E[XY] < 0$

They tend to have same sign

tend to have opposite sign

$\text{Cov}(X,X) = \text{var}(X)$
$= E[X^2] - (E[X])^2$

- $\text{cov}(X,Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y]$

independent → these are =

- $\text{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{var}(X_i) + 2 \sum_{(i,j):i \neq j} \text{cov}(X_i, X_j)$

need if dependency

$(X_1 + \cdots + X_n)^2 = \sum_i X_i^2 + 2\sum_{i \neq j} X_i X_j$

- $\boxed{\text{independent} \Rightarrow \text{cov}(X,Y) = 0 \text{ (converse is not true)}}$

Unit is strange

better to have dimensionless quantity

Correlation coefficient

- Dimensionless version of covariance:

$$\rho = \mathbf{E}\left[\frac{(X - \mathbf{E}[X])}{\sigma_X} \cdot \frac{(Y - \mathbf{E}[Y])}{\sigma_Y}\right]$$
$$= \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y}$$

- $-1 \leq \rho \leq 1$

extreme case

- $|\rho| = \pm 1 \Leftrightarrow (X - \mathbf{E}[X]) = c(Y - \mathbf{E}[Y])$ (linearly related)

linear relationship

if $\rho = \pm 1$ knowing X always gives you Y

- Independent $\Rightarrow \rho = 0$ (converse is not true) ←

Slide 2

$g(x) = x^3 = y$

$y = x^3$

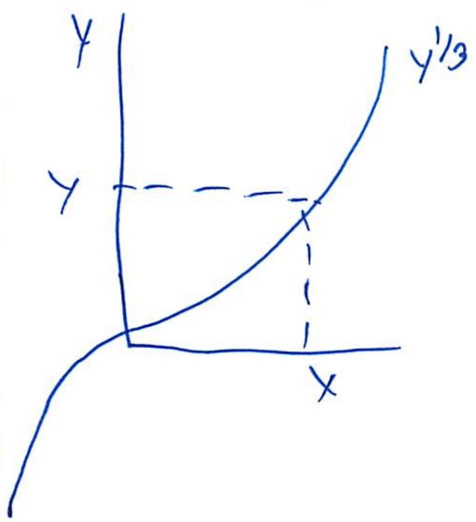
$x = y^{1/3}$

$f_x(x) = f_y(y) \cdot 3x^2$

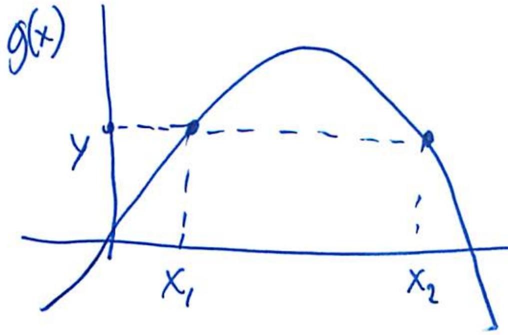
x, y are related by formula

$f_x(y^{1/3}) = f_y(y)\, 3y^{2/3}$

$f_y(y) = \dfrac{f_x(y^{1/3})}{3y^{2/3}}$ density of y as function of y

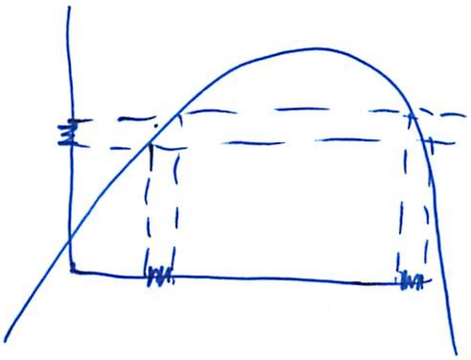


Suppose not 1-1



multiple x

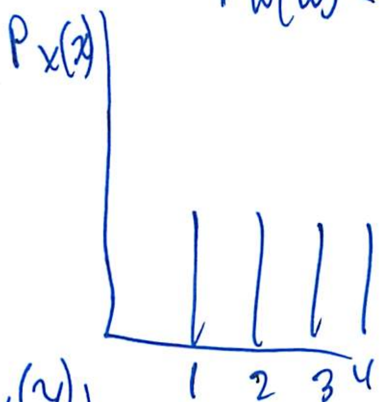
2 But intervals

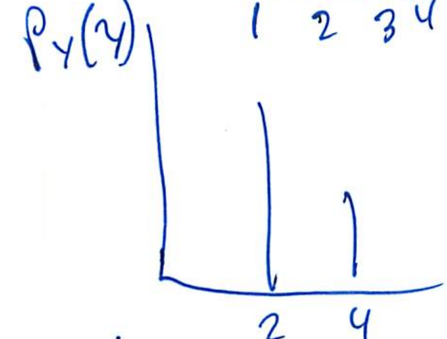


function of multiple ... (did not hear)

Slide 3

$$P_W(w) = \cancel{\#} \sum_x p_X(x)\, p_Y(w-x)$$





take $w = 0$

y is just negitive so just flip it

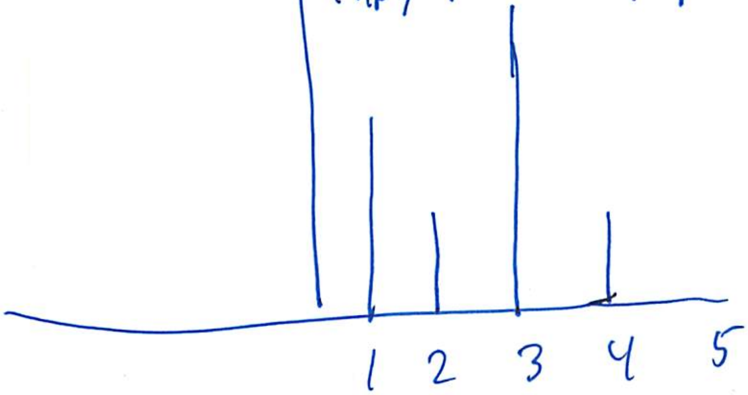
nothing to add
normalize

take $w = 5$

since $5 \cancel{\#} - x$

flip y + shift y plot 5 units →

add one on top of other
normalize



Recitation 12
October 19, 2010

1. Show $\rho(aX + b, Y) = \rho(X, Y)$.

2. Romeo and Juliet have a date at a given time, and each, independently, will be late by amounts of time, X and Y, respectively, that are exponentially distributed with parameter λ.

   (a) Find the PDF of $Z = X - Y$ by first finding the CDF and then differentiating.

   (b) Find the PDF of Z by using the total probability theorem.

3. Problem 4.16, page 248 in text.
   Let X and Y be independent standard normal random variables. The pair (X, Y) can be described in polar coordinates in terms of random variables $R \geq 0$ and $\Theta \in [0, 2\pi]$, so that

$$X = R\cos\Theta, \quad Y = R\sin\Theta.$$

   Show that R and Θ are independent (i.e. show $f_{R,\Theta}(r, \theta) = f_R(r) f_\Theta(\theta)$).

   (a) Find $f_R(r)$.

   (b) Find $f_\Theta(\theta)$.

   (c) Find $f_{R,\Theta}(r, \theta)$.

4. Problem 4.20, page 250 in text. **Schwarz inequality.**
   Show that for any random variables X and Y, we have

$$(\mathbf{E}[XY])^2 \leq \mathbf{E}[X^2]\mathbf{E}[Y^2].$$

Recitation 12 10/ 19

Derived Distributions

- given 2 RV
- derive a 3rd one

X, Y are RVs

$Z = g(X, Y)$

Now want – PDF of Z
└ PMF

So need to find the CDF of Z

General Procedure

1. Find CDF $F_Z(z) = P(g(X,Y) \le z) = \dots$

   and differenitate F_Z to get f_Z

   or difference to get p_Z

   (since CDF always exists – discrete
   └ continuous)

Special Case – if $Z = X + Y$

and X, Y independent

– Convolution Formula

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

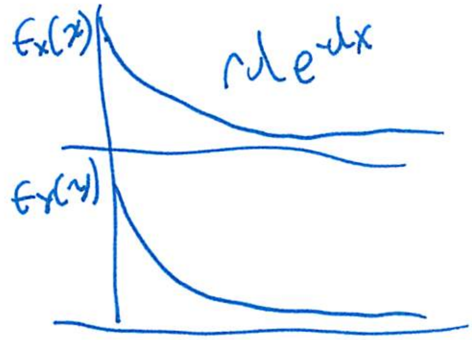
②

Example Romeo + Juliet

- can do both ways

X = Amt of time Romeo is late
Y = " " " Juliet " "

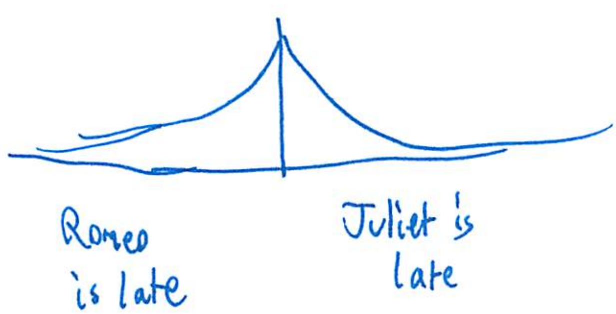
) independent exponentially distributed ↳ parameter λ



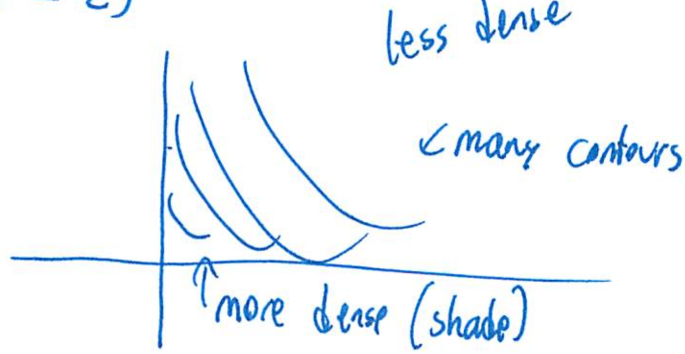
$Z = X - Y$ = amt of time Juliet waits

$Z > 0$ if $Y < X$

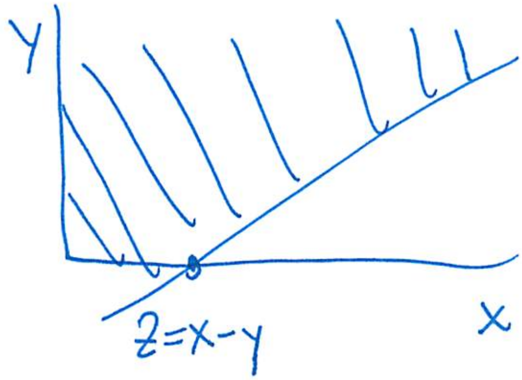
$Z < 0$ if $Y > X$



$F_Z(z) = P(X - Y \le z)$



③ $z \geq 0$



$F_Z(z) = P(X - Y \leq z) = 1 - P(X - Y \geq z)$

$= 1 - \int_0^{\infty} \left(\int_{z+y}^{\infty} f_{X,Y}(x,y)\, dx \right) dy$ double integration

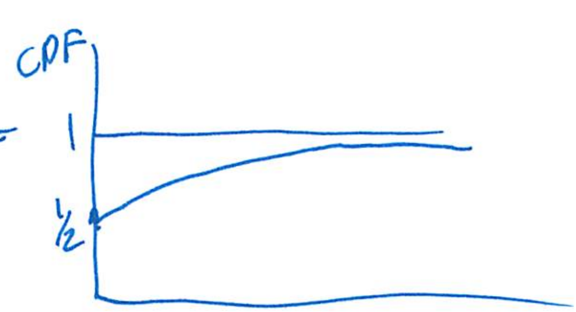
$= 1 - \int_0^{\infty} \lambda e^{-\lambda y} \underbrace{\left(\int_{z+y}^{\infty} \lambda e^{-\lambda x}\, dx \right)}_{e^{-\lambda(z+y)}} dy$

integrate

$= 1 - e^{-\lambda z} \underbrace{\int_0^{\infty} \lambda e^{-2\lambda y}\, dy}_{1/2}$

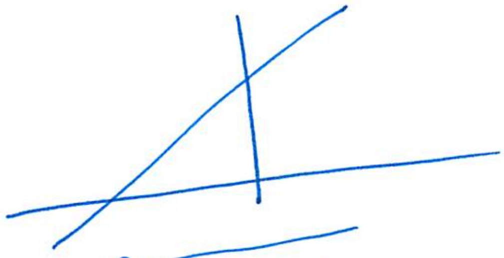
$= 1 - \frac{1}{2} \cdot e^{-\lambda z}$

=



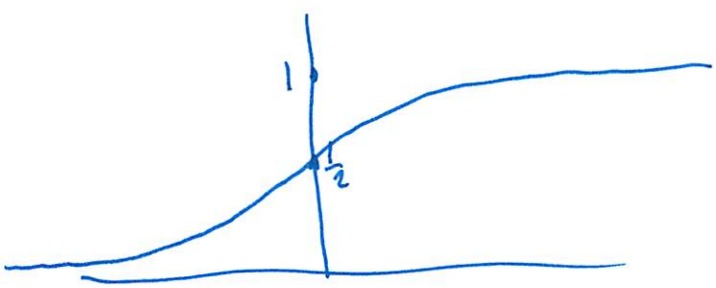
④

For $z < 0$



Same except shifted over

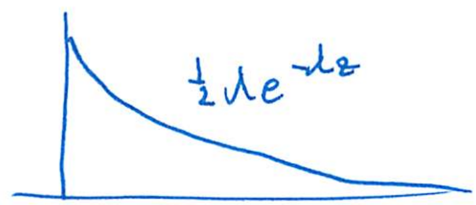
$$F_Z(z) = \frac{1}{2} e^{\lambda z}$$ by symmetry



or calculate PDF of $z > 0$

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \frac{1}{2} \lambda e^{-\lambda z}$$



reflection

exponential on both sides
= Laplace distribution

$$f_Z(z) = \frac{1}{2}\lambda e^{-\lambda|z|}$$

(5)

Now w/ convolution formula

$Z = X + (-Y)$

flip axis/PDF

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_{-Y}(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} \underbrace{f_X(x)}_{\lambda e^{-\lambda x}} \ \underbrace{f_Y(x-z)\, dx}_{\lambda e^{-\lambda(x-z)}}$$

$=0$ for $x \le 0$ $\qquad$ $=0$ for $x \le z$

$$= \int_{z}^{\infty} \lambda e^{-\lambda x}\, \lambda e^{-\lambda(x-z)}\, dx$$

← skipping integration

$$= \frac{\lambda}{2} e^{-\lambda z}$$

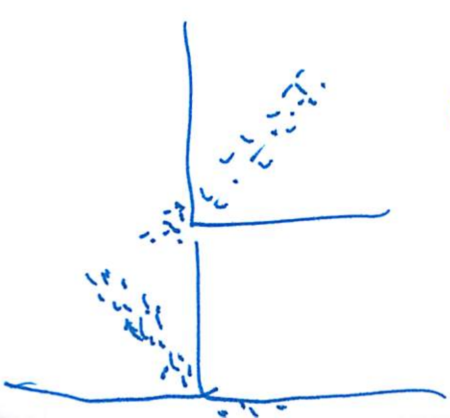
argue other side by symmetry

Convolution more direct here - since this is a special case

Covariance

Gives us idea how similar two RV are

$$\operatorname{cov}(X,Y) = E\big[(X - E[X])(Y - E[Y])\big]$$



cov > 0

cov < 0

can move 0 w/o change

high value of X leads to likely high value of Y

⑥

$$= E[XY] - E[X] \cdot E[Y]$$

Main properties

$$\begin{cases} cov(aX, Y) = a\, cov(X,Y) \\ cov(X+Z, Y) = cov(X,Y) + cov(Z,Y) \\ cov(aX+b, Y) = a\, cov(X,Y) + \cancel{cov(b,Y)}^{=0} \\ cov(X, \cancel{X}) = var(X) \end{cases}$$

$\uparrow X, X$

to Normalize → <u>correlation coefficent</u>

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{var(X) \cdot var(Y)}}$$

show $-1 \le \rho(X,Y) \le 1$

$$\rho(aX+b, Y) = \begin{cases} \rho(X,Y) & \text{if } a \ge 0 \\ -\rho(X,Y) & \text{if } a < 0 \end{cases}$$

---

example

X = Temp at MIT °C

$Y = \begin{cases} 1 & \text{if rain} \\ 0 & \text{if no rain} \end{cases}$ (Bernouli)

Correlation

high temp is less rain

$$Z = \frac{9}{5} X + 32$$

7

$\rho(x, y) = \rho(z, y)$

since ρ is dimensionless

Proof

$$\rho(ax+b, Y) = \frac{\operatorname{cov}(aX+b, Y)}{\sqrt{\underbrace{\operatorname{var}(aX+b)}_{a^2\operatorname{var}(x)} \cdot \operatorname{var}(Y)}}$$

$$= \frac{a \operatorname{cov}(X, Y)}{|a|\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}}$$

$$= \underbrace{\frac{a}{|a|}}_{\substack{1 \text{ for } a>0 \\ -1 \text{ for } a<0}} \rho(X, Y)$$

so equivilant

Show that $-1 \le \rho(X, Y) \le 1$

$(\rho(x,y))^2 \le 1$

$(\operatorname{Cov}(x, y))^2 \le \operatorname{var}(x)\operatorname{var}(y)$

$\operatorname{cov}(\tilde{X}, \tilde{Y})^2 \le \operatorname{var}(\tilde{x})\operatorname{var}(\tilde{y})$

translated so centered at the mean

$\tilde{X} = X - E[x]$

$\tilde{Y} = Y - E[Y]$

8

$$\left(E[\tilde{X}\tilde{Y}]\right)^2 \leq E[\tilde{X}^2] \cdot E[\tilde{Y}^2]$$

