DAGs + Scheduling

3/11

Don't really need axioms
- interesting
- will do on other day

Course prerecs
- kinda old
- indirect prereqs : prereqs of the prereqs

Look at path of arrows

$u \to v$

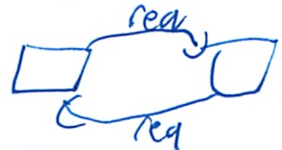
u is earlier/smaller than v

$u R^+ v$

↑ positive (aka non-0) lenght path

minimal - no prereqs needed
- nothing smaller than it
- can be multiple (not minimum)
  - absolute smallest one
    indirect prereq of everything
    nothing here
-

Cycle - prereq - but can never get in and take both
so can't graduate

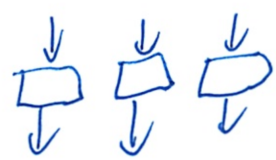


Strategy: Greedy: take as many as you can each semester
- first: all the minimals at same time, remove

Now what are the new minimal elements.
(Not paying attention to how many in a semester)
Then eliminate those, repeat

What other ways are there to get through?
- want to balance load

<u>Antichain</u> - set of courses to take simultaneous
since no indirect preqs
- no arrows at all b/w them
- <u>incomperable</u>
    - nothing is incomprobable to itself
- the horizontal ~~xx~~ rows of the chart



- or diagonal
- could take them at once

<u>Chain</u> - seq of subjects that must be taken in order
- vertical columns

- no choice in order to take

(3)

- also sideways / diagonal (see slides)
- does not need to have every course in it
- ~~[illegible]~~ - more general than path

- longest chain
  - not a property of the greedy schedule
  - that lenght is the smallest amt of time to graduate

$\boxed{\text{min parallel time} = \text{max chain size}}$

max term load ≥

$\boxed{\#\text{ processors for min time} \le \text{max antichain}}$

How long to graduate taking 1 class at a time
- topological ~~[illegible]~~ sort
- take 1 after other

- go through antichain horizontally

Could you ~~[illegible]~~ gratuate on time w/ ≥ 3 subjects term

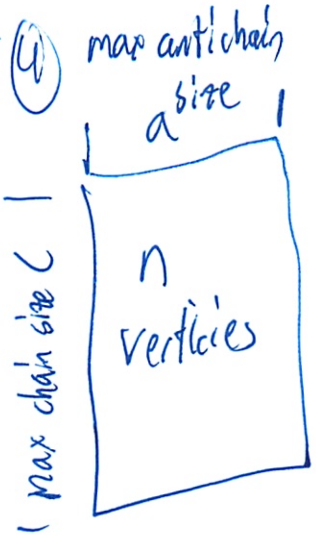
13 sub

max chain size

$\geq \left\lceil \frac{13}{5} \right\rceil = 3$

avg load is lower bound on # processors need

4



$n \leq C \cdot a$

Dillworth's Lemma

Impossible for both of them to be Small

- every DAG has
  - a ch . . . (see slides)

In-Class Problems Week 6, Fri.

Problem 1.
The table below lists some prerequisite information for some subjects in the MIT Computer Science program (in 2006). This defines an indirect prerequisite relation, $\prec$, that is a DAG with these subjects as vertices.

$18.01 \to 6.042$	$18.01 \to 18.02$
$18.01 \to 18.03$	$6.046 \to 6.840$
$8.01 \to 8.02$	$6.001 \to 6.034$
$6.042 \to 6.046$	$18.03, 8.02 \to 6.002$
$6.001, 6.002 \to 6.003$	$6.001, 6.002 \to 6.004$
$6.004 \to 6.033$	$6.033 \to 6.857$

Smaller than on graph
not integer, is less then

more classes than board

(a) Explain why exactly six terms are required to finish all these subjects, if you can take as many subjects as you want per term. Using a *greedy* subject selection strategy, you should take as many subjects as possible each term. Exhibit your complete class schedule each term using a greedy strategy.

(b) In the second term of the greedy schedule, you took five subjects including 18.03. Identify a set of five subjects not including 18.03 such that it would be possible to take them in any one term (using some nongreedy schedule). Can you figure out how many such sets there are?

(c) Exhibit a schedule for taking all the courses—but only one per term.

(d) Suppose that you want to take all of the subjects, but can handle only two per term. Exactly how many terms are required to graduate? Explain why.

(e) What if you could take three subjects per term?

Problem 2.
A pair of Math for Computer Science Teaching Assistants, Oshani and Oscar, have decided to devote some of their spare time this term to establishing dominion over the entire galaxy. Recognizing this as an ambitious project, they worked out the following table of tasks on the back of Oscar's copy of the lecture notes.

1. **Devise a logo** and cool imperial theme music - 8 days.
2. **Build a fleet** of Hyperwarp Stardestroyers out of eating paraphernalia swiped from Lobdell - 18 days.
3. **Seize control** of the United Nations - 9 days, after task #1.
4. **Get shots** for Oshani's cat, Tailspin - 11 days, after task #1.
5. **Open a Starbucks chain** for the army to get their caffeine - 10 days, after task #3.
6. **Train an army** of elite interstellar warriors by dragging people to see *The Phantom Menace* dozens of times - 4 days, after tasks #3, #4, and #5.

7. **Launch the fleet** of Stardestroyers, crush all sentient alien species, and establish a Galactic Empire - 6 days, after tasks #2 and #6.

8. **Defeat Microsoft** - 8 days, after tasks #2 and #6.

We picture this information in Figure 1 below by drawing a point for each task, and labelling it with the name and weight of the task. An edge between two points indicates that the task for the higher point must be completed before beginning the task for the lower one.

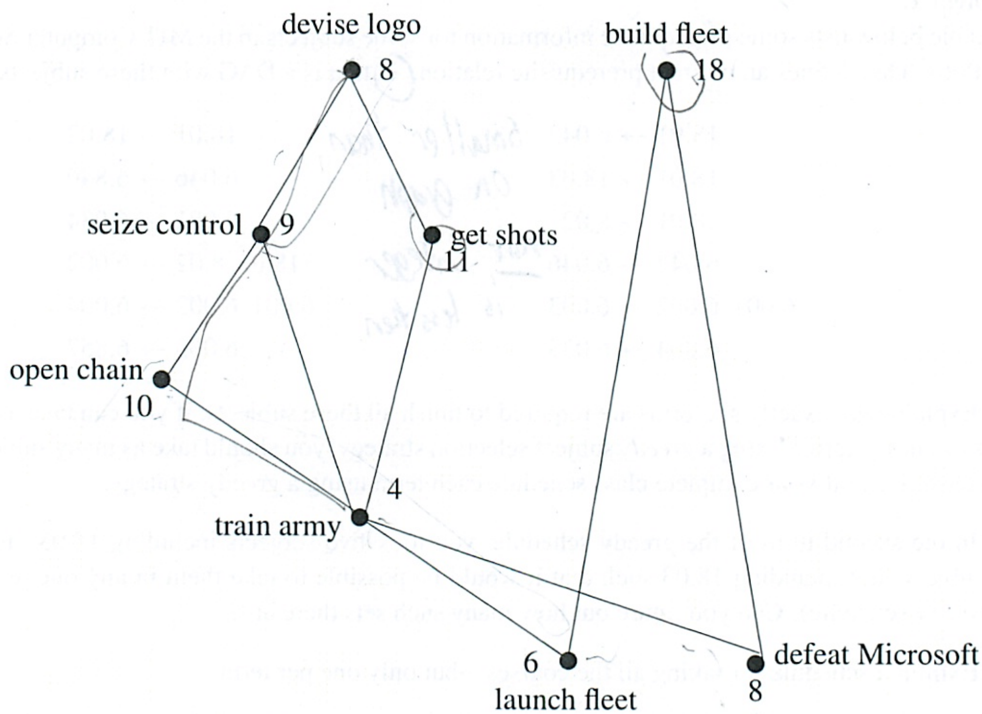


Figure 1 Graph representing the task precedence constraints.

(a) Give some valid order in which the tasks might be completed.

Oshani and Oscar want to complete all these tasks in the shortest possible time. However, they have agreed on some constraining work rules.

- Only one person can be assigned to a particular task; they can not work together on a single task.
- Once a person is assigned to a task, that person must work exclusively on the assignment until it is completed. So, for example, Oshani cannot work on building a fleet for a few days, run to get shots for Tailspin, and then return to building the fleet.

(b) Oshani and Oscar want to know how long conquering the galaxy will take. Oscar suggests dividing the total number of days of work by the number of workers, which is two. What lower bound on the time to conquer the galaxy does this give, and why might the actual time required be greater?

(c) Oshani proposes a different method for determining the duration of their project. She suggests looking at the duration of the "critical path", the most time-consuming sequence of tasks such that each depends on the one before. What lower bound does this give, and why might it also be too low?

(d) What is the minimum number of days that Oshani and Oscar need to conquer the galaxy? No proof is required. 39

Problem 3.
Let R be a binary relation on a set A and C^n be the composition of R with itself n times for $n \geq 0$. So $C^0 ::= \mathrm{Id}_A$, and $C^{n+1} ::= R \circ C^n$.[1] Regarding R as a digraph, let R^n denote the length-n walk relation in the digraph R, that is,

$$a \; R^n \; b ::= \text{there is a length-}n\text{ walk from } a \text{ to } b \text{ in } R.$$

Prove that

$$R^n = C^n \tag{1}$$

for all $n \in \mathbb{N}$.