Special case of Schwarz inequality

For any RV X, Y

$$\left(E[XY]\right)^2 \leq E[X^2]\,E[Y^2]$$

<u>Proof</u> shortest but least intuitive proof
trick proof

Have $0 \leq E\left[\left(X - \frac{E[XY]}{E[Y^2]} \cdot Y\right)^2\right]$

$\boxed{E[\text{pos RV}] \text{ is } \oplus}$

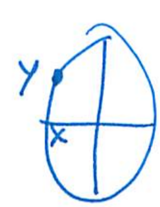
$$0 \leq^{\#} E[X^2] - 2\frac{E[XY]}{E[Y^2]} E[XY] + \frac{(E[XY])^2}{(E[Y^2])^2} E[Y^2]$$

$$0 \leq^{\#} E[X^2] - \frac{(E[XY])^2}{E[Y^2]}$$

$\rho \sim .999$ = almost aligned

$\rho \approx 0$ = Un correlated

<u>not</u> nessarily independent



10/20

Last lecture on ~~exam~~ quiz 2 — bit abstract

LECTURE 12

- **Readings:** Section 4.3; parts of Section 4.5 (mean and variance only; no transforms)

Lecture outline

- Conditional expectation — $E[X|Y]$
  - Law of iterated expectations
  - Law of total variance — $var(X|Y)$
- Sum of a random number of independent r.v.'s
  - mean, variance

Conditional expectations

- Given the value y of a r.v. Y: & if continuous - density instead

$$E[X \mid Y = y] = \sum_x x p_{X|Y}(x \mid y)$$

(integral in continuous case)

new universe

stick, break at random place Y then break again X

- Stick example: stick of length ℓ
  break at uniformly chosen point Y
  break again at uniformly chosen point X
- $E[X \mid Y = y] = \frac{y}{2}$ (number) after 1st break

↓ rewriting form

$E[X \mid Y] = \frac{Y}{2}$ (r.v.) but is a random variable

define notation

- **Law of iterated expectations:**

*view Y as random too

$$E[E[X \mid Y]] = \sum_y E[X \mid Y = y] p_Y(y) = E[X]$$

total expectation theorem

expected value rule ↳ divide + conquer

first expected value of Y then can calculate expected value of X

- In stick example:
  $E[X] = E[E[X \mid Y]] = E[Y/2] = \ell/4$

$$E[g(Y)] = \sum_y p_Y(y) g(y)$$

$$\frac{\ell/2}{2} = \frac{\ell}{4}$$

$E[X|Y] = g(Y)$

↑ same function

$\text{var}(X \mid Y)$ and its expectation

difference from mean?
- but which mean?
- the conditional mean

- $\text{var}(X \mid Y = y) = \mathbf{E}\left[(X - \mathbf{E}[X \mid Y = y])^2 \mid Y = y\right]$
- $\text{var}(X \mid Y)$: a r.v. with value $\text{var}(X \mid Y = y)$ when $Y = y$

don't know conditional variance until you know Y
Want the realized value of that RV

- **Law of total variance:**

$$\boxed{\text{var}(X) = \mathbf{E}[\text{var}(X \mid Y)] + \text{var}(\mathbf{E}[X \mid Y])}$$

Proof:

(a) Recall: $\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$

(b) $\text{var}(X \mid Y) = \mathbf{E}[X^2 \mid Y] - (\mathbf{E}[X \mid Y])^2$

law of iterated expectation

(c) $\mathbf{E}[\text{var}(X \mid Y)] = \mathbf{E}[X^2] - \mathbf{E}[\,(\mathbf{E}[X \mid Y])^2\,]$

(d) $\text{var}(\mathbf{E}[X \mid Y]) = \mathbf{E}[\,(\mathbf{E}[X \mid Y])^2\,] - (\mathbf{E}[X])^2$

Sum of right-hand sides of (c), (d):
$\mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \text{var}(X)$

Section means and variances

X = quiz score
Y = Section # (1 or 2)

Two sections:
$y = 1$ (10 students); $y = 2$ (20 students)

$$y = 1: \ \frac{1}{10}\sum_{i=1}^{10} x_i = 90 \qquad y = 2: \ \frac{1}{20}\sum_{i=11}^{30} x_i = 60$$

$$\mathbf{E}[X] = \frac{1}{30}\sum_{i=1}^{30} x_i = \frac{90 \cdot 10 + 60 \cdot 20}{30} = 70$$

$$\mathbf{E}[X \mid Y = 1] = 90, \quad \mathbf{E}[X \mid Y = 2] = 60$$

$$\mathbf{E}[X \mid Y] = \begin{cases} 90, & \text{w.p. } 1/3 \\ 60, & \text{w.p. } 2/3 \end{cases}$$

now abstraction
RV - don't know which section will pick

$$\mathbf{E}[\mathbf{E}[X \mid Y]] = \tfrac{1}{3} \cdot 90 + \tfrac{2}{3} \cdot 60 = 70 = \mathbf{E}[X]$$

the section average is random since section is random

$$\begin{aligned} \text{var}(\mathbf{E}[X \mid Y]) &= \frac{1}{3}(90 - 70)^2 + \frac{2}{3}(60 - 70)^2 \\ &= \frac{600}{3} = 200 \end{aligned}$$

weight according to possibilities
- did before but w/ more elementary variables

How different are the sections from each other?

How far is each value from mean of functions

Section means and variances (ctd.)

$$\frac{1}{10}\sum_{i=1}^{10}(x_i-90)^2 = 10 \qquad \frac{1}{20}\sum_{i=11}^{30}(x_i-60)^2 = 20$$

↳ variance inside each section

$\text{var}(X \mid Y=1) = 10 \qquad \text{var}(X \mid Y=2) = 20$

in each section

$$\text{var}(X \mid Y) = \begin{cases} 10, & \text{w.p. } 1/3 \\ 20, & \text{w.p. } 2/3 \end{cases}$$

variance when we <u>don't</u> know section

$\text{E}[\text{var}(X \mid Y)] = \frac{1}{3}\cdot 10 + \frac{2}{3}\cdot 20 = \frac{50}{3}$

$$\begin{aligned} \text{var}(X) &= \text{E}[\text{var}(X \mid Y)] + \text{var}(\text{E}[X \mid Y]) \\ &= \frac{50}{3} + 200 \end{aligned}$$

← previous slide

= (average variability **within** sections) ← this slide
+ (variability **between** sections)

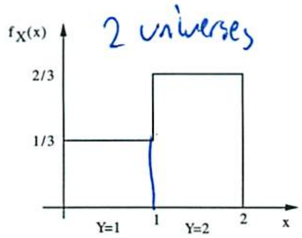
Example

$\text{var}(X) = \text{E}[\text{var}(X \mid Y)] + \text{var}(\text{E}[X \mid Y])$

$Y = 1$ if $0 \le x < 1$

$Y = 2$ if $1 \le x \le 2$

2 universes



$\text{E}[X \mid Y=1] = \frac{1}{2} \qquad \text{E}[X \mid Y=2] = \frac{3}{2}$ ← midpoint of uniform distribution

$\text{var}(X \mid Y=1) = \frac{1}{12} \qquad \text{var}(X \mid Y=2) = \frac{1}{12}$ in table

$\text{E}[X] = \frac{1}{3}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{3}{2} = \frac{7}{6}$

$\text{var}(\text{E}[X \mid Y]) =$

$\frac{1}{3}\left(\frac{1}{2}-\frac{7}{6}\right)^2 + \frac{2}{3}\left(\frac{3}{2}-\frac{7}{6}\right)^2 + \frac{1}{12}$

↙ other term
– same for $Y=1$, $Y=2$
does not always happen like that

Sum of a random number of independent r.v.'s

find distribution of sums w/ convolution

- N: number of stores visited (N is a nonnegative integer r.v.)
- X_i: money spent in store i
  - X_i assumed i.i.d.
  - independent of N

spend fresh in each store

- Let $Y = X_1 + \cdots + X_N$ total amt spent in all bookstores

if N was known + fixed we already know how to deal with that

divide + conquer

$$\begin{aligned} \mathrm{E}[Y \mid N = n] &= \mathrm{E}[X_1 + X_2 + \cdots + X_n \mid N = n] \\ &= \mathrm{E}[X_1 + X_2 + \cdots + X_n] \\ &= \mathrm{E}[X_1] + \mathrm{E}[X_2] + \cdots + \mathrm{E}[X_n] \\ &= n\,\mathrm{E}[X] \end{aligned}$$

← sum of fixed # RV

- $\mathrm{E}[Y \mid N] = N\,\mathrm{E}[X]$ equality b/w RV

? don't know how many stores you will visit...

#visited store · Expected value

$$\begin{aligned} \mathrm{E}[Y] &= \mathrm{E}[\mathrm{E}[Y \mid N]] \\ &= \mathrm{E}[N\,\mathrm{E}[X]] \\ &= \mathrm{E}[N]\,\mathrm{E}[X] \end{aligned}$$

this case answer is very intuitive - ok to reason w/ averages but not everywhere!

Variance of sum of a random number of independent r.v.'s

- harder → $\mathsf{var}(Y) = \mathrm{E}[\mathsf{var}(Y \mid N)] + \mathsf{var}(\mathrm{E}[Y \mid N])$
- $\mathrm{E}[Y \mid N] = N\,\mathrm{E}[X]$
  $\mathsf{var}(\mathrm{E}[Y \mid N]) = (\mathrm{E}[X])^2\,\mathsf{var}(N)$
- $\mathsf{var}(Y \mid N = n) = n\,\mathsf{var}(X)$ ← just sum of variances
  $\mathsf{var}(Y \mid N) = N\,\mathsf{var}(X)$ ← abstract notation
  $\mathrm{E}[\mathsf{var}(Y \mid N)] = \mathrm{E}[N]\,\mathsf{var}(X)$

$$\begin{aligned} \mathsf{var}(Y) &= \mathrm{E}[\mathsf{var}(Y \mid N)] + \mathsf{var}(\mathrm{E}[Y \mid N]) \\ &= \mathrm{E}[N]\,\mathsf{var}(X) + (\mathrm{E}[X])^2\,\mathsf{var}(N) \end{aligned}$$

variability in how much you spend in each store

variability in # stores

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Recitation 13
October 21, 2010

For the problems below, recall the Law of Iterated Expectations and the Law of Total Variance:

$$\mathbf{E}[X] = \mathbf{E}\left[\mathbf{E}[X|Y]\right]$$

$$\text{var}(X) = \mathbf{E}\left[\text{var}(X|Y)\right] + \text{var}\left(\mathbf{E}[X|Y]\right).$$

1. Let X, Y, and Z be discrete random variables. Show the following generalizations of the law of iterated expectations.

   (a) $\mathbf{E}[Z] = \mathbf{E}\big[\mathbf{E}[Z \mid X, Y]\big]$.

   (b) $\mathbf{E}[Z \mid X] = \mathbf{E}\big[\mathbf{E}[Z \mid X, Y] \mid X\big]$.

   (c) $\mathbf{E}[Z] = \mathbf{E}\big[\mathbf{E}\big[\mathbf{E}[Z \mid X, Y] \mid X\big]\big]$.

2. Example 4.17, page 223 in text.

   We start with a stick of length ℓ. We break it at a point which is chosen randomly and uniformly over its length, and keep the piece that contains the left end of the stick. We then repeat the same process on the piece that we were left with.

   (a) What is the expected value of the length of the piece that we are left with after breaking twice?

   (b) What is the variance of the length of the piece that we are left with after breaking twice?

3. Widgets are stored in boxes, and then all boxes are assembled in a crate. Let X be the number of widgets in any particular box, and N be the number of boxes in a crate. Assume that X and N are independent integer-valued random variables, with expected value equal to 10, and variance equal to 16. Evaluate the expected value and variance of T, where T is the total number of widgets in a crate.

Recitation 13

10/21

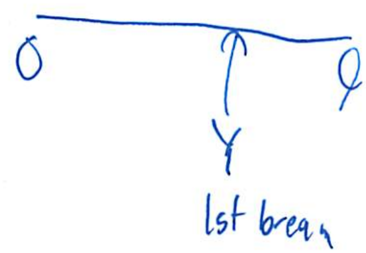
Using conditioning to help find expectations + variances
? expected value

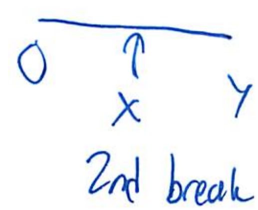
Law of Iterated Expressions / Law of Total variance

- need some time to understand
- once you understand, helpful

Law Iterated Expectations: $E[X] = E[\underbrace{E[X|Y]}_{\text{function of } Y \text{ aka } g(y)}]$

example





Could find PDF of X
- will be squed to left
- not so easy

$E[X] = E[\underbrace{E[X|Y]}_{\frac{Y}{2}}]$

②

$$= \frac{E[Y]}{2} = \frac{\ell/2}{2} = \frac{\ell}{4}$$

Example

Consider functions $g(x)$, $h(Y)$ of X and Y

claim $E[\underbrace{g(x) \cdot h(Y)}_{fn(x)} | X] = \underbrace{g(x)}_{fn(x)} \cdot \underbrace{E[h(Y) | X]}_{fn(x)}$

verify w/ law of iterated expectations

proof Have for any possible value $X = x$

$$E[g(x) \cdot h(Y) | X = x]$$

$$= \int_{-\infty}^{\infty} g(x) h(y) f_{Y|X}(y|x)\, dy$$

$$= g(x) \cdot \underbrace{E[h(Y) | X = x]}_{\substack{\text{Constant} \\ \text{Comes out}}}$$

For any value they are the same

- take same numeric value for every possible x

Use to prove 1 more thing

if $E[h(Y) | X]$ is constant (ind. of x) $= E[h(Y)]$ Conditional = unconditional

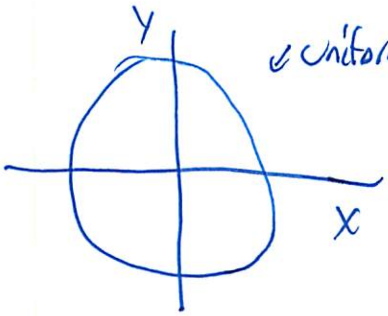
so $E[E[g(x) \cdot h(Y) | X]] = E[g(x) \cdot E(h(Y))]$

3

By iterated expectations

$$E[g(x) \cdot h(y)] = E[g(x)]\, E[h(y)]$$

So means $g(x)$ and $h(y)$ are <u>un</u>correlated



uniformly distributed on circle

Not independent

- info on one gives info on other

<u>Uncorrelated</u>

- all symetric
- or above property
- * knowing one does not tell us anything about $E[Y]$ but <u>does</u> tell us about PMF/distribution (see above w/ independence) *
- $E[Y|X] = E[Y]$
- tells you less about the other

$\boxed{\text{Correlated implies dependence}}$

$$Var(X) = E[\overbrace{Var(X|Y)}^{f(y)}] + var(\overbrace{E[X|Y]}^{f(y)})$$

$$E\, var|_Y + var\, E|_Y$$

(4)

Apply to stick example

$$\mathrm{var}(X) = E[\underbrace{\mathrm{var}(X|Y)}_{\frac{Y^2}{12}}] + \mathrm{var}(E[X|Y])$$

↑ since uniform

$0 \text{---} Y$

$$= \frac{E[Y^2]}{12} + \frac{Y}{2}$$

↑ Integrate

$$= \int_0^\ell \frac{1}{\ell} \cdot Y^2 \, dy \Big/ 12 + \frac{\mathrm{var}(Y)}{4}$$

$$= \frac{1}{\ell} \frac{Y^3}{3} \Big|_0^\ell \Big/ 12 + \frac{1}{4} \frac{\ell^2}{12}$$

$$= \frac{\ell^3}{3\ell} \Big/ 12 + \ldots$$

$$= \frac{\ell^2}{36} + \frac{\ell^2}{48}$$

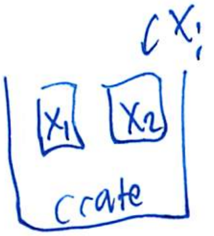
$$= \frac{7\ell^2}{144}$$

whithin + between sections not so clear here

5

Random Sums

- Sums of RV
- shows variance b/w and within clearer



N boxes (RV)

X_i # of items in Box i (IID)

$T = X_1 + \dots + X_n$

$$E[T] = E[\underbrace{E[X_1 + \dots + X_n \mid N]}_{N \cdot E[x]}]$$

if fix N, problem becomes easy

Note
$E[X_i] = E[x]$
$var(X_i) = var(x)$

$$= E[N] \cdot E[x]$$

↑ but N is RV
- substituted in E
- X does not even need to be independent

$$var(T) = E[\underbrace{var(T \mid N)}_{\substack{\text{fixed } N \\ \text{Sum of} \\ Var(T)}}] + var(E[T \mid N])$$

$$= E[N \cdot var(x)] + var(N \cdot E[x])$$

$$= E[N]\, var(x) + (E[x])^2 + var(N)$$

Varability due to RV in a box — within section

Variability in # boxes — between section

then just plug in #

Explaining Law of Total Variance

$$Var(X) = E[var(X|Y)] + var(E[X|Y])$$

conditional mean given info
- estimator given observation

$$\hat{X} = E[X|Y] = f(Y)$$

$$\tilde{X} = X - \hat{X}$$

estimation error

1. X is the sum of 2 RV

$$X = \tilde{X} + \hat{X}$$

2. - uncorrelated
   - estimate and estimation error

So $var(X) = var(\tilde{X}) + var(\hat{X})$ — proved

needs further proof

$$Var(X) = E[var(X|Y)] + var(E[X|Y])$$

7

Have ~~cov(x,y)~~ $var(x+y) = var(x) + var(y) + 2\,cov(x,y)$

↳ expanding the square

$\uparrow$ = to 0 if uncorrelated

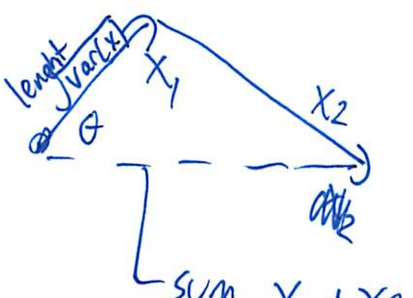
$$E\left[(x - E[x]) + (y - E[y])\right]^2$$

Geometric view of Random Values

$\sqrt{var(x)}$ = lenght of x

look as 2 RV as vectors

Correlation coefficent = cos(angle b/w the two)



Get 3rd side

$$(\text{Lenght}(X_1 + X_2))^2 = (\text{lenght}(X_1))^2 + (\text{lenght}(X_2))^2 + 2\,\text{lenght}(X_1)\,\text{lngh}(X_2)\cos(X_1 X_2)$$

if interperate lengh as $\sqrt{var(x)}$

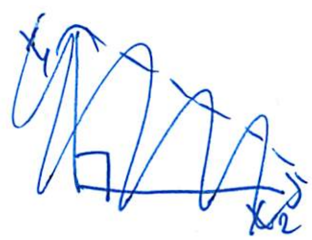
$$var(x_1 + x_2) = var(x_1) + var(x_2) + 2\,cov(x_1 x_2)$$

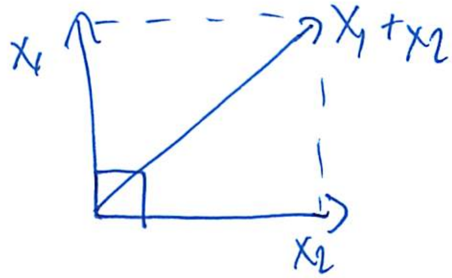
translates to formula of var of sum of 2 RV

8

Uncorrelated = orthagonal

Case of uncorrelated x_1, x_2



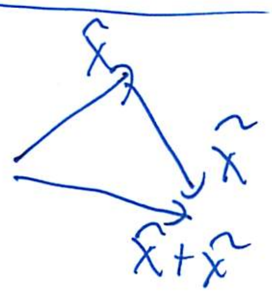


becomes pythagorian theorm

$$(\text{lengh}(x_1 + x_2))^2 = (\text{lengh } x_1)^2 + (\text{lengh } x_2)^2$$

$$\text{var}(x_1 + x_2) = \text{var}(x_1) + \text{var}(x_2)$$

a special case



$$\text{var}(x) = \underbrace{E[\text{var}(x|y)]}_{\text{var estimation error}} + \longrightarrow \underbrace{\text{var}(E[x|y])}_{\text{var error}}$$

Need the overhead to think abstractly
then every thing connects
w/ few principles

Fact Sheet

10/26

Derived Dist (cont.)

if X, Y independent $Z = X + Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(t)\, f_Y(z-t)\, dt$$
$$= \int_{-\infty}^{\infty} f_Y(t)\, f_X(z-t)\, dt$$

convolution

- graphical method
  ↳ if give graphical flip + slide

Iterated Expectations Law

$$E[X] = E[\underbrace{E[X \mid Y]}_{g(Y)}]$$

Law of Total Var

$$Var(X) = E[Var(X \mid Y)] + var(E[X \mid Y])$$

Random Sum of RV

$$Y = X_1 + X_2 + X_3 + \ldots + X_N$$

X_i's iid

N indep. of X_is ← no need to rederive

(2)

$E[Y] = E[N]\,E[X_i]$

$Var(Y) = E[N]\,var(X) + (E[X_i])^2\,var(N)$

<u>Note</u> $E[X \mid Y = y] = f(y) = \#$

$E[X \mid Y] = f(Y) = RV$

Tutorial 6
October 21/22, 2010

1. Let X be a discrete random variable with PMF p_X and let Y be a continuous random variable, independent from X, with PDF f_Y. Derive a formula for the PDF of the random variable $X+Y$.

2. The random variables X and Y are described by a joint PDF which is constant within the unit area quadrilateral with vertices $(0,0)$, $(0,1)$, $(1,2)$, and $(1,1)$. Use the law of total variance to find the variance of $X+Y$.

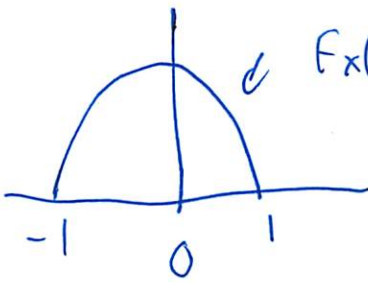
3. (a) You roll a fair six-sided die, and then you flip a fair coin the number of times shown by the die. Find the expected value and the variance of the number of heads obtained.

   (b) Repeat part (a) for the case where you roll two dice, instead of one.

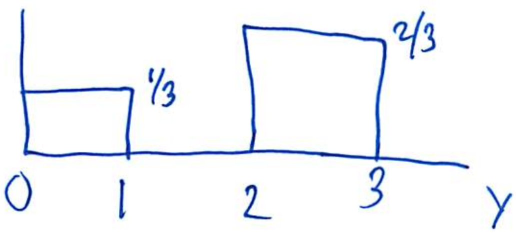
Tutorial 6

10/22

P-Set #2



$f_X(x) = \frac{3}{4}(1-x^2)$

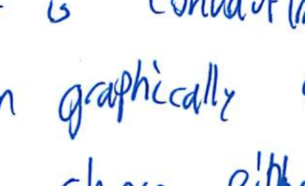


Answer is convolution
Given graphically so flip + slide
Can choose either one

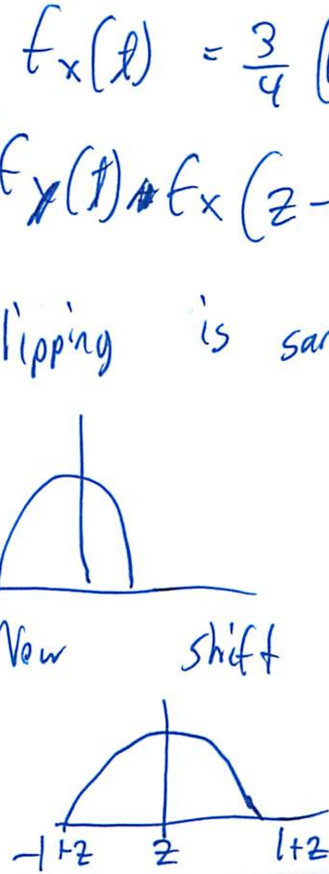
$$f_X(t) = \frac{3}{4}(1-t^2)$$

$$\int f_Y(t)\, f_X(z-t)\, dt$$

flipping is same

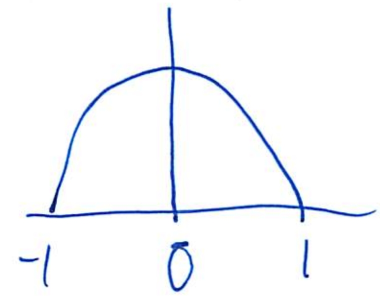


Now shift → by z

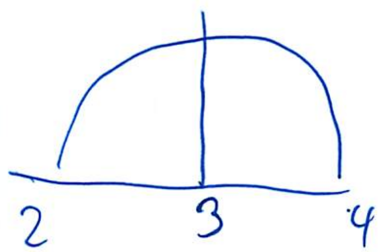


(2)

Multiply and integrate



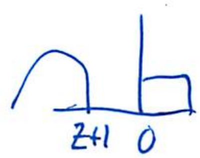
if $z = 0$



if $z = 3$

Need to solve in piecewise way for z from $-\infty$ to ∞

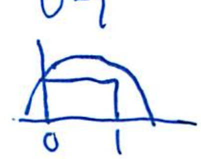
$f_Z(z) = \begin{cases} 0 & z+1 < 0 \\ \int_0^{z+1} \frac{1}{3} \cdot \frac{3}{4}(1-t)^2 & -t \leq z < 0 \\ \int_0^1 \frac{1}{3} \cdot \frac{3}{4}(1-t^2) & \end{cases}$

$\rightarrow$  product 0

$\rightarrow$  product now non 0

as long as sliding have same integral till $1 = z+1$

then intersection only b/w $0-1$

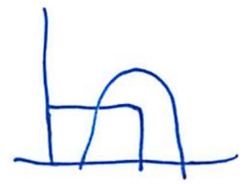


3

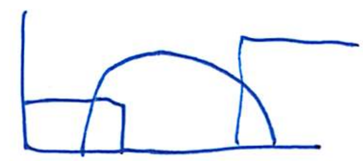
~~then~~

$$\int \quad + \quad \int$$

then dome starts merging

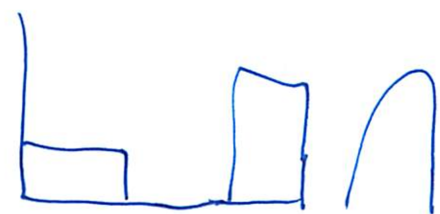


then will intersect both



~~then~~ '' ''

until dome is out



Tutorial #1 $\qquad$ X + Y independent

$$Z = \underset{\substack{\uparrow \\ \text{discrete} \\ P_X}}{X} + \underset{\substack{\uparrow \\ \text{contineous} \\ f_Y}}{Y}$$

PDF of Z?

Use convolution formula

$$f_Z(z) = \underbrace{\sum_t P_X(t) \cdot f_Y(z-t)}_{\substack{\text{don't flip} \\ \text{+ slide}}} = \int f_Y(t)\, P_X(z-t)\, dt$$

determines partition of scenarios

or
- can do either
- whatever is caiser
- first one easier here

4

Prove intuition

Go back to how got convolution

$\downarrow$

derived dist

$\downarrow$

CDF

$$F_Z(z) = P(Z \le z) = P(\underbrace{X + Y}_{\text{want to seperate these}} \le z)$$

total prob theorm

$$= \sum_x P(X + Y \le z \mid X = x) \cdot P_X(x)$$

$$= \sum_x P(\underbrace{x + Y \le z}_{Y \le z - x} \mid X = x) \cdot P_X(x)$$

$$= \sum_x P(Y \le z - x) \cdot P_X(x)$$

$$= \sum_x F_Y(z - x) P_X(x)$$

(5)

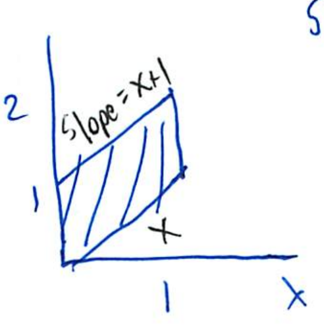
PDF $f_Z(z) = \frac{dF_Z(z)}{dz}$

$= \frac{d}{dz} \sum_x F_Y(z-x) \cdot p_X(x)$

$= \sum_x \frac{d}{dz} \Big(\underbrace{F_Y(z-x)}_{\text{take deriv} \hookrightarrow \text{the PDF}} \underbrace{p_X(x)}_{\text{constant}} \Big)$

Answer $= \sum_x f_Y(z-x) \cdot p_X(x)$

2.



Similar to P-set
- assume result
- ask for more

$var(x) = \frac{1}{12}$

$var(X+Y) = ?$

$E[var(X+Y|\_\_)] + var(E[X+Y|\_\_])$

which to condition on does not make much of a difference

if condition on $X \to$ uniform from $0 \to 1$

$Y \to$ piecewise $0 \to x$, $x+1 \to 1$

If they are no independent like last p set $Z = X - Y$ took CDF draw joint PDF

$Y \leq Z - X$

Shift by Z - translate line, calculate area under curve

⑥

So condition on X

$$E[\mathrm{var}(X+Y|X)] + \mathrm{var}(E[X+Y|X])$$

$$E[X+Y|X=x] = E[x+Y|X=x]$$
$$= x + E[Y|X=x)$$
$\quad$ * Y uniform b/w X and X+1
$$= X + X + .5$$
$$= 2X + .5$$

$$E[X+Y|X] = 2X+.5$$

$$\mathrm{var}(x+y|x) =$$

$$\mathrm{var}(X+Y|X=x) = \mathrm{var}(x+Y|X=x)$$
$$= \mathrm{var}(y|X=x)$$
$$= \frac{1}{12}$$

$$\mathrm{var}(X+Y|X) = \frac{1}{12}$$

$$\mathrm{var}(x+y) = \cancel{E[2X+.5]}\ E\left[\frac{1}{12}\right] + \mathrm{var}(2X+.5)$$

7

$$= \frac{1}{12} + 4\,\text{var}(x)$$
$$= \frac{5}{12}$$

~~can't do~~

if were ind. $E[X]E[Y]$ etc

Avoid PDF - gets messy calculations

---

3. Die roll

N

$$P_N(n) = \begin{cases} \frac{1}{6} & \text{for } n = 1,2,3,4,5,6 \\ 0 & \text{else} \end{cases}$$

result of die roll is # times to flip coin

Y = # heads in n tosses

$E[Y] = ?$

$\text{var}(Y) = ?$

Use sum of RV

- where is sum here?
- sum of bernouli RV

$$X_i = \begin{cases} 1 & \text{if toss = heads} \\ 0 & \text{if toss = tails} \end{cases}$$

8

$$Y = X_1 + X_2 + \dots + X_n$$

~~M~~ ~~Move~~

$$E[Y] = E[N]\, E[X_i]$$
$$= 3.5 \cdot .5$$
$$= \frac{7}{4}$$

$$Var(Y) = E[N]\, var(X_i) + (E[X_i])^2\, var(N)$$
$$= \frac{7}{2} \cdot \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \cdot \frac{35}{12}$$
$$= \frac{77}{48}$$

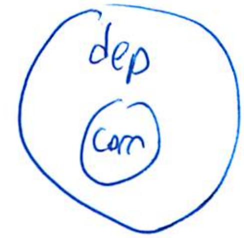
Clarification

independent $\rightarrow$ uncorrelated

uncorrelated $\overset{?}{\rightarrow}$ independent

~~dep~~.
correlated $\rightarrow$ dependent

dependent $\overset{?}{\rightarrow}$ correlated



$Var(X+Y) = Var(X) + Var(Y)$ if uncorrelated actually, and of course ind.

(9)

Uncorrelated

$$\mathrm{cov}(x, y) = 0$$

$$\mathrm{var}(x+y) = \mathrm{var}(x) + \mathrm{var}(y) \pm 2\,\mathrm{cov}(x, y)$$

~~Uncorrelated~~

Chap 4 More RVs

10/23

- a few more advanced RV topics
  - deriving the distribution of a function of 1 or more RV
  - dealing w/ sum of independent RV
    - inc where # of RVs is random
  - quantifying the degree of dependence b/w 2 RV
  - transformations
  - convolutions
  - need for more advanced wory

4.1 Derived Distributions

$Y = g(X)$ ← continuous RV, given PDF

Y ← want PDF

2 step approach

1. Calculate the CDF F_Y of Y

$$F_Y(y) = P(g(X) \le y) = \int_{\{x \mid g(x) \le y\}} f_X(x)\, dx$$

2. Differentiate to obtain PDF

$$f_Y(y) = \frac{dF_Y}{dy}(y)$$

(2)

example

X = uniform $[0,1]$

$Y = \sqrt{X}$

So for every $y \in [0,1]$ ~~we~~ we have

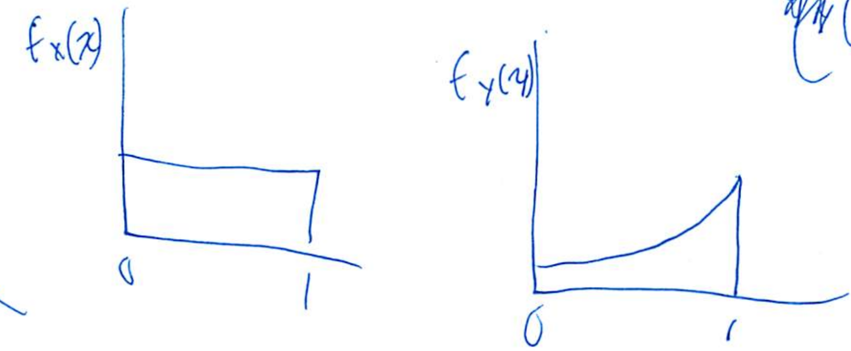
$$F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2) = y^2$$

↑ the CDF

Now differenitate

$$f_Y(y) = \frac{dF_Y}{d}(y) = \frac{dy^2}{dy} = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

he trying wish book would show



example 2

~~A.~~ Traveling 180 miles b/w 30-60 mph

PDF of trip

X = speed

Y = duration $= g(x) = \frac{180}{X}$

$$P(Y \le y) = P\left(\frac{180}{X} \le y\right) = P\left(\frac{180}{y} < X\right)$$

PDF of X $\quad f_X(x) = \begin{cases} 1/30 & \text{if } 30 \le x \le 60 \\ 0 & \text{else} \end{cases}$

③

CDF of x

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 30 \\ (x-30)/30 & \text{if } 30 \leq x \leq 60 \\ 1 & \text{if } 60 \leq x \end{cases}$$

$$F_Y(y) = P\left(\frac{180}{Y} \leq X\right)$$

$$= 1 - F_X\left(\frac{180}{Y}\right)$$

$$= \begin{cases} 0 & \text{if } Y \leq 180/60 \\ 1 - \dfrac{\frac{180}{Y} - 30}{30} & \text{if } \frac{180}{60} \leq Y \leq \frac{180}{30} \\ 1 & \text{if } 180/30 \leq Y \end{cases}$$