---

[1] Id_A is the equality relation on A

$$a \; \mathrm{Id}_A \; b \quad \text{iff} \quad a = b.$$

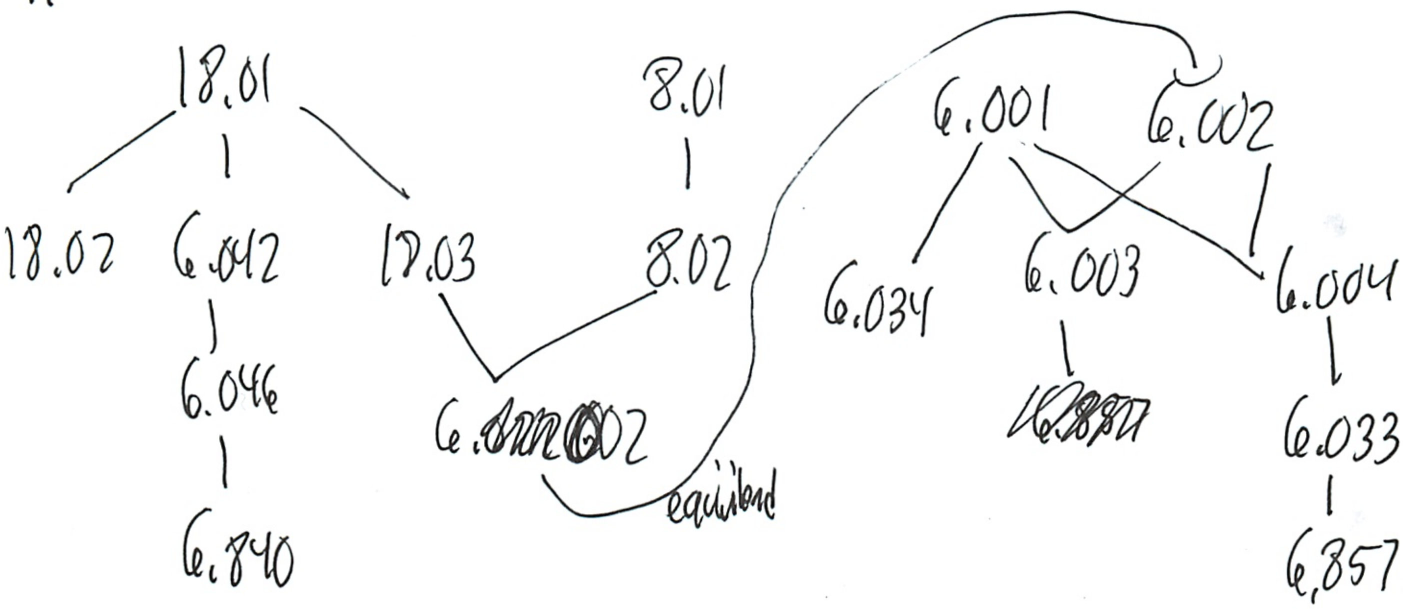
For binary relations $R : M \to B$ and $S : A \to M$, the composition of R with S is the binary relation $(R \circ S) : A \to B$ defined by the rule

$$a \; (R \circ S) \; b ::= \exists m \in M. \; (a \; S \; m) \text{ AND } (m \; R \; b).$$

In Class

3/1

1.



a) 6 is the longest ~~the~~ chain

18.01
18.03
6.002
~~6.003~~
~~6.857~~
6.004
6.033
6.857

b) ? How many antichains of lenght 5 are possable

No easy way of doing it, just try

②

c) ~~III~~ 1 class per term
This is the longutudel sort
just go through each antichain

18.01 → 8.01 ⇒ 18.02 → 6.042 → 18.03 ⇒ 8.02
etc

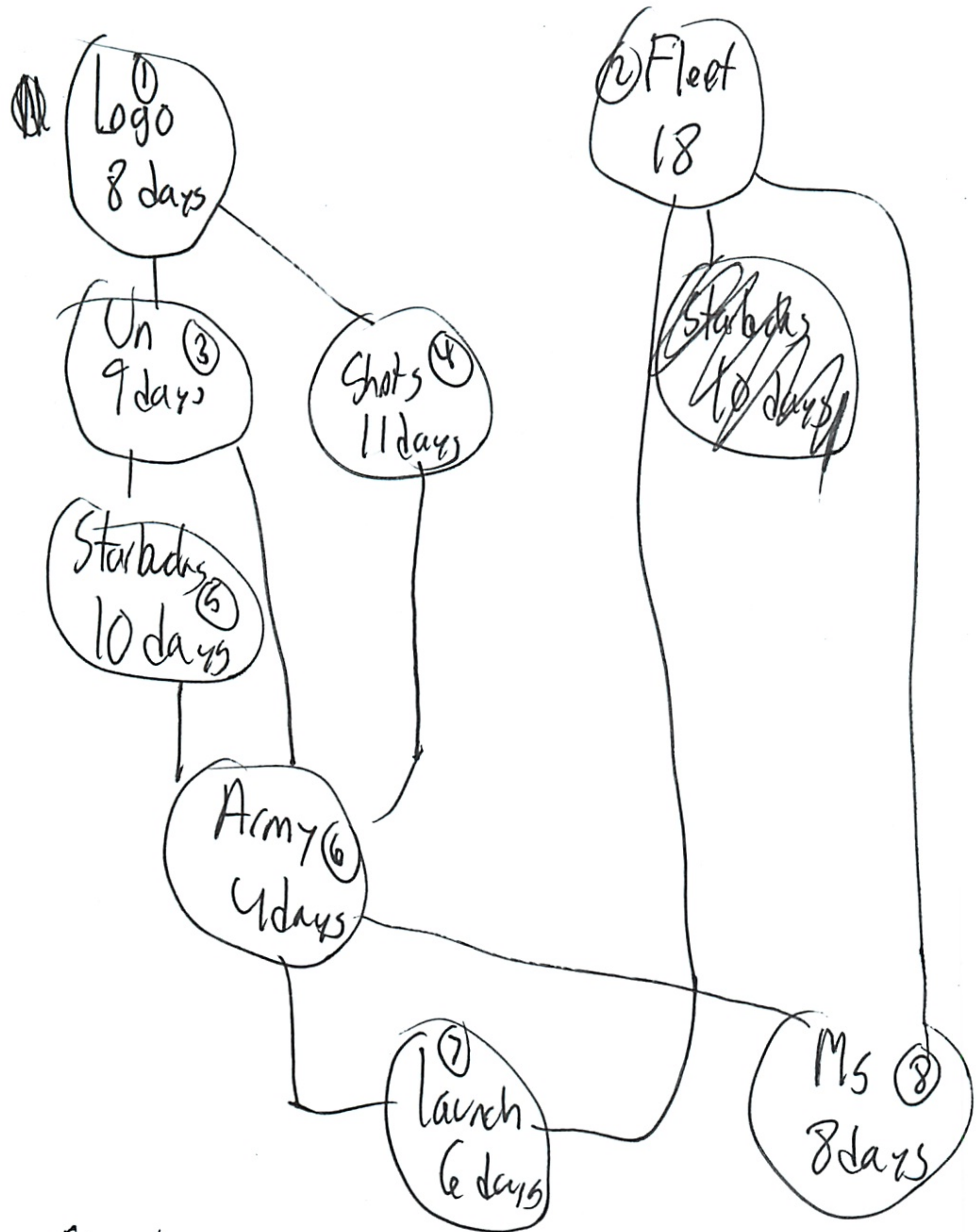
d) If only 2 per term how many ~~terms~~ terms?
Similar to c
∴ just divide in half
Since can always do 2 in parallel
$\left\lceil \frac{15}{2} \right\rceil = 8$

e) 3 sub per term
Same assuming no bottlenecks
- should prove somehow?
$\left\lceil \frac{15}{3} \right\rceil = 5$

Could actually write up to show if possible

3

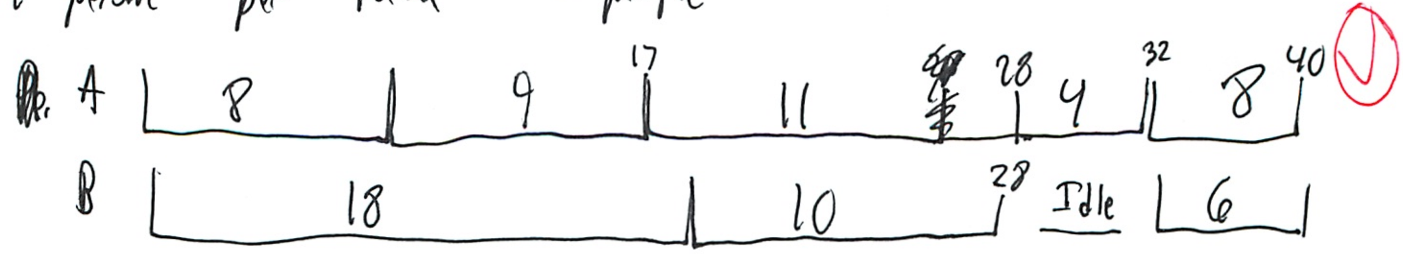
Problem 2



Oh they gave pic

a) A Valid order

1 person per task 2 people



4) Here: There is no better way to test if shorter
- another satisfibility problem
- proof by cases

How to do fastest

Look at longest sequence

$8 \to 9 \to 10 \to 4 \to 8$

How do know can plan it in

$18 < 8+9+10+4$

But $18+11 \leq 8+9+10+4$

$29 \leq 18+13$

$29 \leq 31$

Must have done wrong

b) Why may actual time be greater?