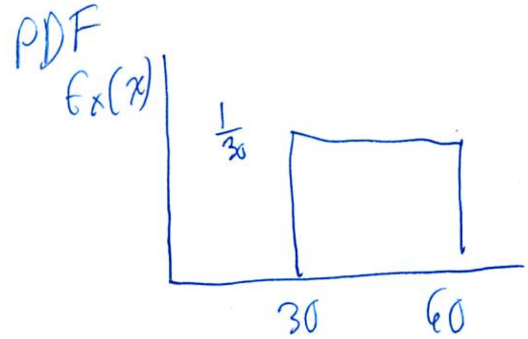
$$= \begin{cases} 0 & \text{if } Y \leq 3 \\ 2 - (6/y) & \text{if } 3 \leq Y \leq 6 \\ 1 & \text{if } 6 \leq Y \end{cases}$$

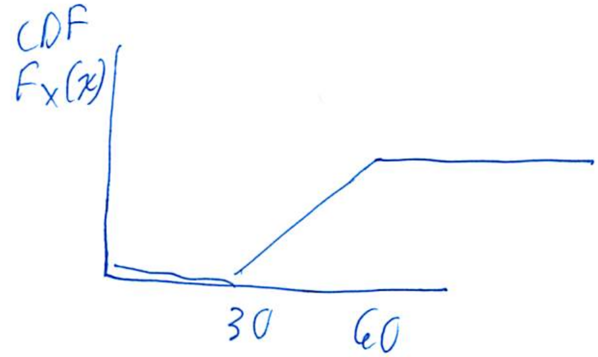
differeitate to PDF

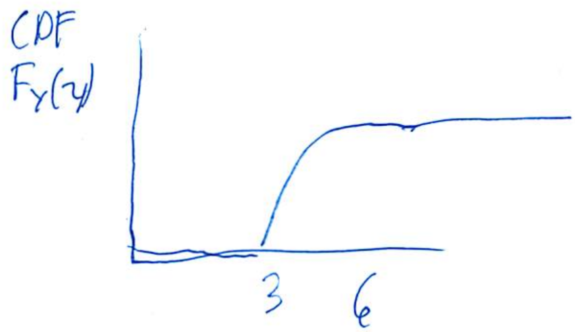
$$f_Y(y) = \begin{cases} 0 & \text{if } Y < 3 \\ 6/Y^2 & \text{if } 3 < Y < 6 \\ 0 & \text{if } 6 < Y \end{cases}$$

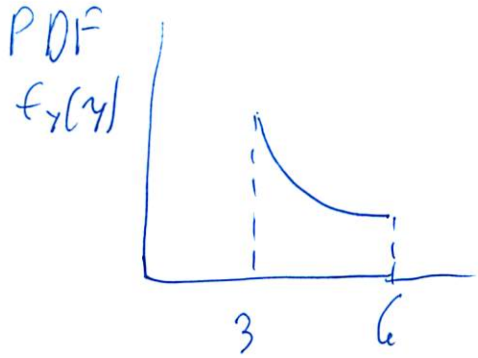
example 2 graphs

$$Y = g(x) = \frac{180}{x}$$









example 3

$$Y = g(x) = X^2$$

$$F_Y(y) = P(Y \le y)$$

$$= P(X^2 \le y)$$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

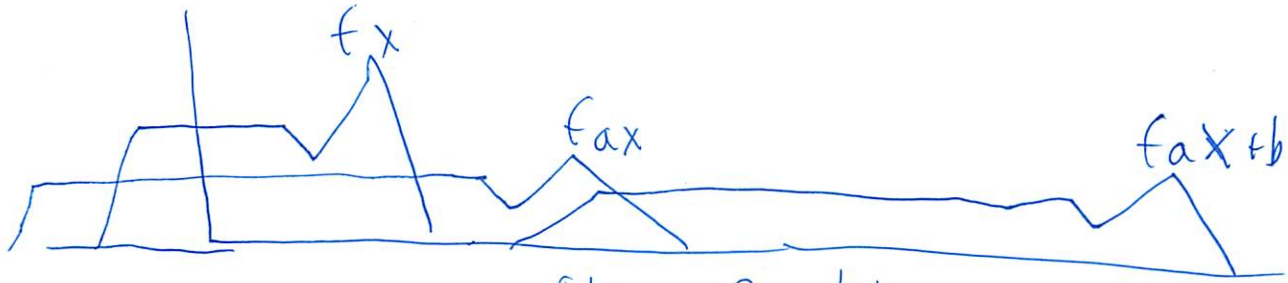
differentiate (use chain rule)

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) \quad y \ge 0$$

(5)

The Linear Case

- when Y is a linear function of X



here $a = 2$ stretch
$b = 5$ move → (translate)

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)} \qquad a \neq 0$$

if $a < 0$ ($-$) then "Flip" f_X

to verify

$$F_Y(y) = P(Y \leq y)$$
$$= P(aX + b \leq y)$$
$$= P\left(X \leq \frac{y-b}{a}\right)$$
$$= F_X\left(\frac{y-b}{a}\right)$$

differenitate

$$f_Y(y) = \frac{dF_Y}{dy}(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

example linear function of exponential RV

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$$

$Y = aX + b$ $\quad f_Y(y) = \begin{cases} \frac{\lambda}{|a|} e^{-\lambda(y-b)/a} & \text{if } (y-b)/a \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Y need not be exp

6

example 2: linear function of normal RV is normal

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

$Y = aX + b$ $\quad f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

$$= \frac{1}{|a|} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\left(\frac{y-b}{a}-\mu\right)^2 / 2\sigma^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,|a|\sigma} \exp\left\{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}\right\}$$

Monotonic case

can generalize linear case to case where g is monotonic

$Y = g(X)$ where g is strictly monotonic over interval I

a) $g(x) < g(x')$ for all $x, x' \in I$

satisfying $x < x'$ (monotonically increasing)

b) $g(x) > g(x')$ for all $x, x' \in I$

satisfying $x < x'$ (monotonically decreasing)

assume g is differentiable

$+$ or 0 in increasing case

$-$ or 0 in decreasing case

⑦

Can be inverted if $y = g(x)$

$x = h(y)$

- basically h is inverse of g?

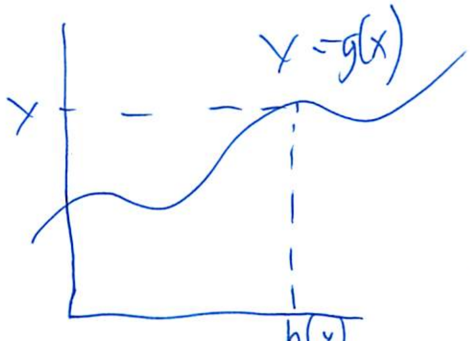
$$g(x) = e^{ax} \qquad h(y) = \frac{\ln y}{a}$$

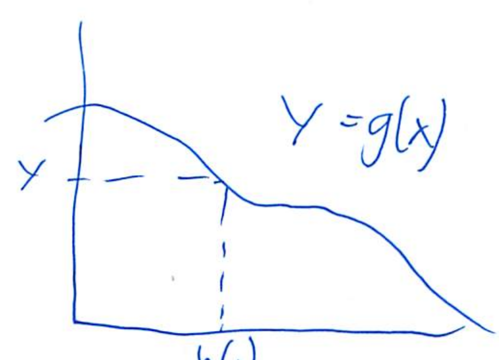
$$g(x) = ax + b \qquad h(y) = \frac{y-b}{a}$$

So PDF of Y

$$f_Y(y) = f_X(h(y)) \left| \frac{dh}{dy}(y) \right|$$

(skipping verification)





Event $\{X \le h(y)\}$ $\{g(X) \le y\}$ } same

Event $\{X \ge h(y)\}$ $\{g(x) \le y\}$

8

So w/ example earlier w/ driving

$$f_X(h(y)) = \frac{1}{30} \qquad \left|\frac{dh}{dy}(y)\right| = \frac{180}{y^2}$$

So $f_Y(y) = \frac{1}{30} \cdot \frac{180}{y^2} = \frac{6}{y^2}$ ← same as before

Function of 2 RV

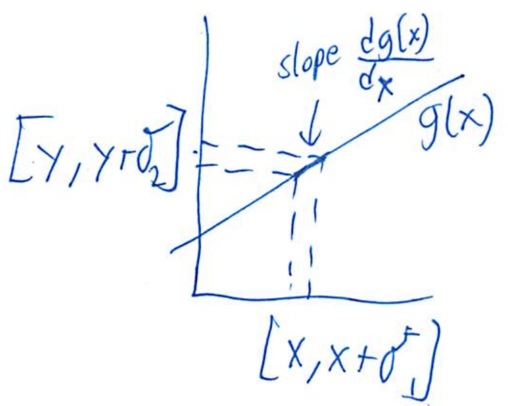
- Same process
- example

2 archers shoot at target

X, Y = distance from center $[0,1]$ evenly distributed

Z = distance of losing shot = $\max\{X, Y\}$

$$P(X \le z) = P(Y \le z) = z$$



$$\frac{\delta_2}{\delta_1} \approx \frac{dg(x)}{dx}$$

$$\frac{\delta_1}{\delta_2} = \frac{dh(x)}{dy}$$

events same $\begin{cases} \{x \le X \le x + \delta_1\} \\ \{y \le Y \le y + \delta_2\} \end{cases}$

(9)

$$f_Y(y)\,\delta_2 \approx P(y \le Y \le y+\delta_2)$$
$$= P(x \le X \le x+\delta_1)$$
$$\approx f_X(x)\,\delta_1$$

Move δ_1 to left $\to$ use $\frac{\delta_2}{\delta_1}$ formula

$$f_Y(y)\frac{dg}{dx}(x) = f_X(x)$$

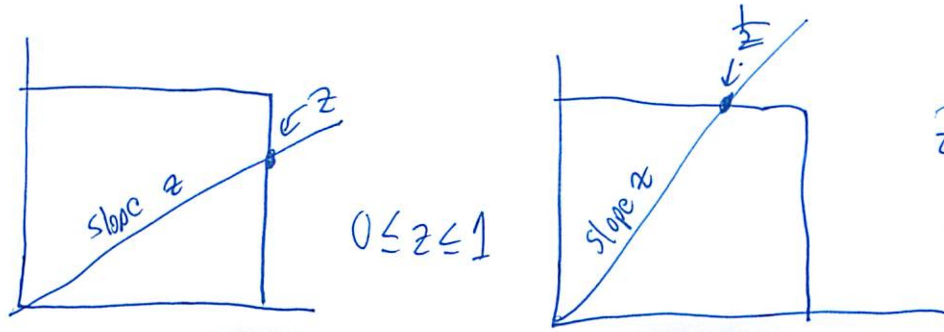
Or move δ_2 to right side $\frac{\delta_1}{\delta_2}$

$$f_Y(y) = f_X(h(y))\,\frac{dh}{dy}(y)$$

Since X, Y independent $\quad z \in [0,1]$

$$F_Z(z) = P(\max\{X, Y\} \le z)$$
$$= P(X \le z,\, Y \le z)$$
$$= P(X \le z)\,P(Y \le z)$$
$$= z^2$$

CDF



$0 \le z \le 1$ $\quad z = \frac{Y}{X}$ $\quad z \ge 1$

10

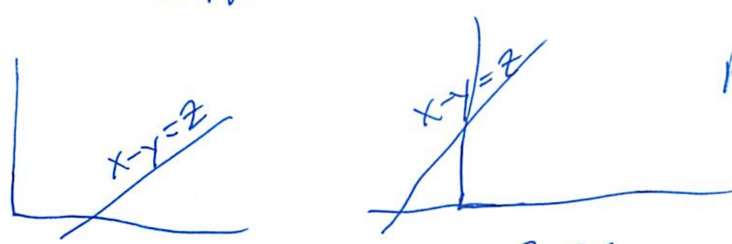
Differentiating

$$f_Z(z) = \begin{cases} 2z & \text{if } 0 \leq z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- (oh that is what we did in class)

Romeo + Juliet example

- did in recitation



PDF of lateness

$z<0$

$z \geq 0$

- is 2 sided exponential PDF = Laplace PDF

Sum of Independent RV = Convolution

$Z = X + Y$

independent

say X, Y are discrete

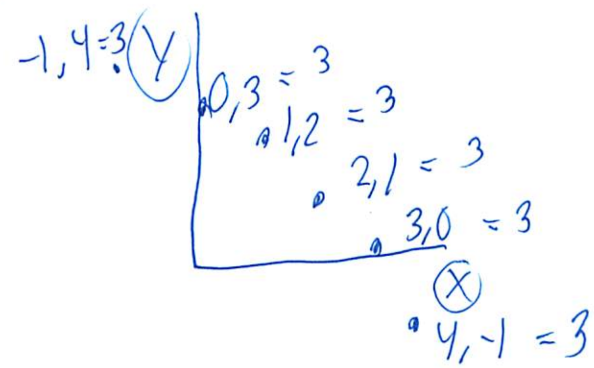
$$p_Z(z) = P(X+Y=z)$$

$$= \sum_{\{(x,y) | x+y=z\}} P(X=x, Y=y)$$

$$= \sum_x P(X=x, Y=z-x)$$

$$= \sum_x p_X(x)\, p_Y(z-x)$$

(11)



if ~~X~~ X, Y contineous, want to find PDF of Z

- so find joint PDF of X and Z ← ; typo Y
- integrate ~~dx~~ to find PDF of Z

$$P(Z \le z \mid X = x) = P(X + Y \le z \mid X = x)$$

$$= P(x + Y \le z \mid X < x)$$

$$= P(x + Y \le z) \quad \leftarrow \text{since } X, Y \text{ independent}$$

$$= P(Y \le z - x)$$

differentiate both sides w/ respect to z

$$f_{Z|X}(z \mid x) = f_Y(z - x)$$

so w/ multiplication rule

$$f_{X,Z}(x, z) = f_X(x)\, f_{Z|X}(z \mid x)$$

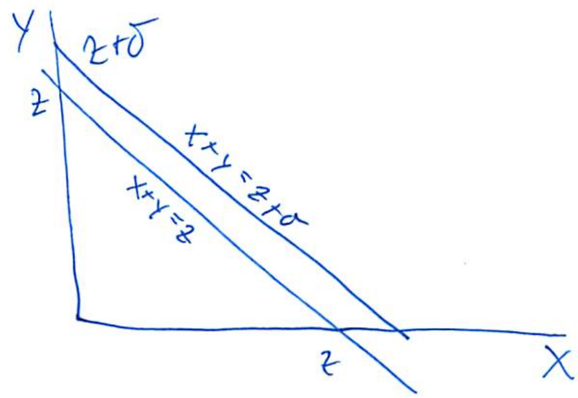
$$= f_X(x)\, f_Y(z - x)$$

So $f_Z(z) = \int_{-\infty}^{\infty} f_{X,Z}(x, z)\, dx$

$$= \int_{-\infty}^{\infty} f_X(x)\, f_Y(z - x)\, dx$$

{very abstract, hard to understand

(12)



for small δ → probablity of strip

$P(z \le X+Y \le z+\delta) \approx f_Z(z)\delta$

(is this graphically just add?)

(flip and slide?) - described fully in lecture

↳ yeah

$z > 0$ →

$z < 0$ ←

integrate the products of the 2 plots

4.2 Covariance + Correlation

streght + ~~dela~~ direction of relationship b/w 2 RV

Used in Chap 8+9 estimation

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

$= 0$ ← uncorrelated

$\approx .5$ tend to have same sign 

$\approx -.5$ " " " opposite "

$$= E[XY] - E[X]E[Y]$$

(13)

$\mathrm{cov}(X,X) = \mathrm{var}(X)$

$\mathrm{cov}(X, aY+b) = a\,\mathrm{cov}(X,Y)$

$\mathrm{cov}(X, Y+Z) = \mathrm{cov}(X,Y) + \mathrm{cov}(X,Z)$

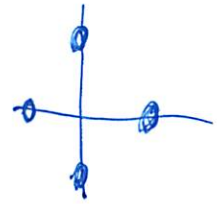
if X, Y independent

$$E[XY] = E[X]E[Y]$$

$$\mathrm{cov}(X,Y) = 0$$

example uncorrelated but not ind.