Since not paying attention to # of servers here

(I think this does count as constrained)

c) We did

d) 39 - from critical path

other stuff falls in around

↑ show this

Other group members is there any way to get in 39 days (that's why they say ≤)

General purpose PC's search each permutation

(5)

R#3

~~$A \; R \; C^n$~~ Put parentheses in qu

~~↑ composition of R w/ itself n times~~

Correct $\begin{cases} C^0 = I_{dA} \\ C^{n+1} = R \circ C^n \end{cases}$

R relation on A

C^n comp of R w/ itself n times

R^n is just n steps

each time multiple R take one more step

R^{n+1} = 1 more step

~~Prove that~~ So $a \, R^n \, b$ ::= length n- walk from $a \to b$ in A

Prove that $R^n = C^n$

- sort of by def
- induction proof

Something about the composition

$a \, (R \circ S) \, b ::= \exists b \in B. \; (a \, S \, b)$ AND $(b \, R \, c)$

Not multiplying matricies here

↳ Composing binary relations

⑥

Just that there is
not how many

$a R^n q$

$aRb \quad bRc \quad cRd \ldots \quad pRq$

So $\exists$ $b, c, d, \ldots p$

No cycles, → no repeating verticies

→ Can't have more than n verticies

So at most n distinct verticies

Is a walk – no repeating

Our board Proof by ind. $P(n) = R^n = C^n$

Base $n=0$

$R^0 = C^0$ b/c the only length 0 walks in R

are from an el to itself

Ind Assume $P(\cancel{n+1})$ when $n=0$

$R^{\cancel{n+1}} = C^{\cancel{n+1}} \quad a R^{\cancel{n+1}} b$ iff $c\, C^{\cancel{n+1}} b \quad \forall a, b \in A$

$C^{n+1} = R \circ C^{\cancel{n+1}}$

7) Could not do iff
Try in both directions

a $a \, C^{n+1} \, b$ iff $\exists x \in A,\ (a \, C^n \, x)$ and $(x R b)$

...

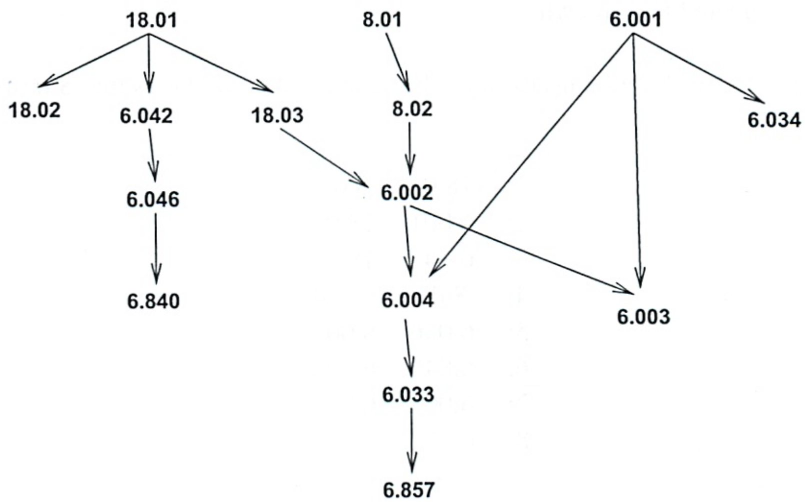
Solutions to In-Class Problems Week 6, Fri.

Problem 1.
The table below lists some prerequisite information for some subjects in the MIT Computer Science program (in 2006). This defines an indirect prerequisite relation, $\prec$, that is a DAG with these subjects as vertices.

$$
\begin{array}{rr}
18.01 \rightarrow 6.042 & 18.01 \rightarrow 18.02 \\
18.01 \rightarrow 18.03 & 6.046 \rightarrow 6.840 \\
8.01 \rightarrow 8.02 & 6.001 \rightarrow 6.034 \\
6.042 \rightarrow 6.046 & 18.03, 8.02 \rightarrow 6.002 \\
6.001, 6.002 \rightarrow 6.003 & 6.001, 6.002 \rightarrow 6.004 \\
6.004 \rightarrow 6.033 & 6.033 \rightarrow 6.857
\end{array}
$$

(a) Explain why exactly six terms are required to finish all these subjects, if you can take as many subjects as you want per term. Using a *greedy* subject selection strategy, you should take as many subjects as possible each term. Exhibit your complete class schedule each term using a greedy strategy.

Solution. It helps to have a diagram of the direct prerequisite relation:



There is a $\prec$-chain of length six:

$$8.01 \prec 8.02 \prec 6.002 \prec 6.004 \prec 6.033 \prec 6.857$$

So six terms are necessary, because at most one of these subjects can be taken each term.

There is no longer chain, so with the greedy strategy you will take six terms. Here are the subjects you take in successive terms.

1:	6.001	8.01	18.01		
2:	6.034	6.042	8.02	18.02	18.03
3:	6.002	6.046			
4:	6.003	6.004	6.840		
5:	6.033				
6:	6.857				

■

(b) In the second term of the greedy schedule, you took five subjects including 18.03. Identify a set of five subjects not including 18.03 such that it would be possible to take them in any one term (using some nongreedy schedule). Can you figure out how many such sets there are?

Solution. We're looking for an antichain in the $\prec$ relation that does not include 18.03. Every such antichain will have to include 18.02, 6.003, 6.034. Then a fourth subject could be any of 6.042, 6.046, and 6.840. The fifth subject could then be any of 6.004, 6.033, and 6.857. This gives a total of nine antichains of five subjects. ■

(c) Exhibit a schedule for taking all the courses—but only one per term.

Solution. We're asking for a topological sort of $\prec$. There are many. One is 18.01, 8.01, 6.001, 18.02, 6.042, 18.03, 8.02, 6.034, 6.046, 6.002, 6.840, 6.004, 6.003, 6.033, 6.857. ■

(d) Suppose that you want to take all of the subjects, but can handle only two per term. Exactly how many terms are required to graduate? Explain why.

Solution. There are $\lceil 15/2 \rceil = 8$ terms necessary. The schedule below shows that 8 terms are sufficient as well:

1:	18.01	8.01
2:	6.001	18.02
3:	6.042	18.03
4:	8.02	6.034
5:	6.046	6.002
6:	6.840	6.004
7:	6.003	6.033
8:	6.857	

■

(e) What if you could take three subjects per term?

Solution. From part (a) we know six terms are required even if there is no limit on the number of subjects per term. Six terms are also sufficient, as the following schedule shows:

1:	18.01	8.01	6.001
2:	6.042	18.03	8.02
3:	18.02	6.046	6.002
4:	6.004	6.003	6.034
5:	6.840	6.033	
6:	6.857		

■

Problem 2.
A pair of Math for Computer Science Teaching Assistants, Oshani and Oscar, have decided to devote some of their spare time this term to establishing dominion over the entire galaxy. Recognizing this as an ambitious project, they worked out the following table of tasks on the back of Oscar's copy of the lecture notes.

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We picture this information in Figure 1 below by drawing a point for each task, and labelling it with the name and weight of the task. An edge between two points indicates that the task for the higher point must be completed before beginning the task for the lower one.

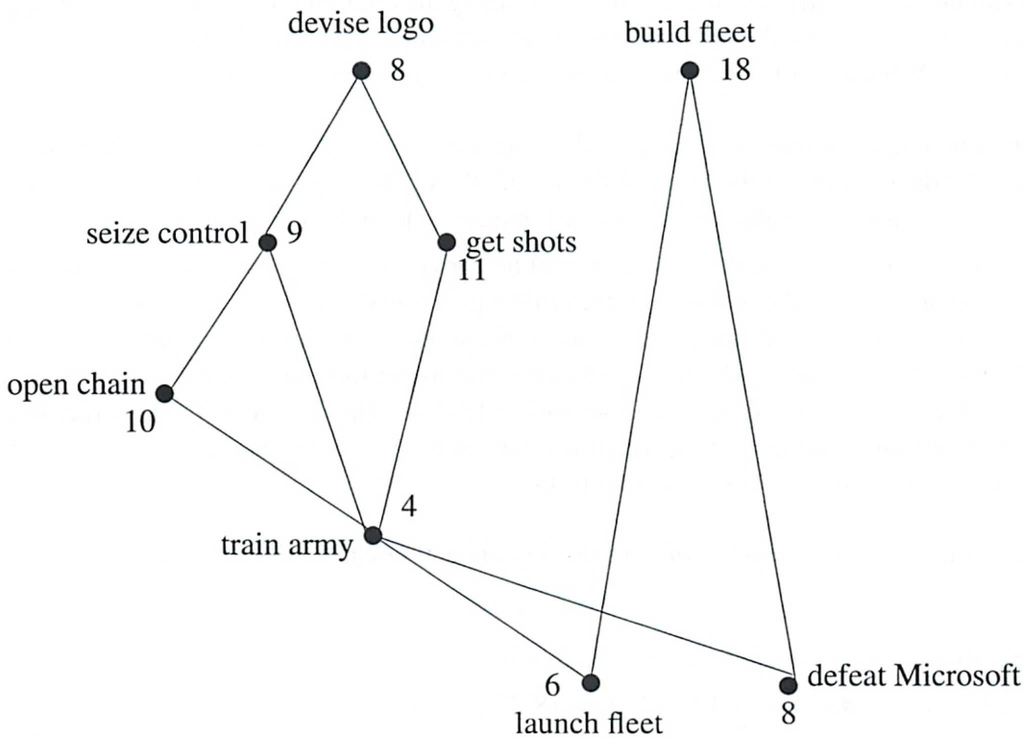


Figure 1 Graph representing the task precedence constraints.

(a) Give some valid order in which the tasks might be completed.

Solution. We can easily find several of them. The most natural one is valid, too: #1, #2, #3, #4, #5, #6, #7, #8.

■

Oshani and Oscar want to complete all these tasks in the shortest possible time. However, they have agreed on some constraining work rules.

- Only one person can be assigned to a particular task; they can not work together on a single task.
- Once a person is assigned to a task, that person must work exclusively on the assignment until it is completed. So, for example, Oshani cannot work on building a fleet for a few days, run to get shots for Tailspin, and then return to building the fleet.

(b) Oshani and Oscar want to know how long conquering the galaxy will take. Oscar suggests dividing the total number of days of work by the number of workers, which is two. What lower bound on the time to conquer the galaxy does this give, and why might the actual time required be greater?

Solution.

$$\frac{8+18+9+11+10+4+6+8}{2} = 37 \text{ days}$$

If working together and interrupting work on a task were permitted, then this answer would be correct. However, the rules may prevent Oshani and Oscar from both working all the time. For example, suppose the only task was building the fleet. It will take 18 days, not 18/2 days, to complete, because only one person can work on it and the other must sit idle. ■

(c) Oshani proposes a different method for determining the duration of their project. She suggests looking at the duration of the "critical path", the most time-consuming sequence of tasks such that each depends on the one before. What lower bound does this give, and why might it also be too low?

Solution. The longest sequence of tasks is devising a logo (8 days), seizing the U.N. (9 days), opening a Starbucks (10 days), training the army (4 days), and then defeating Microsoft (8 days). Since these tasks must be done sequentially, galactic conquest will require at least 39 days.

If there were enough workers, this answer would be correct; however, with only two workers, Oshani and Oscar may be unable to make progress on the critical path every day. For example, suppose there were only four tasks: devise logo, build fleet, seize control, get shots. Now the critical path consists of two critical tasks: devise logo, get shots, which take 19 days. But to get through this path in 19 days, some worker must be working on a critical task at all times for the 19 days. This leaves only one worker free to complete building the fleet and seizing control, which will take at least 27 days. So in fact, 27 days is the minimum time for two workers to complete these four tasks. ■

(d) What is the minimum number of days that Oshani and Oscar need to conquer the galaxy? No proof is required.

Solution. 40 days. Tasks could be divided as follows:

Oscar: #1 (days 1-8), #3 (days 9-17), #4 (days 18-28), #8 (days 33-40).

Oshani: #2 (days 1-18), #5 (days 19-28), #6 (days 29-32), #7 (days 33-38).

It takes some care to verify that 40 days is the best you can do. If someone comes up with a simple proof of this, tell the course staff.

■

Problem 3.
Let R be a binary relation on a set A and C^n be the composition of R with itself n times for $n \geq 0$. So $C^0 ::= \mathrm{Id}_A$, and $C^{n+1} ::= R \circ C^n$.[1] Regarding R as a digraph, let R^n denote the length-n walk relation in the digraph R, that is,

$$a \; R^n \; b ::= \text{there is a length-}n\text{ walk from } a \text{ to } b \text{ in } R.$$

Prove that

$$R^n = C^n \tag{1}$$

for all $n \in \mathbb{N}$.

Solution. *Proof.* By induction on n with equation (1) as induction hypothesis.

Base case $n = 0$: $C^0 = \mathrm{Id}_A$ by definition, and R^0 is the length-0 walk relation which also equals Id_A by definition.

Inductive step: Suppose (1) holds for some $n \geq 0$. We want to prove it holds with "n" replaced by "$n + 1$."

We begin by showing that

$$a \; R^{n+1} \; b \quad \text{implies} \quad a \; C^{n+1} \; b. \tag{2}$$

So suppose $a \; R^{n+1} \; b$, that is, there is a length-$(n + 1)$ walk

$$a = a_0 \; \langle a_0 \to a_1 \rangle \; a_1 \; \langle a_1 \to a_2 \rangle \ldots \langle a_n \to a_{n+1} \rangle \; a_{n+1} = b$$

in R. In particular, there is a length-n walk from a to some vertex a_n such that $a_n \; R \; b$. By induction hypothesis, we have that $a \; C^n \; a_n$. Therefore,

$$a \; (C^n \circ R) \; b$$

by the definition of composition. This proves (2).

Conversely, we show that

$$a \; C^{n+1} \; b \quad \text{implies} \quad a \; R^{n+1} \; b. \tag{3}$$

So suppose $a \; C^{n+1} \; b$, that is $a \; (C^n \circ R) \; b$. By definition of composition, there must be an $a_n \in A$ such that

$$a \; C^n \; a_n \text{ AND } a_n \; R \; b.$$

So there is a length-n walk from a to a_n by induction hypothesis, and $\langle a_n \to b \rangle \in E(R)$. Merging the walk with this edge yields a length-$(n + 1)$ walk from a to b. That is, $a \; R^{n+1} \; b$. This proves (3)

Combining (2) and (3) we conclude that $R^{n+1} = C^{n+1}$, which is the exactly (1) with "n" replaced by "$n + 1$," completing the proof by induction. ■

---

[1] Id_A is the equality relation on A

$$a \; \mathrm{Id}_A \; b \quad \text{iff} \quad a = b.$$

For binary relations $R : M \to B$ and $S : A \to M$, the composition of R with S is the binary relation $(R \circ S) : A \to B$ defined by the rule

$$a \; (R \circ S) \; b ::= \exists m \in M. \; (a \; S \; m) \text{ AND } (m \; R \; b).$$

6.042 Review Session

3/6

Miniquiz #2

#1. $\overline{A-B} = (\bar{A} - \bar{C}) \cup (B \cap C) \cup [(\bar{A} \cup B) \cap \bar{C}]$

$x \in A - B$ iff $x \in A$ and $x \notin B$

↑ forgot this part

↑ I said = instead

Notation of sets

$x \notin B$ iff $x \in \bar{B}$

Prove the set inequality

$x \in LHS$ iff $x \in RHS$

↑ left hand set

Then finish w/ proposition equivilance

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

$x \in \bar{A} \cap \bar{B}$ iff $x \notin A$ and $x \notin B$

$\boxed{\text{De Morgan's Law} \quad NOT(P \text{ AND } Q) = \bar{P} \text{ and } \bar{Q}}$

$P ::= x \in A$

$Q ::= x \in B$

iff $x \notin (A \cup B)$

iff $x \in \overline{A \cup B}$

② Need to know

$x \in \overline{A} \Leftrightarrow x \notin A$

$x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

$x \in A \cup B \Leftrightarrow x \in A$ and $x \in B$.

$x \in \overline{A - B}$

iff $x \notin A - B$

iff $x \in A \cap \overline{B}$

$x \in (\overline{A} - \overline{C}) \cup (B \cap C)$

iff $x \in (\overline{A} - \overline{C})$ or $x \in (B \cap C)$

iff $(x \notin \overline{A}$ and $x \in C)$ or $(x \in B$ and $x \in C)$

$(P$ and $R)$ or $(Q$ and $R)$

Say the prop. equivilence must always hold b/w the two pieces.

Don't

$P ::= x \in A$

$Q ::= x \in B$

$R = x \in (A \cup B)$ ← since this is not independent

Need to go far enough where everything is ind

Don't

Not, and, or are Boolean variables

$\cup \cap -$ are sets

$x \in (A$ and $B)$ → instead → $x \in A$ and B

↑ does not make any sese

$x \in A \cap B)$

(3)

#2 Two countably ∞ sets

$A = \{a_0, a_1, a_2, \ldots\}$

$B = \{b_0, b_1, b_2, \ldots\}$

Prove that $A \times B$ is countaby infinite

$A \times B$) (Matt helped me here)

$S_1 \times S_2 \times \ldots S_n$ is set of all seq of length n whose ith term is drawn from S_i

example $\{1, 2\} \times \{3, 4\}$

$= \{(1,3), (1,4), (2,3), (2,4)\}$

$X \in \mathbb{R}$

$(x, y) \in \mathbb{R}^2$ plane
$= \mathbb{R} \times \mathbb{R}$

Countable)
- finite
- Countable infinite

Don't talk about size of sets

Not intuition "seems bigger"

A is countably infinite iff ~~M~~A bij $\mathbb{N}$
non neg ints

(4)

$$\begin{array}{llllll} \mathbb{N} & 0 & 1 & 2 & 3 & \cdots \\ A & \downarrow & \downarrow & \downarrow & \downarrow & \\ & a_0 & a_1 & a_2 & a_3 & \cdots \end{array}$$

Can prove countable by listing
- obvious if ~~in~~finite
- if C.I. can index them

$$A \times B = \left\{ \begin{array}{lll} (a_0, b_0), & (a_1, b_0) & \cdots \\ (a_0, b_1), & (a_1, b_1) & \cdots \\ (a_0, b_2), & (a_1, b_2) & \cdots \\ \vdots & \vdots & \ddots \end{array} \right\}$$

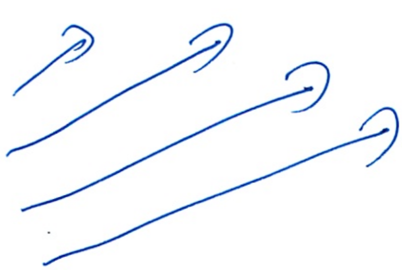
This is ~~[illegible]~~ not a list, need to index, be linear

Say go

But that is <u>not</u> a list

Goes → forever. Will never get to second line

So go



Would cover the whole thing

5

Or could 

Induction

3. Fibinacchi

$F_0 = 0$

$F_1 = 1$

$F_{n+1} = F_n + F_{n-1}$

Make sure clear understanding what trying to prove + steps to do

Decide what inducting over

Induction hyp is what you are trying to prove

Trying to prove

$F_0^2 + F_1^2 + \dots + F_n^2 = F_n F_{n+1}$

(1) Show Base Case $n=0$

$F_0^2 = F_0 \cdot F_1$

$0^2 = 0 \cdot 1$ ✓

(2) Inductive Case

Assume, proved $P(n-1)$

$n = k$, ↓same thing

$F_0^2 + F_1^2 + \dots + F_k^2 = F_k F_{k+1}$

Prove for $n = k+1$

Show $F_0^2 + \dots + F_{k+1}^2 = F_{k+1} F_{k+2}$

6

So can assume $n-1$ prove n } same thing
or assume n prove $n+1$

$$F_k \cdot F_{k+1} + F_{k+1}^2 = F_{k+2} F_{k+2}$$

[Make sure intermediate steps are equivilant of course!]

$$F_{k+1} \cdot (F_k + F_{k+1}) = F_{k+1} \cdot F_{k+2}$$

Same thing
base on how fib # defined

(I need to really work out how to do this!)