X, Y take $(1,0)$ $(-1,0)$ each w/ $p = 1/4$
$(0,1)$ $(0,-1)$



so $XY = 0$ so $E[XY] = 0$

so $\mathrm{cov}(X,Y) = E[XY] - E[X]E[Y] = 0$

but not indep. since non-zero X means $Y = 0$

generally

$$E[X \mid Y = y] = E[X] \text{ for all } y$$

so assuming X, Y discrete

$$E[XY] = \sum_y y\, p_Y(y) E[X \mid Y = y]$$

$$= E[X] \sum_y y\, p_Y(y)$$

$$= E[X]E[Y]$$

argument for continuous similar

(14)

Correlation coefficent ρ

$$\rho = \frac{cov(X,Y)}{\sqrt{var(X)\,var(Y)}}$$

↳ like a normalized cov

↳ result $[-1,1]$

$\rho > 0$ $\left.\begin{matrix} X - E[X] \\ Y - E[Y] \end{matrix}\right\}$ tend to have the same sign

$\rho < 0$ " $\}$ tend to have opposet sign

as ρ gets closer to $-1, 1$ correlation becomes stronger

when $\rho = 1, -1$ then

$$\cancel{Y}\ Y - E[Y] = c(X - E[X])$$

example

n tossed coins $\quad$ $\left.\begin{matrix} X = \\ Y = \end{matrix}\right\}$ # of heads

$\rho = p$ head

$X + Y = n$

$E[X] + E[Y] = n$

$$X - E[X] = -(Y - E[Y])$$

$\uparrow c = -1$

$$cov(X,Y) = E[(X - E[X])(Y - E[Y])]$$
$$= -E[(X - E[X])^2]$$
$$= -var(X)$$

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{var(X)\,var(Y)}} = \frac{-var(X)}{\sqrt{var(X)\,var(Y)}} = -1$$

(15)

Variance of Sum of RV

$$Var(X_1 + X_2) = Var(X_1) + var(X_2) + 2\,cov(X_1, X_2)$$

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} var(X_i) + \sum_{\{(i,j) \mid i \neq j)\}} cov(X_i, X_j)$$

skipping proof

4.3 Conditional Expectation + Variance Revisited

law of iterated expectations

- reformulation of total expectation theorm
- biased coin $X = \#$ heads

  $Y =$ prob of heads

  toss n times

So $E[X \mid Y = y] = ny$

$E[X \mid Y] = nY$ ← a RV

has expectation $E[E[X \mid Y]]$

$$E[E[X \mid Y]] = \begin{cases} \sum_{y} E[X \mid Y = y]\, p_Y(y) & Y \text{ discrete} \\ \int_{-\infty}^{\infty} E[X \mid Y = y]\, f_Y(y)\, dy & Y \text{ continuous} \end{cases}$$

$$\boxed{E[E[X \mid Y]] = E[X]}$$

(16) back to the example

$$E[Y] = \frac{1}{2} \quad Y \text{ distributed evenly } [0,1]$$

uniformally

$$E[X] = E[E[X \mid Y]] = E[nY] = nE[Y] = \frac{n}{2}$$

the expected value of all of the expected values

Conditional Expectation as an Estimator

$$\hat{X} = E[X \mid Y]$$

estimator of X given Y

estimation error $\tilde{X} = \hat{X} - X$

- RV satisfying

$$E[\tilde{X} \mid Y] = E[\hat{X} - X \mid Y] = E[X \mid Y] = \hat{X} - \hat{X} = 0$$

- so identically 0

$$E[\tilde{X} \mid Y = y] = 0$$

- law of iterated expectations

$$E[\tilde{X}] = E[E[\tilde{X} \mid Y]] = 0$$

- aka estimation error has no bias

- also $\hat{X}$ and $\tilde{X}$ are uncorrelated

$$E[\hat{X}\tilde{X}] = E[E[\tilde{X}\hat{X} \mid Y]] = E[\hat{X}E[\tilde{X} \mid Y]] = 0$$

Since $\hat{X}$ determined by Y

$$E[\hat{X}\tilde{X} \mid Y] = \hat{X}E[\tilde{X} \mid Y] = 0$$

(17)

Can check that uncorrelated

$$\text{Cov}(\hat{X}, \tilde{X}) = E[\hat{X}\tilde{X}] - E[\hat{X}]E[\tilde{X}]$$
$$= 0 - E[X]\cdot 0$$
$$= 0$$

So that means

$$\text{var}(X) = \text{var}(\tilde{X}) + \text{var}(\hat{X})$$

Conditional Variance

$$\text{Var}(X|Y) = E[(X - E[X|Y])^2 | Y] = E[\tilde{X}^2 | Y]$$

So

$$\text{var}(X|Y=y) = E[\tilde{X}^2 | Y=y]$$

and since $E[\tilde{X}] = 0$

$$\text{var}(\tilde{X}) = E[\tilde{X}^2] = E[E[\tilde{X}^2|Y]]$$
$$= E[\text{var}(X|Y)]$$

$$\text{var}(X) = \text{var}(\tilde{X}) + \text{var}(\hat{X})$$

$$\boxed{\text{var}(X) = E[\text{var}(X|Y)] + \text{var}(E[X|Y])}$$

law of total variance

(18) # 4.4 Transforms

transforms associated w/ a RV
alternate representation of probability law
covienient for some math manipulation
aka moment generating function

$$M_X(s) = E[e^{sX}]$$

↑ s = scalar parameter

$$M(s) = \sum_x e^{s\cancel{x}} p_X(x)$$

← when cleared RV X from context
So when discrete RV

$$= \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx$$

example

$$p_X(x) = \begin{cases} 1/2 & x = 2 \\ 1/6 & x = 3 \\ 1/3 & x = 5 \end{cases}$$

$$M(s) = E[s^{sx}] = \frac{1}{2} e^{2s} + \frac{1}{6} e^{3s} + \frac{1}{3} e^{5s}$$

- basically the Laplace transform of its PDF
  - but positive

- Z transform is $M(z) = \sum_x z^x p_X(x)$

(19) From transforms to moments

Moment of RV easy to generate w/ transform ← aka expected value

take derivitive of both sides of

$$M(s) = \int_{-\infty}^{\infty} e^{sx} f_x(x)\, dx$$

to get

$$\frac{d\, M(s)}{ds} = \frac{d}{ds} \int_{-\infty}^{\infty} e^{sx} f_x(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{d}{ds} e^{sx} f_x(x)\, dx$$

$$= \int_{-\infty}^{\infty} x e^{sx} f_x(x)\, dx$$

holds for all values of s

Special case $s=0$

$$\frac{d}{ds} M(s) \Big|_{s=0} = \int_{-\infty}^{\infty} x f_x(x)\, dx = E[x]$$

example

builds on last example

$$E[X] = \frac{d}{ds} M(s) \Big|_{s=0}$$

$$= \frac{1}{2} \cdot 2e^{2s} + \frac{1}{6} \cdot 3e^{3s} + \frac{1}{3} 5e^{5s} \Big|_{s=0}$$

$$= \frac{1}{2} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{3} \cdot 5$$

$$= \frac{19}{6}$$

20

Then $E[X^2] = \frac{1}{2} \cdot 4e^{2s} + \frac{1}{6} \cdot 9e^{3s} + \frac{1}{3} \cdot 25e^{5s} \Big|_{s=0}$

$$= \frac{1}{2} \cdot 4 + \frac{1}{6} \cdot 9 + \frac{1}{3} \cdot 25$$

$$= \frac{71}{6}$$

Also

$$M_X(0) = E[e^{0X}] = E[1] = 1$$

if X is nonnegitive integers then

$$\lim_{s \to -\infty} M_X(s) = P(X=0)$$

Inversion of Transfomations

- $M_X(s)$ can be inverted
- ie can be used to find prob. law of X
- Uniquely determines the CDF of X
- assuming $M_X(s)$ is finite for all s in interval $[a, a]$ where a is positive
- hard to do
- can also recover PMF/PDF of RV from its transfoms
  - but very hard to do "pattern matching"

(21)

Sum of Independent RVs

- transform methods useful when have sums of RVs
- addition of ind. RV corresponds to multiplication of transforms

$$Z = X + Y$$

$$M_Z(s) = E[e^{sZ}] = E[e^{s(X+Y)}] = E[e^{sX}e^{sY}]$$

indep. RV
for any fixed s

$$= E[e^{sX}]E[e^{sY}] = M_X(s)\,M_Y(s)$$

for any # of X_is

$$Z = X_1 + \dots + X_n$$

$$M_Z(s) = M_{X_1}(s) \cdots M_{X_n}(s)$$

example Binomial

$$M_{X_i}(s) = (1-p)e^{0s} + pe^{1s} = 1 - p + pe^s \quad \text{for all } i$$

$$Z = X_1 + \dots + X_n$$

$$M_Z(s) = (1 - p + pe^s)^n$$

Transforms for Common Discrete RVs

Bernoulli (p) $\quad (k = 0, 1)$

$$p_X(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \end{cases} \qquad M_X(s) = 1 - p + pe^s$$

Binomial (n, p) $\quad (k = 0, 1, \dots, n)$

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad M_X(s) = (1 - p + pe^s)^n$$

22

Geometric (p) $k = 1, 2, 3, \dots$

$p_X(k) = p(1-p)^{k-1}$ $M_X(s) = \dfrac{pe^s}{1-(1-p)e^s}$

Poisson (λ), $k = 0, 1, \dots$

$p_X(k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$ $M_X(s) = e^{\lambda(e^s-1)}$

Uniform (a,b) $(k = a, a+1, \dots, b)$

$p_X(k) = \dfrac{1}{b-a+1}$ $M_X(s) = \dfrac{e^{sa}(e^{s(b-a+1)}-1)}{(b-a+1)(e^s-1)}$

Transforms for Common Contineous RVs

Uniform (a,b) $(a \le x \le b)$

$f_X(x) = \dfrac{1}{b-a}$ $M_X(s) = \dfrac{e^{sb}-e^{sa}}{s(b-a)}$

Exponential (λ) $(x \ge 0)$

$f_X(x) = \lambda e^{-\lambda x}$ $M_X(s) = \dfrac{\lambda}{\lambda - s}$ $s < \lambda$

Normal (μ, σ^2) $(-\infty < x < \infty)$

$f_X(x) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$ $M_X(s) = e^{(\sigma^2 s^2/2)+\mu s}$

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Transforms Associated w/ Joint Distributions

- each one associated w/ a transform $M_X(s)$ $M_Y(s)$
  - marginal distributions
  - do not convey dependence

need a multivariate transform

have n random variables $X_1, \ldots, X_n$

$s_1, \ldots, s_n$ be scalar free parameters

$$M_{X_1 \ldots X_n}(s_1, \ldots, s_n) = E[e^{\underbrace{s_1 X_1 + \ldots + s_n X_n}_{\text{all superscript to } e}}]$$

inversion property applies

4.5 Sum of Random # of Independent RVs

$Y = X_1 + \ldots + X_n$

N = non-negitive integer

$X_1, X_2, \ldots X_n$ independent identically distributed RVs

if $N=0$ $Y=0$

$E[X]$ $var(X)$ are common mean, variance for all X_i

Want $E[Y \mid N=n]$ etc

So first fix a n

$$E[Y \mid N=n] = E[X_1 + \ldots + X_n \mid N=n]$$

$$= E[X_1 + \dots + X_n \mid N = n]$$
$$= E[X_1 + \dots + X_n]$$
$$= n\,E[X]$$

true for every nonnegitive integer n

$$E[Y \mid N] = N\,E[X]$$

law of iterated expectations

$$E[Y] = E[E[Y \mid N]] = E[N\,E[X]]$$
$$= E[N]\,E[X]$$

$$\mathrm{var}(Y \mid N = n) = \mathrm{var}(X_1 + \dots + X_n \mid N = n)$$
$$= \mathrm{Var}(X_1 + \dots + X_n)$$
$$= n\,\mathrm{var}(X)$$

now law of total variance

$$\mathrm{var}(Y) = E[\mathrm{var}(Y \mid N)] + \mathrm{var}(E[Y \mid N])$$
$$= E[N\,\mathrm{var}(X)] + \mathrm{var}(N\,E[X])$$
$$= E[N]\,\mathrm{var}(X) + (E[X])^2\,\mathrm{var}(N)$$

calculation of transformation similar

$$E[e^{sY} \mid N = n] = E[e^{sX_1} \dots e^{sX_N} \mid N = n]$$
$$= E[e^{sX_1} \dots e^{sX_n}]$$
$$= E[e^{sX_1}] \dots E[e^{sX_n}]$$
$$= (M_X(s))^n$$

(25)

$M_X(s)$ is transformation associated with X_i for each i,
Law of iterated expectations

$$M_Y(s) = E[e^{sY}] = E[E[e^{sY} | N]] = E[(M_X(s))^N]$$
$$= \sum_{n=0}^{\infty} (M_X(s))^n p_N(n)$$

Use observation

$$(M_X(s))^n = e^{\log (M_X(s))^n} = e^{n \log M_X(s)}$$

have

$$M_Y(s) = \sum_{n=0}^{\infty} e^{n \log M_X(s)} p_N(n)$$

$$= M_N(\log M_X(s))$$

Chap over
skipped a bunch of examples at end

8c

Now multiply and integrate

- product depends on z

I had messed up a lot of the variables - OH made it clear

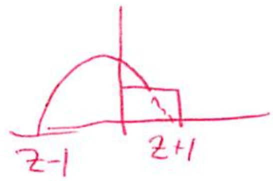
$$f_Z(z) = \begin{cases} 0 & z+1 \le 0,\ z \le -1 \\ \text{take product + integrate} & \\ \int_0^{z+1} \frac{1}{3} \cdot \frac{3}{4}(1-t^2)\,dt & \\ \int_0^{z+1} \frac{1}{3} \cdot \frac{3}{4}(1-(z-t)^2)\,dt & z+1 \le 1,\ z \le 2,\ z+1 \ge 0,\ -1 \le z \le 0 \\ \int_0^1 \frac{1}{3} \cdot \frac{3}{4}(1-(z-t)^2)\,dt & 0 \le z,\ z+1 \le 2 \\ \int_{z-1}^1 \frac{1}{3} \cdot \frac{3}{4}(1-(z-t)^2)\,dt + \int_2^{z+1} \frac{2}{3} \cdot \frac{3}{4}(1-(z-t)^2)\,dt & \end{cases}$$

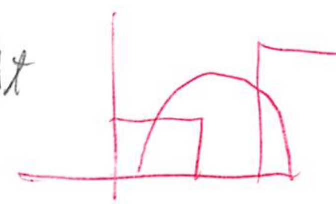
express as since flipping

intersection

slides in

no intersection





integrate

piecewise answer as function of z

assumption $f_X(t) = \frac{3}{4}(1-t^2)$

OH #2

P-set 6

$$\int_{-\infty}^{\infty} f_X(t)\, f_Y(z-t)\, dt$$

or

$$\boxed{\int_{-\infty}^{\infty} f_Y(t)\, f_X(z-t)\, dt}$$

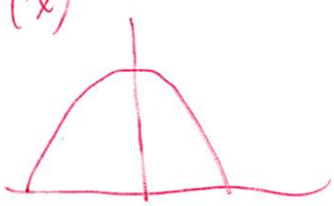
Convolution can be written each way

integration variable integrating over will dissapear

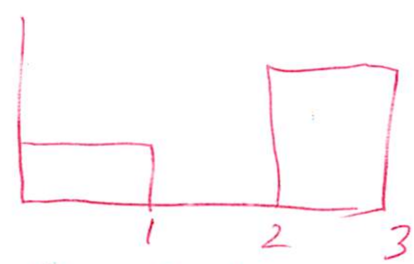
z is like constant
want PDF of z
ans in terms of z

so

$f_X(x)$



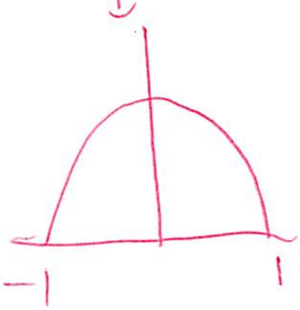
$f_Y(y)$



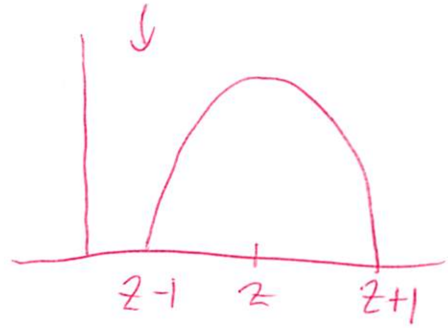
f_Y flip $f(-t)$
slide $f_Y(z-t)$

flip + slide this easier

$f_X(-t)$



$f_X(z-t)$



$1 \to k$

3. $\_\ \_\ \_\ \_\ \_\ \_$ n tosses

X_i = # of tosses that result in i

a) correlated

b) $cov(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2]$

$E[X_1] = \frac{n}{k}$ binomial

$E[X_2] = \frac{n}{k}$

but what is $E[X_1 X_2]$?

- could do total expectation theorm
- how did we get this?
  - proof of Bernouli
  - repeat the proof
  - used indicator RVs

X_2 = # of 2s (# sucesses where sucess is a 2)

$X_1 = A_1 + A_2 + \dots + A_n$

$A_i = \begin{cases} 1 & \text{if ith roll} = 1 \\ 0 & \text{else} \end{cases}$

- just 0 and 1, easy to work with

$B_i = \begin{cases} 1 & \text{if ith roll} = 2 \\ 0 & \text{else} \end{cases}$

$\boxed{E[A_i B_j] = E[A_i]E[B_j]}$

$= E[(A_1 + A_2 + \dots + A_n)(B_1 + \dots + B_n)]$

product nice - always a 1 or 0

$= E[\sum_{i,j} A_i B_j]$ linearty of expectation

$$= \sum_{i,j} E[A_i B_j]$$

just calculate this

$$\frac{n(n-1)}{k^2}$$

So

$$Cov(X_1, X_2) = \frac{n(n-1)}{k^2} - \frac{n^2}{k^2}$$

$$= \frac{n^2 - n - n^2}{k^2}$$

$$= -\frac{n}{k^2}$$

5.04. N members

$$p_N(n) = p^{n-1} \qquad n = 1, 2, 3, \ldots$$

each member attends w/ prob q

$$f_M(m) = \lambda e^{-\lambda m} \qquad m \; x$$

a) Bernouli Something

Indicator RV

$X_i = 1$ if ith wombat shows up $= \begin{cases} 1 & \text{w/ prob } q \\ 0 & \text{" " } 1-q \end{cases}$

$\quad\; 0$ else

$$Y = X_1 + X_2 + \ldots + X_n$$

number of indicator variables
— sum of random # of RV

$$E[Y] = E[X_i] \cdot E[N]$$

the indicator

...

b) Get total \$ brought to meeting

- Sum of \$ people who showed up to wombat meeting brought

$$Z = M_1 + M_2 + \ldots\ M_Y$$

new RV

from part a

just need $E[Y]$

and $var(Y)$

Michael Plasmeier

7/10

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Problem Set 6
Due October 27, 2010

1. Random variables X and Y are distributed according to the joint PDF

$$f_{X,Y}(x, y) = \begin{cases} ax, & \text{if } 1 \le x \le 2 \text{ and } 0 \le y \le x, \\ 0, & \text{otherwise.} \end{cases}$$

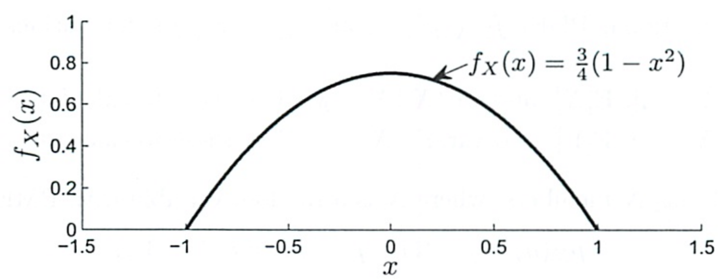
   (a) Evaluate the constant a.