Could also have traced backwards
- Start at goal
- Subtract assumption

Another Fib example (aka the golden ratio)

$$F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$$ phi

phi $\to$ $\varphi = \frac{1+\sqrt{5}}{2}$

Prove that is true

So start w/ Base case

$$F_0 = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(1 - \frac{1+\sqrt{5}}{2}\right)^0}{\sqrt{5}}$$

(7)

$$\frac{1-1}{\sqrt{5}} = 0 \checkmark$$

~~Do othe~~

Don't have to do other base case

Do F_{n+1}

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(1 - \frac{1+\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}$$

But then have to simplify

Assume $n=k$ ← forgot the assume

$$F_k = \frac{\varphi^k - (1-\varphi)^k}{\sqrt{5}}$$

Now $F_{k+2} = \dfrac{\varphi^{k+2} - (1-\varphi)^{k+2}}{\sqrt{5}}$

- Use def. Fib seq that $F_{k+2} = F_{k+1} + F_k$

- So will 2 base cases ← oh ~~[illegible]~~ was right
- can't prove for F_1 - but for wrong reason

$$F_1 = \frac{\varphi - (1-\varphi)}{\sqrt{5}}$$

$$= \frac{2\varphi - 1}{\sqrt{5}} = 1$$

8

Also needed to have assumed

$$F_{k+1} = \frac{\varphi^{k+1} - (1-\varphi)^{k+1}}{\sqrt{5}}$$

Now show

$$F_{k+2} = F_{k+1} + F_k = \frac{\varphi^{k+1} - (1-\varphi)^{k+1}}{\sqrt{5}} + \frac{\varphi^{k} - (1-\varphi)^{k}}{\sqrt{5}}$$

So show $= \dfrac{\varphi^{k+2} - (1-\varphi)^{k+2}}{\sqrt{5}}$

By collecting terms, algebra, etc

Use Quadratic equation, etc

$$\underbrace{\varphi^k(\varphi^2 - \varphi - 1)}_{=0} = \underbrace{(1-\varphi)^k((1-\varphi)^2 - (1-\varphi) - 1)}_{=0}$$

Might have a fancy name, but can just do it

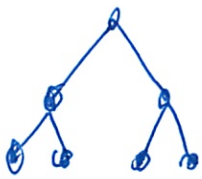
(Don't be scared of special names)

9

A different structural induction proof

Binary trees $\rightarrow \leq 2$ children

Full binary trees $\rightarrow = 2$ children
(FBT) or 0

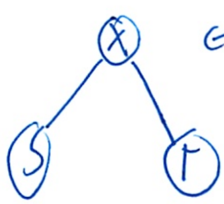


Can also have recursive def of full binary tree

<u>Base</u> Single note is a FBT

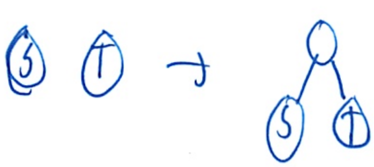
<u>Constuctor</u> ~~∅~~ Have S and T ∈ FBT then is a FBT

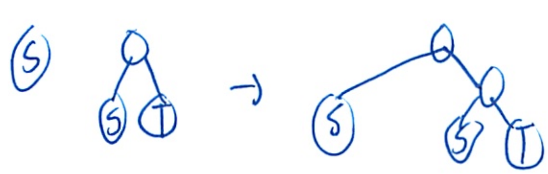
Use construct to build new ones



← add a common parent

← so take 2 trees

So from 

So from 

etc

11

So this is structural induction

If can prove for base cases and that constructer perserves statements can construct anything!

(I think I understand the basic ideas - just need to know how to do/implement)

~~Another Example~~

$T \in FBT$ $\quad T_h = \text{height}$

$T_n = \#$ of nodes

$T_n \leq 2^{T_h+1} - 1$

Using structual induction can prove almost everything about the tree

---

Hyp = what you are trying to prove

$P(T)$

defined on set trying to get relation on

$P(T) ::= (T_n \leq 2^{T_H+1} - 1)$

<u>Base</u> single node

Prove P[illegible] is true for single node $(n=0)$

$P(0)$

$1 \leq 2^{0+1} - 1$

✓

(12)

Constructor

A, B are full binary trees

Assume $P(A)$ & $P(B)$ true

Show $P($  $)$ is also true

↑ call that C

C_n = # of nodes in $C = A_n + B_n + 1$

$C_H = 1 + \max(A_H, B_H)$

$C_n \le 2^{C_H + 1} - 1$

$1 + A_n + B_n \le 2^{2 + \max(A_H, B_H)} - 1$

$$\boxed{\begin{array}{l} \text{Known } A_n \le 2^{A_H+1} - 1 \to P(A) \\ \phantom{\text{Known }} B_n \le 2^{B_H+1} - 1 \to P(B) \end{array}}$$

$1 + A_n + B_n \le 2^{A_H+1} + 2^{B_H+1} - 1 \le 2^{2+\max(A_H/B_H)} - 1$

The -1 cancle at $\quad$ take exponent down $\quad$ ↑ $4 \cdot 2$

$2 \cdot 2^{A_H} + 2 \cdot 2^{B_H} \le 4 \cdot 2^{\max(A_H, B_H)}$

(13)

Diagraphs + DAGs

- abstraction of something, depends on problem
- set of verticies, directed edges

#1 Wed In Class.

Tornement b/w n players

$A \xrightarrow{\text{beats}} B$

Ranking

$a_1 \xrightarrow{\text{beats}} a_2 \xrightarrow{\text{beats}} a_3 \rightarrow \dots \rightarrow a_n$

a) Can give torn w/ more than one ranking?

Cycle

$$\begin{tikzcd} x_2 \arrow[r] & x_3 \arrow[dl] \\ x_1 \arrow[u] & \end{tikzcd}$$

Could say $x_1 \rightarrow x_2 \rightarrow x_3$
$x_2 \rightarrow x_3 \rightarrow x_1$

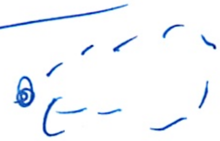
b) DAG = directed, acyclic
- no cycles

Asked to prove not 2 diff rankings

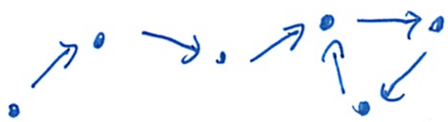
Just asks if there is a ranking, proving that unique
(aka if two are identical)

So assume are 2, prove by induction. See sol.

(14)

a  is a cycle

But may not nessarly be a cycle



(see website)

2 rankings

$a_1, a_2, \ldots, a_n$

$b_1, b_2, \ldots, b_n$

$P(n)$ = when there are 2 rankings, they are the same

<u>Base</u> 2 players

$a \to b$ but can only play once

$b \to a$ So must be one of them

So if two rankings must be same

<u>Assume</u> $P(n)$ is true

$P(n+1)$

$a_0 \to a_1 \to \cdots \to a_{n+1}$

$b_0 \to b_1 \to \cdots \to b_{n+1}$

?

if these are the same

(15)

Then by induction → are same ranking

Case 1 $a_0 = b_0$

If kill a_0 (and b_0 since same) then have same torn. left

So then induction new a_0 = new b_0, repeat

Case 2 $a_0 \neq b_0$

[the hard part]

a_0 is a player

must be somewhere in the bs

The people in front of him

$b_0 \to \dots \to b_g \to \dots b_{n-1}$

In order to not be true must be cont.
So is a cycle
So not a DAG
So Cont.

So must be a loop here
Go through this list till find the guy in ~~b~~ as
So have a cycle

16

Read the sol

No repeating verticies

Alt really easy solution

How to prove is perfect ranking

← biggest loser (or group!)

Will be a biggest loser

How to prove that they exist?

Start at arbitrary vertex, keep walking, pick arbitrary edges

No cycles so can never go back

Since finite # of vertices

Will stop eventually

Thats the biggest loser.

Kill that guy

Now new biggest loser

Repeat above process

~~Proves~~ Only 1 left

This also proves b and c

(I am starting to get hang of how this class <u>and</u> algorithms work!)

Relations Review

3/12

Transitive

$(aRb \text{ AND } bRc) \rightarrow aRc$

Symmetric

Asymetric

$aRb \rightarrow \text{Not}(bRa)$ for $\forall\, a, b \in A$

Antisymmetric (weak)

$aRb \rightarrow \text{Not}(bRa)$ for $\forall\, a \neq b \in A$

Reflexive

aRa for $\forall\, a \in A$

Irreflexive

$\text{Not}(aRa)\ \forall\, a \in A$

Strict Partial Order

transitive + irreflexive

positive path relation of a DAG

Weak Partial Order

transitive + reflexive + antisymetric

Relation R on set A is weak partial order iff Strict partial order S on A such that

aRb iff $(aSb$ or $a = b)$

What is Weak Partial Order

3/12

wikipedia readings

Order theory = studies binary relations that capture intuitive notion of ordering – like "less than" or "precedes"

Partial Order

P set

$\leq$ relation on P

then is partial order if reflexive, antisym, trans

$a \leq a$ reflexivity

$a \leq b$ and $b \leq a$ then $a = b$ antisymmetry

$a \leq b$ and $b \leq c$ then $a \leq c$ transitivity

So less then literally means earlier then?

? That why there is a special symbol?

* Means that one item precedes another *

? Means DAG?

~~Non~~ Partial since not element needs to be related

Total – every set is related

Strict Weak ordering $<$

- (transitive and irreflexive) or asymmetric

-

Problem Set 5

Due: March 14

Reading: Chapter 9–9.10.1, Parallel Task Scheduling.
Skip Chapter 9.11, Equivalence Relations and Chapter 10, Communication Nets, which will not be covered this term.

Problem 1.
How many binary relations are there on the set $\{0, 1\}$?

How many are there that are transitive?, ...asymmetric?, ...reflexive?, ...irreflexive?, ...strict partial orders?, ...weak partial orders?

Hint: There are easier ways to find these numbers than listing all the relations and checking which properties each one has.

Problem 2.
The following procedure can be applied to any digraph, G:

1. Delete an edge that is in a cycle.
2. Delete edge $\langle u \to v \rangle$ if there is a path from vertex u to vertex v that does not include $\langle u \to v \rangle$.
3. Add edge $\langle u \to v \rangle$ if there is no path in either direction between vertex u and vertex v.

Repeat these operations until none of them are applicable.

This procedure can be modeled as a state machine. The start state is G, and the states are all possible digraphs with the same vertices as G.

(a) Let G be the graph with vertices $\{1, 2, 3, 4\}$ and edges

$$\{\langle 1 \to 2 \rangle, \langle 2 \to 3 \rangle, \langle 3 \to 4 \rangle, \langle 3 \to 2 \rangle, \langle 1 \to 4 \rangle\}$$

What are the possible final states reachable from G?

A *line graph* is a graph whose edges are all on one path. All the final graphs in part (a) are line graphs.

(b) Prove that if the procedure terminates with a digraph, H, then H is a line graph with the same vertices as G.

Hint: Show that if H is *not* a line graph, then some operation must be applicable.

(c) Prove that being a DAG is a preserved invariant of the procedure.

by def of line graph + DAG

(d) Prove that if G is a DAG and the procedure terminates, then the path relation of the final line graph is a topological sort of G.

Hint: Verify that the predicate

$$P(u, v) ::= \text{there is a directed path from } u \text{ to } v$$

is a preserved invariant of the procedure, for any two vertices u, v of a DAG.

6.033

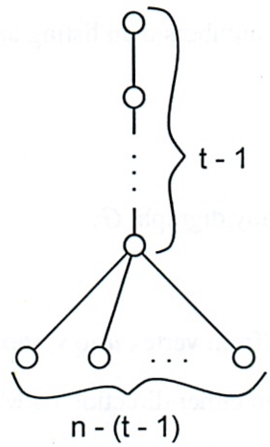
(e) Prove that if G is finite, then the procedure terminates.

Hint: Let s be the number of cycles, e be the number of edges, and p be the number of pairs of vertices with a directed path (in either direction) between them. Note that $p \leq n^2$ where n is the number of vertices of G. Find coefficients a, b, c such that $as + bp + e + c$ is nonnegative integer valued and decreases at each transition.

Problem 3.
We want to schedule n partially ordered tasks.

(a) Explain why any schedule that requires only p processors must take time at least $\lceil n/p \rceil$.

(b) Let $D_{n,t}$ be the strict partial order with n elements that consists of a chain of $t - 1$ elements, with the bottom element in the chain being a prerequisite of all the remaining elements as in the following figure:



What is the minimum time schedule for $D_{n,t}$? Explain why it is unique. How many processors does it require?

(c) Write a simple formula, $M(n, t, p)$, for the minimum time of a p-processor schedule to complete $D_{n,t}$.

(d) Show that *every* partial order with n vertices and maximum chain size, t, has a p-processor schedule that runs in time $M(n, t, p)$.

Hint: Induction on t.

Problem 4.
Let S be a sequence of n different numbers. A *subsequence* of S is a sequence that can be obtained by deleting elements of S.

For example, if

$$S = (6, 4, 7, 9, 1, 2, 5, 3, 8)$$

Then 647 and 7253 are both subsequences of S (for readability, we have dropped the parentheses and commas in sequences, so 647 abbreviates (6, 4, 7), for example). ✓ Reduces (readability)

An *increasing subsequence* of S is a subsequence of whose successive elements get larger. For example, 1238 is an increasing subsequence of S. Decreasing subsequences are defined similarly; 641 is a decreasing subsequence of S.

(a) List all the maximum length increasing subsequences of S, and all the maximum length decreasing subsequences.

Now let A be the *set* of numbers in S. (So $A = \{1, 2, 3, \ldots, 9\}$ for the example above.) There are two straightforward ways to totally order A. The first is to order its elements numerically, that is, to order A with the $<$ relation. The second is to order the elements by which comes first in S; call this order $<_S$. So for the example above, we would have

$$6 <_S 4 <_S 7 <_S 9 <_S 1 <_S 2 <_S 5 <_S 3 <_S 8$$

Next, define the partial order $\prec$ on A defined by the rule

$$a \prec a' \quad ::= \quad a < a' \text{ and } a <_S a'.$$

(It's not hard to prove that $\prec$ is strict partial order, but you may assume it.)

(b) Draw a diagram of the partial order, $\prec$, on A. What are the maximal elements,... the minimal elements?

(c) Explain the connection between increasing and decreasing subsequences of S, and chains and anti-chains under $\prec$.

(d) Prove that every sequence, S, of length n has an increasing subsequence of length greater than $\sqrt{n}$ or a decreasing subsequence of length at least $\sqrt{n}$.

(e) (Optional, tricky) Devise an efficient procedure for finding the longest increasing and the longest decreasing subsequence in any given sequence of integers. (There is a nice one.)

Doing P-Set 5

3/12

Binary relations - just the arrows

Chap 4.4

a R b

But don't you need 2 sets?

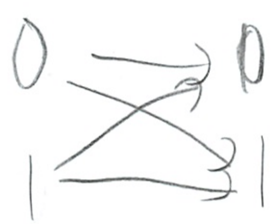
- domain + codomain

Or am I getting notation for {0,1} wrong

Just set w/ 2 elements

Read this Chap!

domain + coadomain are same



Learned iff instead of = from review session

Memorize transitive, symetric, reflexive, etc

What does "weak partial order" really imply?

Oh skipped hint - wonder what it is

#1 This is just a set of 0 and 1, right?
And relation where domain and codomain are the same

4 total relations (arrows)

Another way to draw

Transitive iff $(a R b$ AND $b R c) \to (a R c)$ (4 lines)

Yes, these lines are transitive because there is a line from any point to any point so can replace two lines with one line

like

$0 \to 1$ AND $1 \to 0 \longrightarrow 0 \to 0$
$0 \to 0$ AND $0 \to 0 \longrightarrow 0 \to 0$
$0 \to 0$ AND $0 \to 1 \longrightarrow 0 \to 1$

Assymetric iff $a R b$ implies NOT $(b R a)$

This is not true for all lines because there is always a way to get back (0 lines)

$0 \to 0 \longrightarrow 0 \to 0$
$0 \to 1 \not\longrightarrow 1 \to 0$

②

Reflexive iff aRa for all $a \in A$

All of the verticies are reflexive (2 lines)

Irreflexive iff NOT (aRa) for $\forall a \in A$

The whole graph is reflexive (2 lines)

But two of the lines are irreflexive (What I belie you are asking about here)

$0 \to 1$

$1 \to 0$

Strict partial orders iff transitive + irreflexive

All lines transitive, 2 were irrelexive (2 lines)

$0 \to 1$

$1 \to 0$

Weak partial orders iff transitive + reflexive + antisymetric

Remember antisymetric ≠ asymmetric

So antisymetric $\stackrel{?}{=}$ aRb iff ($a S b$ or $a = b$)
(S strict)

So our two strict partial orders count (4 lines)

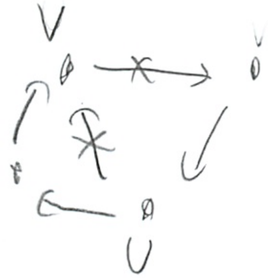
$1 \to 0$

$0 \to 1$

As well as our two that are = to each other $0 \to 0$ $1 \to 1$

②

#2 Procedure



Path = can repeat

So #2 is delete edge if something else

So this is basically force into DAG

But may change meaning?

Can formalize as a state machine

G ∈ start

all other state are same vertices of G

but w/ diff arrows

Did very long + extencive ans for a

b) line graph - edges are al on one path

Prove that end w/ line graph

Good thing hint was there

Or I would have shown that my things were line graphs

which would have been wrong

But how exactly do I prove each case

- a key problem I should think about

(3)

How to do w/o proof by cases on each aspect of line graph-ism
I should be able to argue using the rules

Don't do   Conversely rules can't apply if line graph
- not what asking to prove
- or can I extend it over somehow

If it terminates must be line graph

~~It will not terminate~~

If no cycles then ~~m~~ line graph by def of line graph
If cycles then can't terminate

∴

Can I do this??

Its this very complicated way of thinking!

~~AA~~ What trying to prove?
I'm getting all turned around!
Proving that when stops then it is a line graph
But hint that if Not a line graph can do stuff

Going to submit this
I think it is close
I am learning to prove things from every angle!

(4)

c) Looks easy

Do we have to do equation for preserved invarient

? Or can we just use words

d) Path relation can be topological sort ← this is 1 class per semester

? look up formal def.

What is total again? - every set is related

Hint verify $P(v,v)$

↑ directed path from v to v

I'm proud I was able to build my understanding for this qu

I hope its right!

e) Oh no there is an e on the back!

So for larger Gs how to show that will terminate

- which I have not yet done

So for my inital

$s = 1$ $\quad n = 4$

$e = 5$

$p = 4$ $\quad 4 \leq 4^2$

$a \cdot 1 + b \cdot 4 + 5 + c$ = integer that decreases at each transition

← any move

5

$1a + 4b + 5 + c = k$

1) $0a + 4b + 5 + c = k-1$

so $a = 1$

2) no change as -1 ~~a~~

~~a~~ no e -1, p unchanged $k-1$

but e just there

3) $+2p$, $+2e$, $k-1$ ← may not decrease by one

$b = -1$

Now put together

$1 \cdot 1 + -1 \cdot 4 + 5 + c = k$

$1 \cdot 0 + -1 \cdot 4 + 5 + c = k-1$

$1 \cdot 0 - 1 \cdot 4 + 4 + c = k-2$

$1 \cdot 0 - 1 \cdot 6 + 6 + c = k-3$

$2 + c = k$

$1 + c = k-1$

$0 + c = k-2$

$0 + c = k-3$

What is steady state?