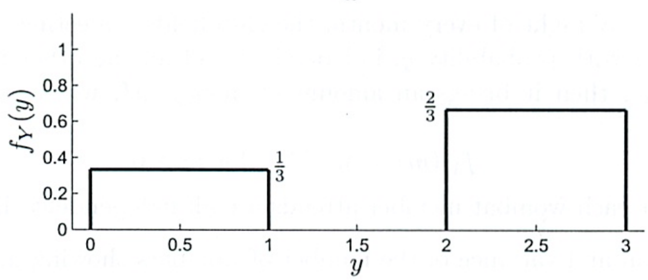
   (b) Determine the marginal PDF $f_Y(y)$.

   (c) Determine the conditional expectation of $1/X$ given that $Y = 3/2$.

   (d) Random variable Z is defined by $Z = Y - X$. Determine the PDF $f_Z(z)$.

2. Let X and Y be two independent random variables. Their probability densities functions are shown below.





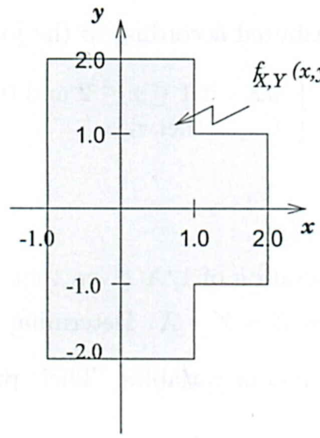
   Let $Z = X + Y$. Determine $f_Z(z)$.

3. Consider n independent tosses of a k-sided fair die. Let X_i be the number of tosses that result in i.

   (a) Are X_1 and X_2 uncorrelated, positively correlated, or negatively correlated? Give a one-line justification.

   (b) Compute the covariance $\text{cov}(X_1, X_2)$ of X_1 and X_2.

4. Random variables X and Y have the joint PDF shown below:



   (a) Find the conditional PDFs $f_{Y|X}(y \mid x)$ and $f_{X|Y}(x \mid y)$, for various values of x and y, respectively.
   (b) Find $\mathbf{E}[X \mid Y = y]$, $\mathbf{E}[X]$, and $\text{var}(X \mid Y = y)$. Use these to calculate $\text{var}(X)$.
   (c) Find $\mathbf{E}[Y \mid X = x]$, $\mathbf{E}[Y]$, and $\text{var}(Y \mid X = x)$. Use these to calculate $\text{var}(Y)$.

5. The wombat club has N members, where N is a random variable with PMF

$$p_N(n) = p^{n-1}(1-p) \quad \text{for } n = 1, 2, 3, \ldots.$$

On the second Tuesday night of every month, the club holds a meeting. Each wombat member attends the meeting with probability q, independently of all the other members. If a wombat attends the meeting, then it brings an amount of money, M, which is a continuous random variable with PDF

$$f_M(m) = \lambda e^{-\lambda m} \quad \text{for } m \geq 0.$$

N, M, and whether each wombat member attends are all independent. Determine:

   (a) The expectation and variance of the number of wombats showing up to the meeting.
   (b) The expectation and variance for the total amount of money brought to the meeting.

G1†. (a) Let $X_1, X_2, \ldots, X_n, X_{n+1}, \ldots, X_{2n}$ be independent and identically distributed random variables.
Find

$$\mathbf{E}[X_1 \mid X_1 + X_2 + \ldots + X_n = x_0],$$

where x_0 is a constant.

(b) Define

$$S_k = X_1 + X_2 + \ldots + X_k, 1 \leq k \leq 2n.$$

Find

$$\mathbf{E}[X_1 \mid S_n = s_n, S_{n+1} = s_{n+1}, \ldots, S_{2n} = s_{2n}],$$

where $s_n, s_{n+1}, \ldots, s_{2n}$ are constants.

Spent like 2x time on this P-Set

†Required for 6.431; optional for 6.041

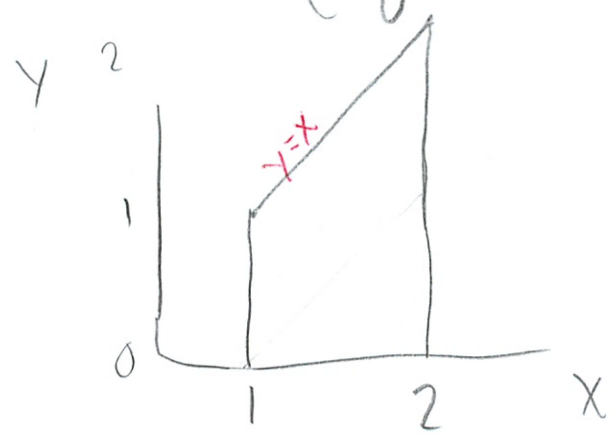
better get good grade

Michael Plasmeier P-Set 6 10/27

1. RV X, Y distributed according to joint PDF

$$f_{X,Y}(x,y) = \begin{cases} ax & \text{if } 1 \le x \le 2 \text{ and } 0 \le y \le x \\ 0 & \end{cases}$$

$y < x$



$\text{area} = \frac{1}{2} \cdot 1 \cdot 1 + 1 \cdot 1 = 3/2$

$a = \frac{1}{\text{area}} = \frac{1}{3/2} = \frac{2}{3}$

probability not! uniformly distributed

Or according to last week's answers format

$$\int_{x=1}^{2} \left(\int_{y=0}^{x} ax \, dx \right) dy$$

← wrong bound

plus messed up dx, dy

$$= \int_{x=1}^{2} ax(x-1)\,dx$$

$$ax^2 - x$$

$$= a(2^2 - 2 - 1^2 - 1) = 1$$

$$a(2 - 0) = 1$$

$$a = \frac{1}{2}$$

1b

$$1 = \int_1^2 \int_0^x ax \, dy dx$$

$$1 = a\int_1^2 xy \Big|_0^x dy$$

$$1 = a\int_1^2 x^2 dx$$

$$1 = a\frac{x^3}{3}\Big|_1^2$$

$$1 = a\left(\frac{8}{3} - \frac{1}{3}\right) =$$

$$1 = \frac{7}{3}a$$

$$a = \frac{3}{7}$$

+0.5

(2)

b) Determine the marginal PDF $f_Y(y)$

$a = \frac{1}{2}$

For $0 \le y \le 1$

$$f_Y(y) = \int_1^2 ax\,dx$$

$$= \frac{ax^2}{2}\Big|_1^2$$

$$= \frac{\frac{3}{7}2^2}{2} - \frac{\frac{3}{7}1^2}{2}$$

$$= \frac{6}{7} - \frac{3}{14}$$

$$= \frac{9}{14}$$

renormalize - should even out since same area?
or not here

For $1 \le y \le 2$

$$f_Y(y) = \int_x^2 ax\,dx$$

$$= \frac{ax^2}{2}\Big|_x^2$$

$$= \frac{\frac{3}{7}2^2}{2} - \frac{\frac{3}{7}x^2}{2}$$

$$= \frac{6}{7} - \frac{3x^2}{14}$$

End # -> non uniform

(3)

$$f_Y(y) = \begin{cases} \frac{9}{14} & \text{if } 0 \le y \le 1 \\ \frac{6}{7} - \frac{3x^2}{14} & \text{if } 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

+1

c) Determine the conditional expectation of f_X given $Y = 3/2$

do formal way like last week's solutions

$$f_{X|Y}(x \mid 3/2) = \frac{f_{X,Y}(x, 3/2)}{f_Y(3/2)}$$

$$\frac{\text{joint PMF from above}}{\text{Use above to fill in}}$$ ← never realized

$$\frac{\frac{3}{7}x}{\frac{6}{7} - \frac{3(3/2)^2}{14}} = \frac{\frac{3x}{7}}{\frac{3}{8}} = \frac{3x}{7} \cdot \frac{8}{3} = \frac{8x}{7}$$

$(-1/2)$ expected value ↓ ?

+1/2

in line w/ last week ans

-oh got the same thing w/ my incorrect # - hmm

d) RV $Z = Y - X$. Find PDF $f_Z(z)$

Derived distribution - now new

~~Well a form of convolution~~

$$\cancel{f_{Z|X}(z \mid x) = f_Y(w - z)}$$

$$\cancel{f_{Z,X}(z, w) = f_X(x)\, f_{Z|X}(x)}$$

$$\cancel{= f_X(x)\, f_Y(z - x)}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z - x)\, dx$$

(4) and note <u>not</u> independent

opps so no convolution

but is monotonic

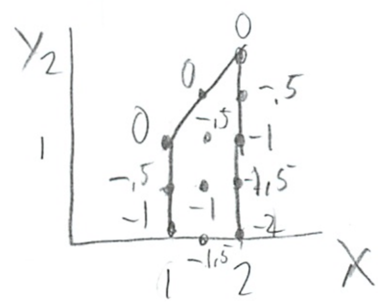
but is joint

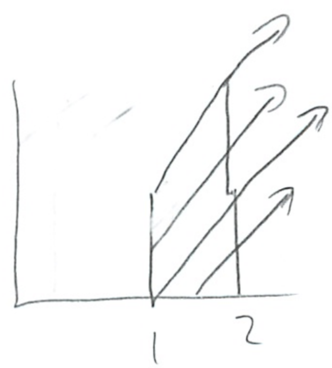
First find CDF of Z and then differentiate

$F_Z(z) = P(Y - X \le z) =$

this is why we did that thing

need to graph 1st





all the same values

for bottom square

$= P(Y - X \le z)$

right???

$= \int_0^1 \left(\int_1^2 \frac{1}{2} x \, dx \right) dy)$

minimum anyway

$\int_0^1 \frac{x^2}{4} \Big|_1^2 \, dy$

$\int_0^1 \frac{3}{4} \, dy$

$\frac{3x}{4} \Big|_0^1 = \frac{3}{4}$

? how to do joint PDF

(5) For top part

$$\int_1^y \int_1^2 \frac{1}{2} x \, dx \, dy$$

$$\int_0^y \frac{3}{4} \, dy$$

$$\frac{3y}{4}$$

$z = y - x$

$z + x = y$

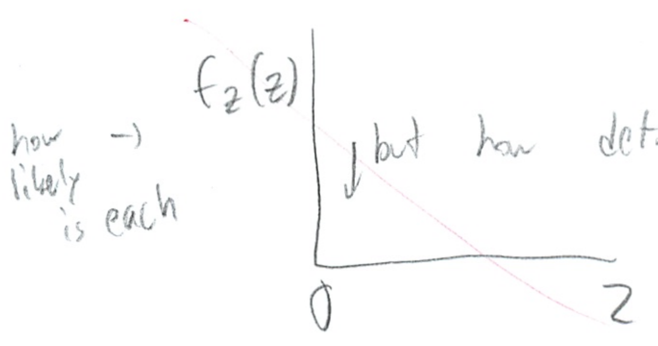
So combined

$$\frac{3}{4} + \frac{3y}{4}$$

$$\frac{3}{4}(1 + y)$$

$$\frac{3}{4}(1 + (z + x))$$

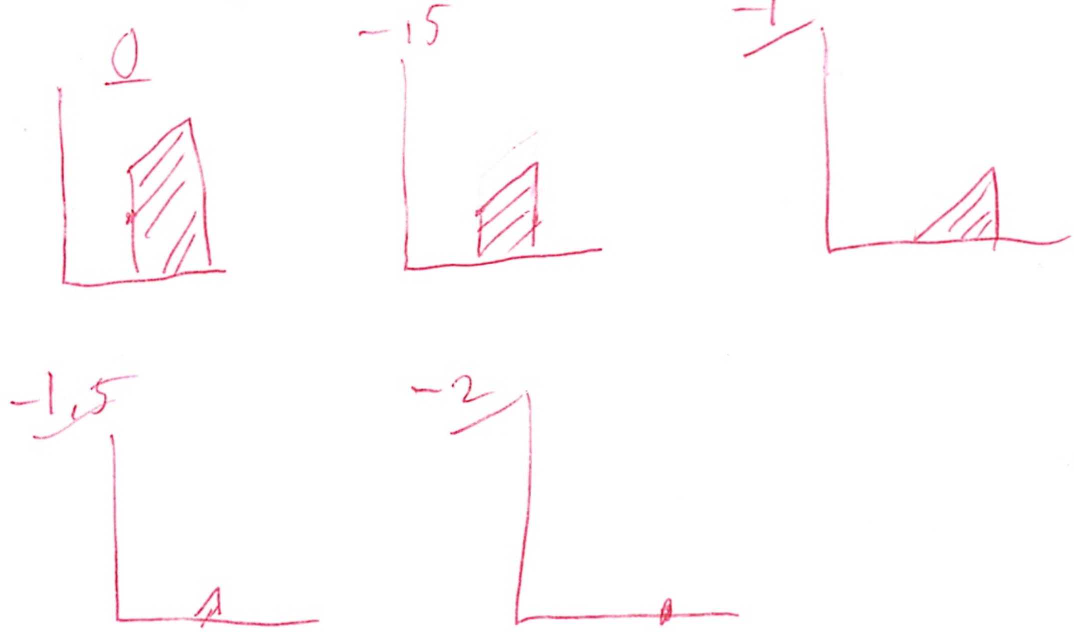
No! Need to do for combined



?? Ask in OH

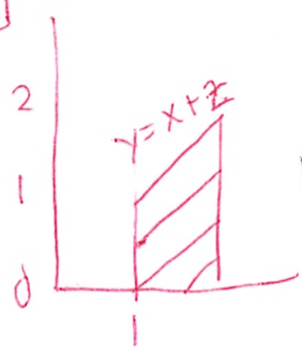
5b OH.

$P(y - x \le z)$



$\boxed{\le \text{ means area underneath}}$

How represent on 1 graph



was correct w/ different lines

need to find CDF of

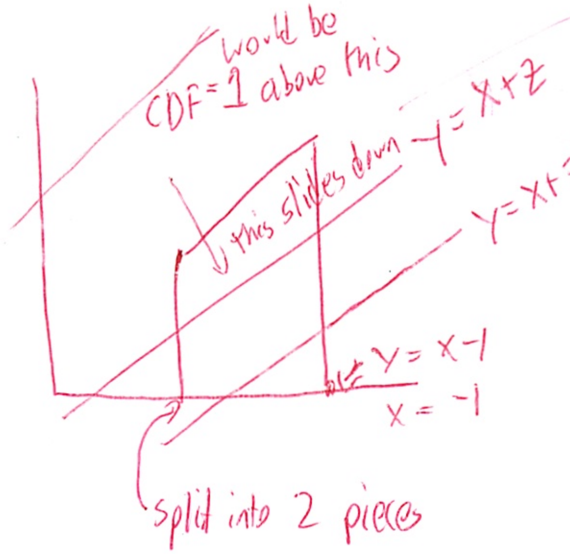
2 pieces $z \ge 1$

$z < 1$ piecewise

more on next pg

5c

joint PDF



direction z is decreasing in sign as coming down
So max z value 0 is at top
So that is CDF of 1

integrate line as it slides down

$z \geq 1$ above

just $f_{x,y}(x,y)$

$$\int_0^{x+z} \int_1^2 ax \, dx \, dy$$

← difficult - want outer bound to be free

can do either

$$\int_1^2 \int_0^{x+z} ax \, dy \, dx$$

which is easier

$z < 1$ lower

$$\int_0^1 \int_{y-z}^2 ax \, dx \, dy$$

← not always 1 dependance taken care of earlier so can say

$$\int_{-z}^2 \int_0^{x+z} ax \, dy \, dx$$

do integration + get expression in terms of z, plot then differentiate (both pieces)

5d

$$= \int_1^2 axy \Big|_0^{x+z} dx$$

$$= \int_1^2 ax(x+z) - ax(0) \; dx$$

$$= \int_1^2 ax^2 + axz \; dx$$

$$= \frac{ax^3}{3} + \frac{ax^2z}{2} \Big|_1^2$$

$$= \frac{a(2)^3}{3} + \frac{a(1)^2 z}{2}$$

$$= \frac{\frac{3}{7}(2)^3}{3} + \frac{\frac{3}{7}(1)^2 z}{2}$$

$$F_Z(z) = \frac{8}{7} + \frac{3z}{14} \quad \text{for } z \geq 1$$

Below

$$= \int_{-z}^2 axy \Big|_0^{z+x} dx$$

$$= \int_{-z}^2 ax(z+x) \; dx$$

(5e).

$$= \int_{-z}^{2} ax^2 + axz \, dx$$

$$= \frac{ax^3}{3} + \frac{azx^2}{2} \Big|_{-z}^{2}$$

$$= \frac{\frac{3}{7}(2)^3}{3} + \frac{\frac{3}{7}z(2)^2}{2} - \frac{\frac{3}{7}(-z)^3}{3} + \frac{\frac{3}{7}z(-z)}{2}$$

$$F_Z(z) = \frac{8}{7} + \frac{3z}{14} + \frac{3z^3}{21} - \frac{3z^2}{14} \quad \text{for } z < 1$$

Now differentitate for PDF

$$\frac{dF_Z(z)}{dz} = f_Z(z) = \begin{cases} \cancel{3/14} \quad 3 & \cancel{0 >} z \geq -1 \\ \frac{3z^2}{7} - \frac{3z}{7} + \frac{3}{14} & z < 1 \end{cases}$$