$a \cdot 0 + b \cdot 6 + 3 + c = k-6$

(6)

$2+c=k$

$1+c=k-1$

$0+c=k-2$

$0+c=k-3$

$-3+c=k-6$

$2+c=k$

$2+c=k$ ← why different?

$3+c=k$

$3+c=k$

Can't seem to find discepency

But how does this prove proc terminates?

Since k must go to certain value

Skip for now

What is # relations

	1	2	3	4
1		X	X	X
2			X	X
3				X
4				

$\frac{n^2}{2} - n/2$

half   not center row

$\frac{n^2-n}{2}$

~~6~~ Discarded 3c ~~[illegible]~~ answer

c) ~~Prove that this means it is a DAG.~~

~~Because we found in B that this will be a line graph, we must show that a line graph is always a DAG.~~

~~A DAG (Directed Acyclic Graph) is a directed graph with no cycles,~~

~~A directed graph is basically a graph with arrows that are directional.~~

~~A line graph is a DAG because~~

~~- it is a directed graph by definition~~
~~- it is acyclic beause we found in B that if there was a cycle, then it would not terminate and thus it would not be a line graph.~~

Preserved invarient once it comes in it never leaves

⑦.

#3. Scheduling

have not fully read yet
But that is on the skip list

Oh this is from that day w/ the ~~[illegible]~~ 6-2 DAG

a) How to prove? - Seems too easy!
Did I write too much/little

b) How to prove uniqueness?

c) Formula, w/ limited processors for min time

First iterate over the $n - (t-1)$ taks

$$\left\lceil \frac{n-(t-1)}{p} \right\rceil + t - 1$$

↑ the other tasks one at a time

d) Now prove?
Induction on t
Induction pratice!
Oh any!
Oh wait my formula was not right
$t = 1$
So say $n=5$ O O O O O time $\lceil \frac{n}{p} \rceil$

8

$t=2$

$n=5$

takes $\lceil \frac{4}{p} \rceil + 1$

? oh $t-1$

Oh I was wrote, just had typo

Any better way to prove?

? Also holds on other dimensions?

(9)

4. S= sequence

take sub sequence

Stupid thing of readability - makes it harder

max lenght =? max possible lenght?

? just try?

6 7 9

4 5 8

1 2 5 8

---

6 4 7

9 5 3

4 1

7 8

7 5 ~~8~~ 3

6 5 3

6 4 3

~~6 5 3~~

~~7 5~~

4 3

4 1 ~~8~~

In 10 chap -> diameter of network

(10)

b) 2 ways to order

- by number

and $<_s$

partial order $a \leq_s$ means $a < ' a$ and $a <_s '$

So both smaller and comes before in set

Does that get custom designated each time?

Now draw diagram

$6 <_s 4$ (x)

$6 <_s 7$ ✓

$6 \leq_s 9$ ✓

$6 <_s 8$

4 1, 2, 3

$7 \to 9, 8$

9 x

1 2, 3

2 , 3

$5 < 8$

$3 < 8$

8 x

? how

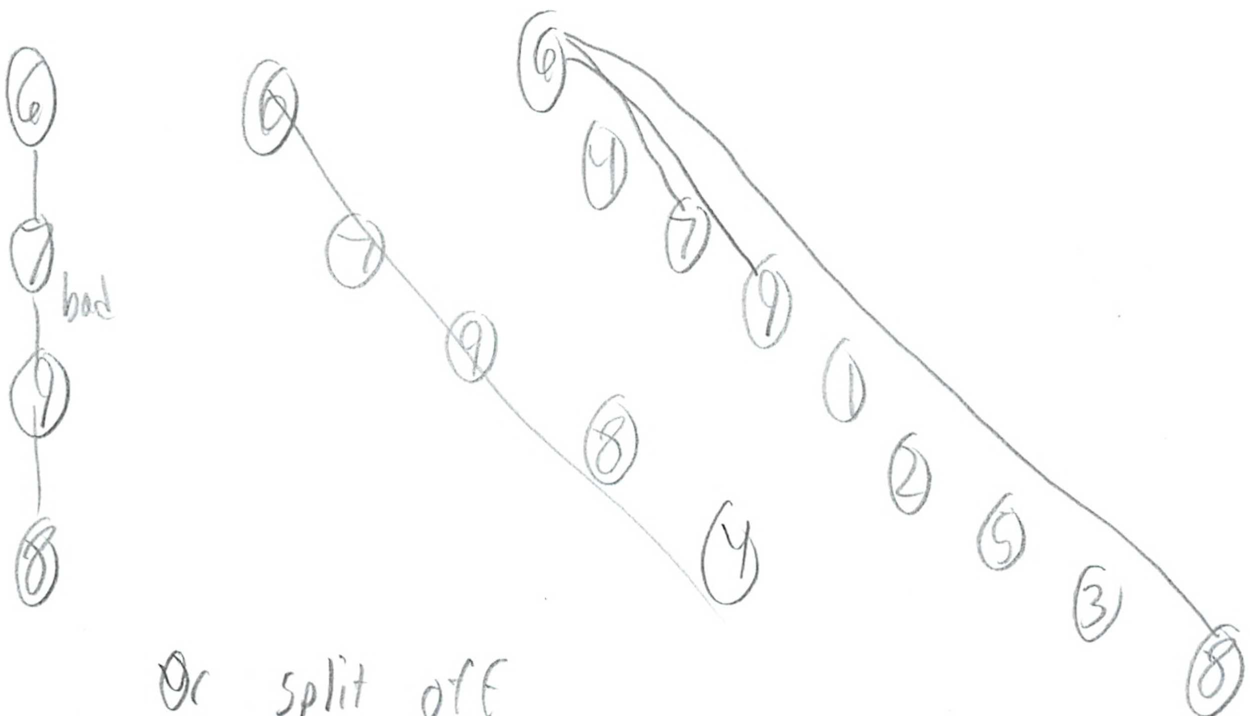
- what order?

← like that

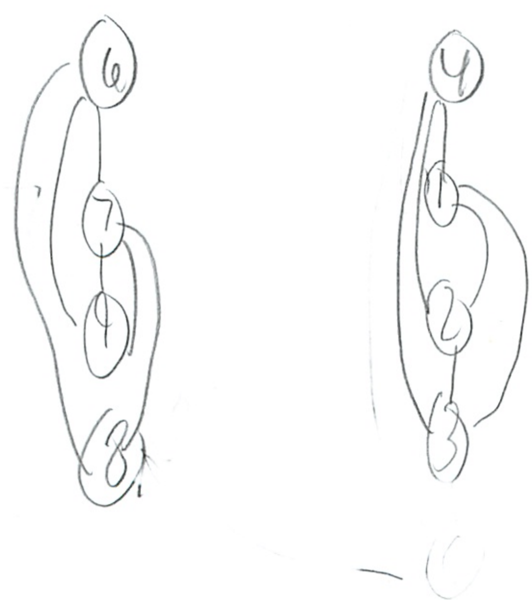
but # only once

11

No do down



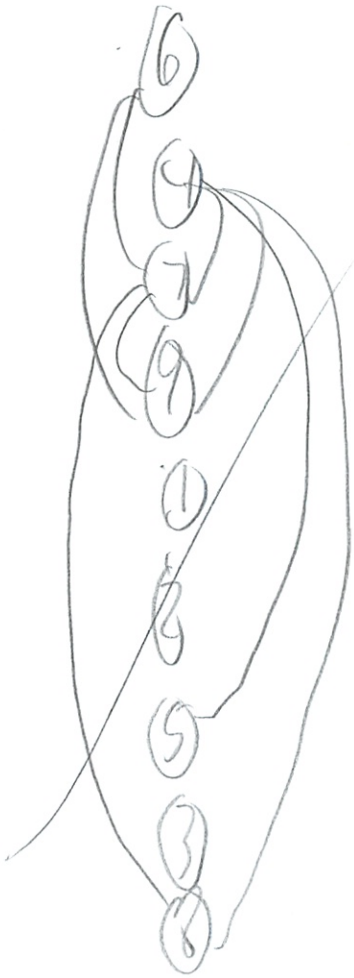
Or split off
no will repeat



Oh had wrong order!

(12)

Do in order



? a clearer way to org?

← Incomple

c) What connection? Oh

chains = sequence in order

Just the arrows ~~a~~ decreasing

Don Looked at incomplete chart

d) Prove that every sequence ~~of~~ S of length n, has increasing sub seq $\geq \sqrt{n}$ and decreasing sub seq $\geq \sqrt{n}$

(13)

e) Oh I see - the ans to my dv tricy means don't try!

So ~~this~~ check

$n = 9$

~~3~~ $3 \geq \sqrt{9}$

✓

$4 \geq \sqrt{9}$

$4 \geq 3$

✓

But how do you prove?

~~M~~ Must be some pattern

Oh one or other

Otherwise 1 2 3 4 5 6 7 8 9

no decreasing, but ✓ increasing

~~M~~ 5 6 4 7 3 8 2 9 1

mix up but then both

If pick random's - try pc

736 236 520

765

264 587 745

245

Won't find a counter example

Ask

(14)

Matts Help

Begining

Mark every time the every int is < previous

Either $\geq \sqrt{n}$ works

Or ~~$\leq \sqrt{n}$~~ don't - so case 2

at least $\sqrt{n}$ b/w 2 corresponding el

If neither

More than $\sqrt{n}$ subsets

So a 7 9

(6) (4 7 9) (1 2 5) (3 8)

either (1) must be size $\geq \sqrt{n}$ - ~~weakly decreasing~~

or (2) ~~more than~~ $\geq \sqrt{n}$ subsets - weakly increasing

or diff change

(6) (4) (2 7) (5 9) (3 8) (1)

$a_i + 1 < a_i$

---

other dir chang (6 4 2) (7 5) (9 3) (8 1)

Confused, told me to ask

After OH much clearer - hope wrote good proof

Oscar OH

3/13

2e) If want to show ends — comes up w/ some quanity
If 0 it stops
Decreases some integers
Then if it reaches 0 it stops
So can show it decreases
invariant - min of 2 #s

---

b) If it ended → it is a line graph
c) That it ended.