$(-1, 5)$

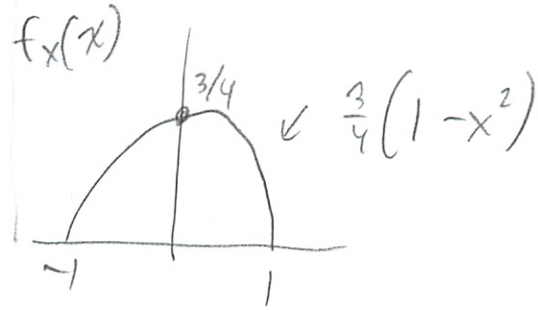
See solutions

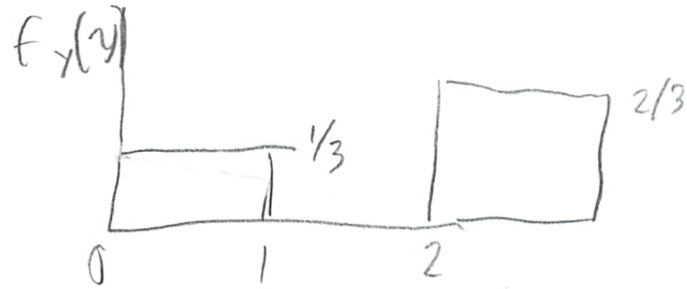
2/4

6

2. Let X, Y be independent RV, PDF shown

$Z = X + F$ determine $f_Z(z)$





Started in tutorial

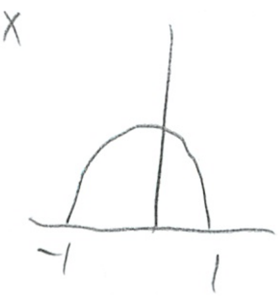
Convolution

Given graphically so flip + slide
↑ eaiser

Can choose either one to flip + slide
and keep static

Pick X to flip + slide

Flip x



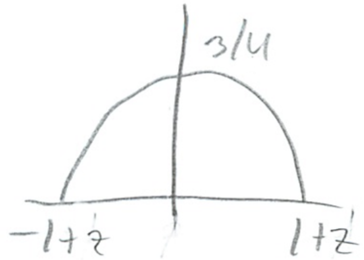
same

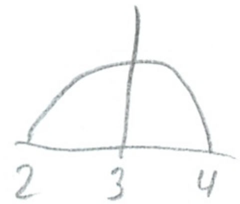
tip flip expression inside $\int$

domain $\frac{3}{4}\left(1-(z-x)^2\right)$

(7) Now shift by t

ie if $t=3$





Need to multipy and integrate
Need to do in a piecewise way
for z from $-\infty \to \infty$

$$f_Z(z) = \begin{cases} 0 & \text{if } z+1 \le 0, \quad z \le -1 \\ \int_0^{z+1} \frac{1}{3} \cdot \frac{3}{4}\left(1-(z-t)^2\right) dt & \text{if } z+1 \le 1,\ z \le 2,\ z+1 \ge 0,\ -1 \le z \le 0 \\ \int_0^1 \frac{1}{3} \cdot \frac{3}{4}\left(1-(z-t)^2\right) dt & \text{if } 0 \le z,\ z+1 \le 2,\ z \le 1,\ 0 \le z \le 1 \\ \int_0^1 \frac{1}{3} \cdot \frac{3}{4}\left(1-(z-t)^2\right) dt & \text{same} \end{cases}$$

product = 0

↑ as long as sliding has same integral till $1 = z+1$

Completly covered

emerging

So these 2 actually same

before entering next One, will always be in this one

(8)

λ

$$\int_{z-1}^{1} \frac{1}{3}\cdot\frac{3}{4}\left(1-(z-t)^2\right)dt + \int_{2}^{z+1} \frac{2}{3}\cdot\frac{3}{4}\left(1-(z-t)^2\right)dt$$

if $z-1 \le 1$ $\quad 1 \le z \le 2$

$z+1 \ge 2$

inside both

$$\int_{2}^{3} \frac{2}{3}\cdot\frac{3}{4}\left(1-(z-t)^2\right)dt$$ if $z-1 \ge 1$

$z-1 \le 2$

$2 \le z \le 3$

Only in second one

$$\int_{z-1}^{3} \frac{2}{3}\cdot\frac{3}{4}\left(1-(z-t)^2\right)dt$$ for $z-1 \ge 2$

$z-1 \le 3$

$3 \le z \le 4$

in second partial

0 $\quad$ for $z-1 > 3$

$z > 4$

Out of both

I realize these are kinda messed up (confusion on t, z variables)

go through and integrate piece wise

integral will depend on z

dt ? - yeah

(8b)

Actually integrate

$$f_Z(z) = \begin{cases} 0 & \text{if } z \le -1 \\ -\frac{t^3}{12} + \frac{t^2 z}{4} + \frac{t z^2}{4} + \frac{t}{4}\Big|_0^{z+1} & -1 < z < 0 \\ -\frac{t^3}{12} + \frac{t^2 z}{4} + \frac{t z^2}{4} + \frac{t}{4}\Big|_0^{1} & 0 < z < 1 \\ -\frac{t^3}{12} + \frac{t^2 z}{4} + \frac{t z^2}{4} + \frac{t}{4}\Big|_{z-1}^{1} + & \\ -\frac{t^3}{6} + \frac{t^2 z}{2} - \frac{t z^2}{2} + \frac{t}{2}\Big|_2^{z+1} & 1 < z < 2 \\ -\frac{t^3}{6} + \frac{t^2 z}{2} - \frac{t z^2}{2} + \frac{t}{2}\Big|_2^{3} & 2 < z < 3 \\ -\frac{t^3}{6} + \frac{t^2 z}{2} - \frac{t z^2}{2} + \frac{t}{2}\Big|_{z-1}^{3} & 3 < z < 4 \\ 0 & z > 4 \end{cases}$$

(8c)

$$f_Z(z) = \begin{cases} 0 & z < -1 \\ -\frac{z^3}{12} + \frac{z}{4} + \frac{1}{6} & -1 \le z < 0 \\ -\frac{z^2}{4} + \frac{z}{4} + \frac{1}{6} & 0 < z < 1 \\ \frac{z^3}{12} - \frac{z^2}{4} + \frac{1}{3} - \frac{z^3}{6} + z^2 - \frac{3z}{2} + \frac{2}{3} & 1 < z < 2 \\ -\frac{z^2}{2} + \frac{5z}{2} - \frac{8}{3} & 2 < z < 3 \\ \frac{z^3}{6} - \frac{3z^2}{2} + 4z - \frac{8}{3} & 3 < z < 4 \\ 0 & z > 4 \end{cases}$$

and no I am not graphing that

+2/2

(9)

hint: indicator RV

3. Consider n independent tosses of a k-sided die

X_i = # of tosses that result in i

? not conventional format

n tosses

X_i = # of tosses that result in i

a) Are X_1, X_2 uncorrelated, pos correlated, ?

? oh not qu expecting

← not RV

~~Say $n = 5$, $k = 5$~~ ~~$X_1 = \frac{1}{5}$~~ ~~$X_2 = \frac{1}{5}$~~ $n=5$ $k=10$ $X_1 = \frac{1}{10}$ $X_2 = \frac{1}{10}$ ← not technically correct

Well assuming $k > 2$, $X_1 = X_2$ no matter n

<u>But</u> a large X_1, means a lower X_2

- so negitivly correlated

b) $cov(X_1, X_2) = E[X_1 X_2] - E[X_1] E[X_2]$

$E[X_1] = E[X_2] = \frac{1}{k}$

$k = 10$, $n = 5$: $E[X_1] = 2$, $E[X_2] = 2$

$k=5$, $n=5$: $E[X_1] = 1 = \cancel{\frac{k}{n}}\ \frac{n}{k}$, $E[X_2] = 1 = \cancel{\frac{k}{n}}\ \frac{n}{k}$

$\cancel{E\left[\frac{k}{n} \cdot \frac{k}{n}\right]}$

take the $n=5$, $k=5$

$E[X_1 X_2] = ?$

$X_1 X_2 = ?$, not 0

so not indepent

say $\frac{1}{5} \cdot \frac{1}{5}$

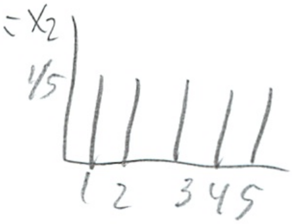
(10)

Was I right in assuming $X_1 = \frac{1}{5}$ $n=5$

$X_2 = \frac{1}{5}$ $k=5$

? no can't say that

Since $X_1 = X_2$



Well flip+slide PDF + multiply ?

? does not normalize

What is $E[X_1 X_2]$?

?? confused

Could do total expectation theorm

but how did we get this ?

- proof of Bernovi
- repeat that proof
- use indicator RV

$X_1 = A_1 + A_2 + \ldots + A_n$

- just 0 or 1 - easy to work w/

$A_i = \begin{cases} 1 & \text{if ith roll} = 1 \\ 0 & \text{else} \end{cases}$

$B_i = \begin{cases} 1 & \text{if ith roll} = 2 \\ 0 & \text{else} \end{cases}$

$\boxed{E[A_i B_j] = E[A_i]E[B_j]}$

$= E[(A_1 + A_2 + \ldots + A_n)(B_1 + \ldots + B_n)]$

- product nice always 1 or 0

$= E[\sum_{i,j} A_i B_j]$ linearity of expectation

$$= \sum_{i,j} E[A_i B_j]$$

just calculate this

$$\frac{n(n-1)}{k^2}$$

So

$$Cov(X_1, X_2) = \frac{n(n-1)}{k^2} - \frac{n^2}{k^2}$$

$$= \frac{n^2 - n - n^2}{k^2}$$

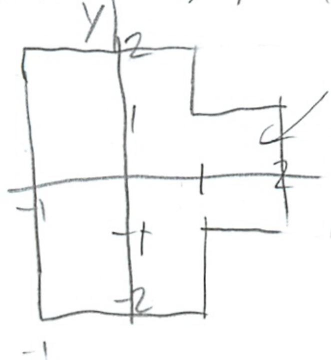
$$= -\frac{n}{k^2}$$

? still confused

?is it p that both occured?

(11)

4. RV X, Y has that joint PDF



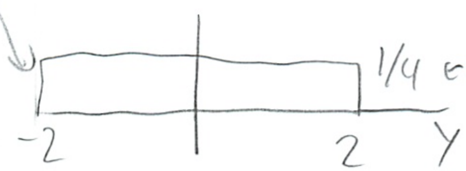
$f_{X,Y}(x,y) = .1$

a) Find the conditional PDFs $f_{Y|X}(y|x)$ $f_{X|Y}(x|y)$ for the various X, Y

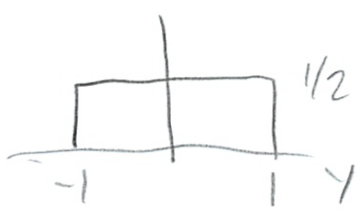
$f_{Y|X}(y \mid -1 \le x < 1)$

$$= \frac{f_{X,Y}(x,y)}{f_Y(y)}$$ ← above graph ← just Y, marginal

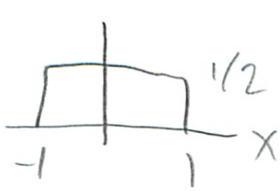
or just graphically p.169



$f_{Y|X}(y \mid 1 \le x \le 2)$

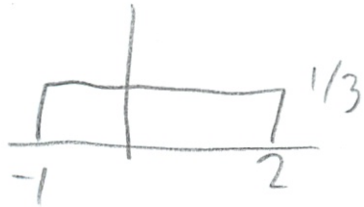


$f_{X|Y}(x \mid -2 \le y < -1)$

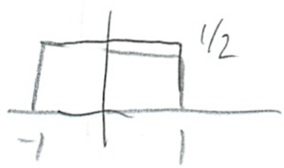


(12).

$f_{X|Y}(x \mid -1 \le y \le 1)$



$f_{X|Y}(x \mid 1 < y \le 2)$



b) Find $E[X \mid Y=y]$, $E[X]$, $var(X \mid Y=y)$, $var(X)$

-section 4.3

Note $E[X \mid Y=y] = f(y) = \#$

$E[X \mid Y] = f(Y) = RV.$

$E[X \mid -2 \le y < -1] = 0$ ← uniform pdf, constant ✓

$E[X \mid -1 \le y \le 1] = \cancel{1/2}\ \frac{1}{3}$; what it is it again formally ?

^ was right before

$E[X \mid 1 < y \le 2] = 0$

$$E[X \mid Y=y] = \begin{cases} 0 & \text{else} \\ 1/3 & -1 \le y \le 1 \end{cases}$$

$E[X] = E[E[X \mid Y]]$

(13) . ? So take the E[ ] from each range · range size ?

$= 0 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{4}$

$= \frac{1}{6}$ ← ? should be a RV

$Var(X \mid Y=y) = \cancel{E[E[X \mid Y]^2 \mid Y=y]}$ $= \frac{(b-a)^2}{12}$

? var of continuous uniform RV

$var(X \mid Y) = E[(X - E[X \mid Y])^2 \mid Y]$

? should be fraction of x, y but w/ real live values

$var(X \mid Y=y) \overset{\text{assuming}}{=} E[(X - E[X \mid Y=y])^2 \mid Y=y]$

$= E[(X-1)^2]$

? no the variance of x at each given y

$Var(X \mid -2 \le y < -1) = \cancel{E[X^2] - E[X]^2}$

$\cancel{(-1)^2 \quad (1)^2}$

duh think continuous

$= \int_{-\infty}^{\infty} (x - E[X \mid Y=y])^2 f_X(x)\, dx$

where in all world did this come from?

$= \int_{-1}^{1} (x-0)^2 \cdot \cancel{X}\, dx$

$= \frac{x^3}{3} \Big|_{-1}^{1}$

$= \frac{2}{3}$ → very different ans

$= \frac{2^2}{12} = \frac{4}{12} = \frac{1}{3}$

(14)

$\text{var}(X \mid -1 \le y \le 1)$

$$= \int_{-1}^{2} \left(x - \tfrac{1}{2}\right)^2 \cancel{.1}\, dx$$

$$= \frac{\left(x - \tfrac{1}{3}\right)^3}{3} \cdot \cancel{.0333} \Big|_{-1}^{2}$$

$$= \frac{7}{3}$$

↙ Var normal distributed RV

$$\frac{(b-a)^2}{12} = \frac{3^2}{12} = \frac{3}{4}$$

$\text{var}(X \mid 1 < y \le +2)$

Same as $-2 \le y < 1$ by symmatry

$$\text{var}(X \mid Y = y) = \begin{cases} \frac{3}{4} & -2 \le y < 1 \cap 1 < y \le 2 \\ \frac{1}{3} & -1 \le y \le 1 \\ \text{Und.} & \text{otherwise} \end{cases}$$

$$\text{Var}(X) = E[\text{var}(X \mid Y)] + \text{var}(E[X \mid Y])$$

$$\frac{1}{4} \cdot \frac{1}{3} + \frac{2}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{3}$$

$$E[\text{var}(X \mid Y = y)] = \frac{13}{24}$$

$$\text{var}(E[X \mid Y = y]) = \int_{-2}^{2} \left(x - E[X \mid Y = y]\right)^2 f_X(x)\, dx$$

$$= \int_{-2}^{2} \left(x - \tfrac{1}{6}\right)^2 \cdot \cancel{.1}\, dx$$

prob messed up w/ PDF somewhere

(15).

=

$$= \frac{(x - \frac{1}{6})^3}{3} \Big|_{-2}^{2} \qquad \frac{(b-a)^2}{12} = \frac{4}{3}$$

$$= \frac{49}{9}$$

$$\mathrm{var}(x) = \frac{13}{49} + \frac{4}{3}$$

$$= \frac{235}{147}$$

c) Find, $E[Y|X=x]$, $E[Y]$, $\mathrm{var}(Y|X=x)$, $\mathrm{var}(Y)$

$$E[Y \mid -1 \le X < 1] = 0$$

$$E[Y \mid 1 \le X \le 2] = 0$$

$$E[Y] = E[E[X|Y]]$$

$$= \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 0$$

$$= 0$$

$$\mathrm{var}(Y \mid -1 \le X < 1) = \int_{-2}^{2} (y - E[Y|X=x])^2 f_Y(y)\, dy$$

$$= \cancel{X}\, y^3 \Big|_{-2}^{2} \qquad \frac{(b-a)^2}{12} = \frac{4^2}{12} = \frac{4}{3}$$

$$= \frac{16}{3}$$

(16).

$$\mathrm{var}(Y \mid 1 \le X \le 2) = \int_{-1}^{1} (y - E[Y \mid X = x])^2 f_Y(y)\,dy$$

$$= \cancel{X} y^3 \Big|_{-1}^{1} \qquad \frac{(b-a)^2}{12} = \frac{1}{3}$$

$$= \frac{2}{3}$$

$$\mathrm{var}(Y \mid X = x) = \begin{cases} \frac{4}{3} & \text{for } -1 \le x < 1 \\ \frac{1}{3} & \text{for } 1 \le x \le 2 \\ \text{und} & \text{else} \end{cases}$$

$$\mathrm{var}(Y) = E[\mathrm{var}(Y \mid X = x)] + \mathrm{var}(E[Y \mid X])$$

$$= \frac{2}{3} \cdot \frac{4}{3} + \frac{1}{3} \cdot \frac{1}{3}$$

$$= \underline{1}$$

$$\mathrm{Var}(E[Y \mid X = x]) = \int_{-1}^{2} (y - E[Y \mid X = x])^2 f_Y(y)\,dy$$

$$\int_{-1}^{2} (y - 0)^2 \cancel{X}\, dy$$

$$.03333\, y^3 \Big|_{-1}^{2} \qquad \frac{(b-a)^2}{12} = \frac{3}{4}$$

$$.3$$

$$\mathrm{var}(Y) = 1 + \frac{3}{4} = \frac{7}{4}$$

this problem made so many # mistakes

(17) .

5. Wombat club has N members

$N = RV$

$p_N(n) = p^{n-1}(1-p)$ for $n = 1, 2, 3, \ldots$

On the 2nd tuesday of every month has a meeting.

Each member attends w/ prob. q - ind. of other members

If attends, brings M independent

$f_M(m) = \lambda e^{-\lambda m}$ $m \geq 0$ contineous

a) Expectation + variance # wombats at meeting (N)

~~$E[N]$ = Straight Geometric Chap 2~~

~~$\frac{1}{1-p}$ but $p, (1-p)$ flipped~~

~~$\text{var}(N) = \frac{p}{(1-p)^2}$~~

depends on # of wombats over →

b) Expectation + variance $ brought to meeting

- Sum of Ind Random # RV Chap 4.5

First need to know how many will come to meeting

$E[Y | N=n] = n\,E[M]$

but n depends # in club and who attends

Y = total $

M_1, M_2 = amt $ each brought

(17b)

Indicator RV

$X_i = \begin{cases} 1 & \text{if } i\text{th wombat shows up} \\ 0 & \text{else} \end{cases}$ binomial

$\begin{cases} 1 & \text{w/ prob } q \\ 0 & \text{" " } 1-q \end{cases}$

$Y = X_1 + X_2 + \cdots + X_n$

# of indicator variables
- sum of Random # of RV

$$E[Y] = E[X_i] \cdot E[N]$$

$$E[N] = \frac{1}{1-p}$$

$$E[X_1] = \frac{1}{q} = q \ ?$$

$$E[Y] = \frac{1}{1-p} \cdot \frac{1}{q} \quad q$$

+1/2 (−1/2)

$$\text{var}[Y] = E[\text{var}(Y|N)] + \text{var}(E[Y|N])$$

$$= E[N] \, \text{var}(X_i) + (E[X])^2 \, \text{var}(N)$$

$$= \frac{1}{1-p} \, q(1-q) + \left(\frac{1}{q}\right)^2 \frac{p}{(1-p)^2}$$

$$= \frac{q-q^2}{1-p} + \frac{p}{q^2(1-p)^2}$$

+1/2 (−1/2)

(18)

$$n = \underbrace{p^{n-1}(1-p)}_{\#\text{ members}} \cdot \underbrace{q}_{\text{p of each attending meeting}}$$

so

$$E[Y|N=n] = p^{n-1}(1-p) \cdot q \cdot \underbrace{E[M]}_{\cancel{\text{normal}}\text{ exponential look up}}$$

$$p^{n-1}(1-p) \cdot q \cdot \frac{1}{\lambda}$$

$$E[Y] = E[E[Y|N]] = E[N\,E[X]] = E[N] \cdot E[X]$$

$$= \frac{1}{1-p} \cdot E[q] \quad \frac{1}{\lambda}$$

bernoulli

q

$$= \overbrace{\frac{1}{1-p} \cdot q}^{\text{same as previous}} \cdot \frac{1}{\lambda} \quad +1$$

$$\text{var}(Y|N=n) = \text{var}(X_1 + X_2 + \dots + X_n | N=n)$$

$$= \text{var}(X_1 + X_2 + \dots + X_n)$$

$$= n\text{Var}(X)$$

var of exponential $= \frac{1}{\lambda^2}$

(1.9)

$$\mathrm{Var}(Y) = E[\mathrm{var}(Y|N) + \mathrm{var}(E[Y|N])$$

$$= E[N\,\mathrm{var}(X) + \mathrm{var}(N\,E[X]$$

$$= E[N]\,\mathrm{var}(X) + (E[X])^2\,\mathrm{var}(N)$$ +1/2

Y = \$ of money = m

N = # of wombats that show up (previous answer Y)

X = amt of \$ they bring

$$= \left(\frac{1}{1-p} \cdot \frac{1}{q}\right) \cdot \frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2 \left(\frac{q-q^2}{1-p} + \frac{p}{q^2(1-p)^2}\right)$$

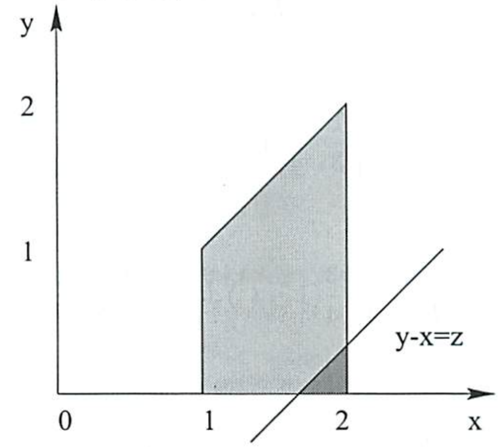
+1/2

3/4

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Problem Set 6: Solutions

1. Let us draw the region where $f_{X,Y}(x,y)$ is nonzero:



The joint PDF has to integrate to 1. From $\int_{x=1}^{x=2}\int_{y=0}^{y=x} ax\,dy\,dx = \frac{7}{3}a = 1$, we get $a = \frac{3}{7}$.

(b) $f_Y(y) = \int f_{X,Y}(x,y)\,dy = \begin{cases} \int_1^2 \frac{3}{7}x\,dx, & \text{if } 0 \le y \le 1, \\ \int_y^2 \frac{3}{7}x\,dx, & \text{if } 1 < y \le 2, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{9}{14}, & \text{if } 0 \le y \le 1, \\ \frac{3}{14}(4-y^2), & \text{if } 1 < y \le 2, \\ 0, & \text{otherwise.} \end{cases}$

(c)

$$f_{X|Y}(x \mid \tfrac{3}{2}) = \frac{f_{X,Y}(x,\frac{3}{2})}{f_Y(\frac{3}{2})} = \frac{8}{7}x, \qquad \text{for } \tfrac{3}{2} \le x \le 2 \text{ and 0 otherwise.}$$

Then,

$$\mathbf{E}\left[\frac{1}{X} \mid Y = \frac{3}{2}\right] = \int_{3/2}^{2} \frac{1}{x}\frac{8}{7}x\,dx = \frac{4}{7}.$$

(d) We use the technique of first finding the CDF and differentiating it to get the PDF.

$$\begin{aligned} F_Z(z) &= \mathbf{P}(Z \le z) \\ &= \mathbf{P}(Y - X \le z) \\ &= \begin{cases} 0, & \text{if } z < -2, \\ \displaystyle\int_{x=-z}^{x=2}\int_{y=0}^{y=x+z} \frac{3}{7}x\,dy\,dx \; = \frac{8}{7} + \frac{6}{7}z - \frac{1}{14}z^3, & \text{if } -2 \le z \le -1, \\ \displaystyle\int_{x=1}^{x=2}\int_{y=0}^{y=x+z} \frac{3}{7}x\,dy\,dx \; = 1 + \frac{9}{14}z, & \text{if } -1 < z \le 0, \\ 1, & \text{if } 0 < z. \end{cases} \end{aligned}$$

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} \frac{6}{7} - \frac{3}{14}z^2, & \text{if } -2 \le z \le -1, \\ \frac{9}{14}, & \text{if } -1 < z \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

2. The PDF of Z, $f_Z(z)$, can be readily computed using the convolution integral:

$$f_Z(z) \;=\; \int_{-\infty}^{\infty} f_X(t) f_Y(z-t)\, dt.$$

For $z \in [-1, 0]$,

$$f_Z(z) \;=\; \int_{-1}^{z} \frac{1}{3} \cdot \frac{3}{4}(1-t^2)\, dt \;=\; \frac{1}{4}\left(z - \frac{z^3}{3} + \frac{2}{3}\right).$$

For $z \in [0, 1]$,

$$f_Z(z) \;=\; \int_{z-1}^{z} \frac{1}{3} \cdot \frac{3}{4}(1-t^2)\, dt \;=\; \frac{1}{4}\left(1 - \frac{z^3}{3} + \frac{(z-1)^3}{3}\right).$$

For $z \in [1, 2]$,

$$f_Z(z) \;=\; \int_{z-1}^{1} \frac{1}{3} \cdot \frac{3}{4}(1-t^2)\, dt + \int_{-1}^{z-2} \frac{2}{3} \cdot \frac{3}{4}(1-t^2)\, dt \;=\; \frac{1}{4}\left(z + \frac{(z-1)^3}{3} - \frac{2(z-2)^3}{3} - 1\right).$$

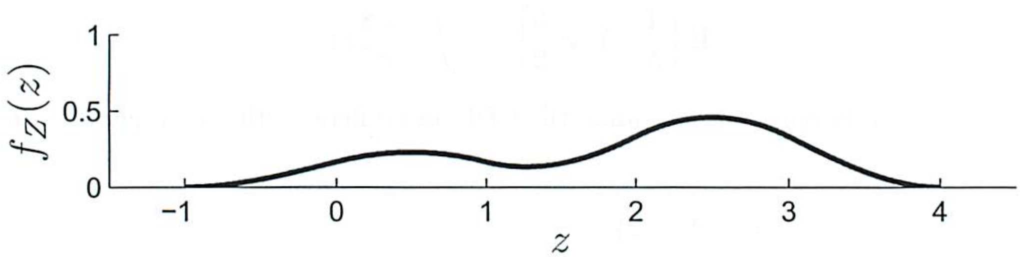
For $z \in [2, 3]$,

$$f_Z(z) \;=\; \int_{z-3}^{z-2} \frac{2}{3} \cdot \frac{3}{4}(1-t^2)\, dt \;=\; \frac{1}{6}\left(3 + (z-3)^3 - (z-2)^3\right).$$

For $z \in [3, 4]$,

$$f_Z(z) \;=\; \int_{z-3}^{1} \frac{2}{3} \cdot \frac{3}{4}(1-t^2)\, dt \;=\; \frac{1}{6}\left(11 - 3z + (z-3)^3\right).$$

A sketch of $f_Z(z)$ is provided below.



3. (a) X_1 and X_2 are negatively correlated. Intuitively, a large number of tosses that result in a 1 suggests a smaller number of tosses that result in a 2.

   (b) Let A_t (respectively, B_t) be a Bernoulli random variable that is equal to 1 if and only if the tth toss resulted in 1 (respectively, 2). We have $\mathbf{E}[A_t B_t] = 0$ (since $A_t \neq 0$ implies $B_t = 0$) and

$$\mathbf{E}[A_t B_s] = \mathbf{E}[A_t]\mathbf{E}[B_s] = \frac{1}{k} \cdot \frac{1}{k} \qquad \text{for} \quad s \neq t.$$

   Thus,

$$\begin{aligned}\mathbf{E}[X_1 X_2] \;&=\; \mathbf{E}\left[(A_1 + \cdots + A_n)(B_1 + \cdots + B_n)\right] \\ &=\; n\mathbf{E}\left[A_1(B_1 + \cdots + B_n)\right] \;=\; n(n-1) \cdot \frac{1}{k} \cdot \frac{1}{k}\end{aligned}$$

and

$$\begin{aligned}\text{cov}(X_1, X_2) &= \mathbf{E}[X_1X_2] - \mathbf{E}[X_1]E[X_2] \\ &= \frac{n(n-1)}{k^2} - \frac{n^2}{k^2} = -\frac{n}{k^2}.\end{aligned}$$

The covariance of X_1 and X_2 is negative as expected.

4. (a) If X takes a value x between -1 and 1, the conditional PDF of Y is uniform between -2 and 2. If X takes a value x between 1 and 2, the conditional PDF of Y is uniform between -1 and 1.

   Similarly, if Y takes a value y between -1 and 1, the conditional PDF of X is uniform between -1 and 2. If Y takes a value y between 1 and 2, or between -2 and -1, the conditional PDF of X is uniform between -1 and 1.

   (b) We have

$$\mathbf{E}[X \mid Y = y] = \begin{cases} 0, & \text{if } -2 \le y \le -1, \\ 1/2, & \text{if } -1 < y \le 1, \\ 0, & \text{if } 1 \le y \le 2, \end{cases}$$

   and

$$\text{var}(X \mid Y = y) = \begin{cases} 1/3, & \text{if } -2 \le y \le -1, \\ 3/4, & \text{if } -1 < y \le 1, \\ 1/3, & \text{if } 1 \le y \le 2. \end{cases}$$

   It follows that $\mathbf{E}[X] = 3/10$ and $\text{var}(X) = 193/300$.

   (c) By symmetry, we have $\mathbf{E}[Y \mid X] = 0$ and $\mathbf{E}[Y] = 0$. Furthermore, $\text{var}(Y \mid X = x)$ is the variance of a uniform PDF (whose range depends on x), and

$$\text{var}(Y \mid X = x) = \begin{cases} 4/3, & \text{if } -1 \le x \le 1, \\ 1/3, & \text{if } 1 < x \le 2. \end{cases}$$

   Using the law of total variance, we obtain

$$\text{var}(Y) = \mathbf{E}[\text{var}(Y \mid X)] = \frac{4}{5} \cdot \frac{4}{3} + \frac{1}{5} \cdot \frac{1}{3} = 17/15.$$

5. First let us write out the properties of all of our random variables. Let us also define K to be the number of members attending a meeting and B to be the Bernoulli random variable describing whether or not a member attends a meeting.

$$\begin{aligned} \mathbf{E}[N] &= \frac{1}{1-p}, & \text{var}(N) &= \frac{p}{(1-p)^2}, \\ \mathbf{E}[M] &= \frac{1}{\lambda}, & \text{var}(M) &= \frac{1}{\lambda^2}, \\ \mathbf{E}[B] &= q, & \text{var}(B) &= q(1-q). \end{aligned}$$

   (a) Since $K = B_1 + B_2 + \cdots B_N$,

$$\begin{aligned} \mathbf{E}[K] &= \mathbf{E}[N] \cdot \mathbf{E}[B] = \frac{q}{1-p}, \\ \text{var}(K) &= \mathbf{E}[N] \cdot \text{var}(B) + (\mathbf{E}(B))^2 \cdot \text{var}(N) = \frac{q(1-q)}{1-p} + \frac{pq^2}{(1-p)^2}. \end{aligned}$$

(b) Let G be the total money brought to the meeting. Then $G = M_1 + M_2 + \cdots + M_K$.

$$\begin{aligned}
\mathbf{E}[G] &= \mathbf{E}[M] \cdot \mathbf{E}[K] = \frac{q}{\lambda(1-p)}, \\
\text{var}(G) &= \text{var}(M) \cdot \mathbf{E}[K] + (\mathbf{E}[M])^2 \text{var}(K) \\
&= \frac{q}{\lambda^2(1-p)} + \frac{1}{\lambda^2}\left(\frac{q(1-q)}{1-p} + \frac{pq^2}{(1-p)^2}\right).
\end{aligned}$$

G1†. (a) Let $X_1, X_2, \ldots X_n$ be independent, identically distributed (IID) random variables. We note that

$$\mathbf{E}[X_1 + \cdots + X_n \mid X_1 + \cdots + X_n = x_0] = x_0.$$

It follows from the linearity of expectations that

$$\begin{aligned}
x_0 &= \mathbf{E}[X_1 + \cdots + X_n \mid X_1 + \cdots + X_n = x_0] \\
&= \mathbf{E}[X_1 \mid X_1 + \cdots + X_n = x_0] + \cdots + \mathbf{E}[X_n \mid X_1 + \cdots + X_n = x_0]
\end{aligned}$$

Because the X_i's are identically distributed, we have the following relationship.

$$\mathbf{E}[X_i \mid X_1 + \cdots + X_n = x_0] = \mathbf{E}[X_j \mid X_1 + \cdots + X_n = x_0], \text{ for any } 1 \le i \le n, 1 \le j \le n.$$

Therefore,

$$\begin{aligned}
n\mathbf{E}[X_1 \mid X_1 + \cdots + X_n = x_0] &= x_0 \\
\mathbf{E}[X_1 \mid X_1 + \cdots + X_n = x_0] &= \frac{x_0}{n}.
\end{aligned}$$

(b) Note that we can rewrite $\mathbf{E}[X_1 \mid S_n = s_n, S_{n+1} = s_{n+1}, \ldots, S_{2n} = s_{2n}]$ as follows:

$$\begin{aligned}
&\mathbf{E}[X_1 \mid S_n = s_n, S_{n+1} = s_{n+1}, \ldots, S_{2n} = s_{2n}] \\
= \; &\mathbf{E}[X_1 \mid S_n = s_n, X_{n+1} = s_{n+1} - s_n, X_{n+2} = s_{n+2} - s_{n+1}, \ldots, X_{2n} = s_{2n} - s_{2n-1}] \\
= \; &\mathbf{E}[X_1 \mid S_n = s_n],
\end{aligned}$$

where the last equality holds due to the fact that the X_i's are independent. We also note that

$$\mathbf{E}[X_1 + \cdots + X_n \mid S_n = s_n] = \mathbf{E}[S_n \mid S_n = s_n] = s_n.$$

It follows from the linearity of expectations that

$$\mathbf{E}[X_1 + \cdots + X_n \mid S_n = s_n] = \mathbf{E}[X_1 \mid S_n = s_n] + \cdots + \mathbf{E}[X_n \mid S_n = s_n].$$

Because the X_i's are identically distributed, we have the following relationship:

$$\mathbf{E}[X_i \mid S_n = s_n] = \mathbf{E}[X_j \mid S_n = s_n], \text{ for any } 1 \le i \le n, 1 \le j \le n.$$

Therefore,

$$\mathbf{E}[X_1 + \cdots + X_n \mid S_n = s_n] = n\mathbf{E}[X_1 \mid S_n = s_n] = s_n \Rightarrow \mathbf{E}[X_1 \mid S_n = s_n] = \frac{s_n}{n}.$$

†Required for 6.431; optional for 6.041