Can apply in any order

4d - that exists
not by constucting
Use partial order - don't think about seq
how can b + c help

---

increasing subseq = ~~longest~~ chain

antichain = decreasing sub seq ?

(2)

big lenght   big   ← of that size

Either chain or anitchain size

Therm in book/class ab.

No induction

Chain - can ~~also~~ be comparable

{b, d} is chain

{a, b} is antichain - not comparable

{a, b, c} neither

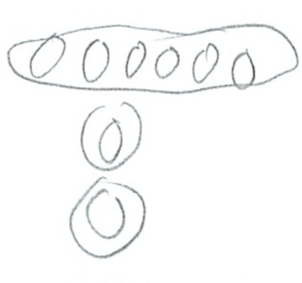
{b, c} antichain

antichain - can do in any order

If do # 2 rules

---

for time schedule



(3)

 also works and then each variation / also 

2c) Perserved invarient - if it becomes true than it stays there

↑

1, Each possible arrow set

$0 \to 1$ does the set meet each criteria

$0 \quad 1$

$0 \quad 1$

$0 \searrow 1$

$0 \quad 1$

$0 \to 1$

Student's Solutions to Problem Set 5

Your name: Michael Plasmeier

Due date: March 14

Submission date: 3/14

Circle your TA/LA: Ali Nick Oscar Oshani

Table 12

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

got help from:[1] Wikipedia: Strict weak ordering
and referred to:[2] Partially ordered set
Order theory

Oscar's OH
+students in Oscar's OH

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	4
2	9.2
3	9
4	10
Total	32.2

– did much better than usual!

[1]People other than course staff.

[2]Give citations to texts and material other than the Spring '11 course materials.

#1 This is just a set of 0 and 1 right?

4

Relation where domain + codomain are the same

a relation will all arrows

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}$$

$2^4 = 16$ possible relations ✓

<u>Transitive</u> iff ($a R b$ and $b R c$) → ($a R c$) for $\forall a,b,c \in A$

<u>Asymetric</u> iff $a R b$ → NOT ($b R a$) for $\forall a,b \in A$

<u>Reflexive</u> iff $a R a$ for all $a \in A$

- that all states are relexive
- not that every arrow is reflexive

<u>Irreflexive</u> iff NOT ($a R a$) for $\forall a \in A$

<u>Strict partial Order</u> iff transitive + irreflexive

<u>Antisymetric</u> ($\neq$ a symmetric) iff $a R b$ → NOT ($b R a$) for all $a \neq b \in A$

<u>Weak partial order</u> iff transitive + reflexive + antisymmetric

iff $a R b$ iff ($a S b$ or $a = b$) for $\forall a,b \in A$

(2) . . .

this more
2nd part must work!

where's the empty relation?

no horiz arrows

~~what about SPO?~~

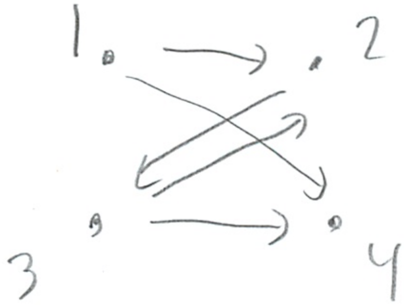
review what transitive means and try this again (or read solutions) :)

	Transitie	Asymetric	Refexive	Irrefl.	Strict	Antisym	Weak
0→0 1 1	✓	X	X	X	X	✓	X
0→0 1 1	✓	X	X	X	X	✓	X
0 0 1 1	X	✓	X	✓	X	✓	X
0 0 1→1	✓	X	X	X	X	✓	X
0 0 1 1	X	✓	X	✓	X	✓	X
0→0 1 1	X	X	X	X	X	✓	X
0 0 1→1	X	X	X	X	X	✓	X
0→0 1→1	✓	X	✓	X	X	✓	✓
0→0 1→1	✓	X	✓	X	X	✓	✓
0→0 1→1	✓	X	✓	X	X	✓	✓
0 0 1→1	X	X	X	X	X	X	X
0→0 1 1	X	X	X	X	X	X	X
0 0 1 1	X	X	X	✓	X	X	X
0 0 1→1	✓	X	X	X	X	✓	X
0→0 1→1	✓	X	✓	X	X	X	X
Total	8 13	2 3	4 ✓	3 4	3 0 !	12	3 ✓

Michaël Plasmeier
Oshani
Table 12

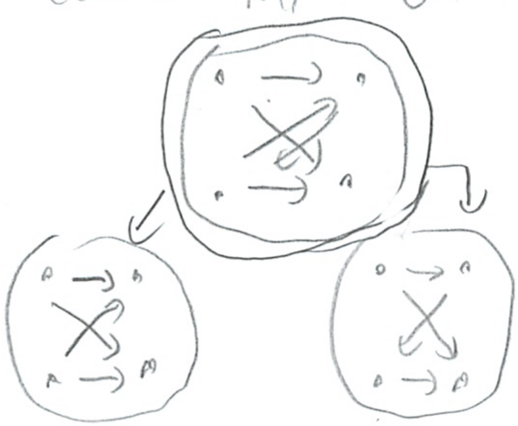
$\frac{9,2}{10}$

#2a. $G = \{1, 2, 3, 4\}$



What are all the things you could do to this?

1.) First cycle $2 \to 3 \to 2$
Delete $2 \to 3$ or $3 \to 2$
First decision tree on SM



2) Delete edge if alternate path

Can't do anything here

Can drop  since $\langle 1 \to 2 \rangle \langle 2 \to 3 \rangle \langle 3 \to 4 \rangle$

$\langle 1 \to 4 \rangle$

 So this is also possible

(2)

3) Add edge if no path in either direction $U \to V$

For the left branch, check each possible path

$\langle 1,2\rangle$ and coverse $\langle 2,1\rangle$ → ✓ exists directly

$\langle 1,3\rangle$ " → X does not exist in either direction add

$\langle 1,4\rangle$ " → ✓ direct

$\langle 2,3\rangle$ " → ✓ direct (reverse)

$\langle 2,4\rangle$ " → X does not exist, add

$\langle 3,4\rangle$ " → ✓ direct

Now right hand side

$\langle 1,2\rangle$ + converse → ✓ direct

$\langle 1,3\rangle$ " → ✓ 2 paths

$\langle 1,4\rangle$ " → ✓ 3 baths

$\langle 2,3\rangle$ " → ✓ direct

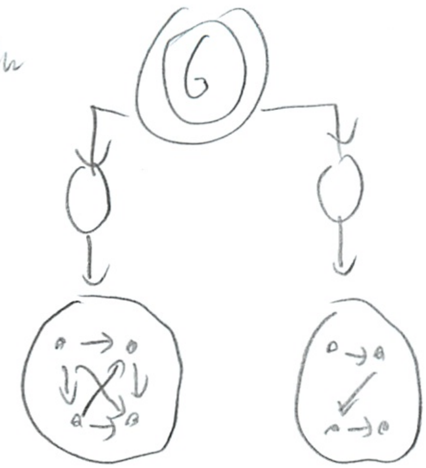
$\langle 2,4\rangle$ " → ✓ 2 paths

$\langle 3,4\rangle$ " → ✓ direct

No changes

(3):

Go to review



Repeat

1) Delete cycles

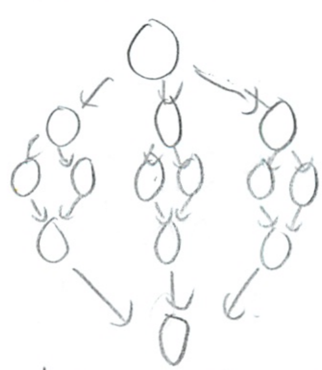
No cycles on either

All have terminal point

2) Delete direct paths

repeat inside each step

Could do in any order



but would end the same

(4)

3. Add paths if not exist

$\langle 1,2 \rangle$ + converse → ✓ 2 paths

$\langle 1,3 \rangle$ " → ✓ direct

$\langle 1,4 \rangle$ " → ✓ 3 paths

$\langle 2,3 \rangle$ " → ✓ reverse

$\langle 2,4 \rangle$ " → ✓ direct

$\langle 3,4 \rangle$ " → ✓ 2 paths

So final states are   Other ones from OH   missing 1

$\frac{1.7}{2}$

So (5)

and every rotation
- but perhaps not specific from this starting state

b) ~~Proof by contridiction. If these procedures could terminate and not be a line graph then there must be a state where no rule can be applied and it is not a line graph~~

$\frac{2}{2}$

Proof. If it terminates, it must be a line graph (w/ the same verticies as G) ~~will always hold, not going to repeat each time)~~
Because if it can not terminate then it can not be a line graph.

(5).

1. If there are cycles left then it can not terminate.

   This is because a line graph requires all edges to be on one path. If there are cycles, then it is not one path, and thus not a line graph.

   If this step does not apply, then there are no cycles which fits the definition of a line graph.

2. If there were two possible paths between verticies then it wold not be a line graph.

   If it was a line graph - where there is only one path which all edges are a part of (and there are no cycles, as above), then this step cold not apply and it wold terminate.

   If it were not a line graph, then this step may still apply.

3. If there was no path between two verticies such as

   Two points coming in does not meet the def. of a line graph because it does not show that there is one single path b/w the points.

Thus, if it terminates, it must be a line graph.

So if it does not terminitate, then an operation may still be possible,

(6)

c) Preserved invarient - Once it becomes true, it will never be false again.

Proof. Once the procedure terminates, it is a line graph (b)). When it becomes a line graph no further steps can take place.

(I did a lot of this in b - put too much in there

1. A line graph has by definition no cycles because all of the edges are on one path. A path by definition has no cycles.
2. A line graph has only 1 way to get to each point because all of the edges follow one path.
3. A line graph is made up of one path between all of the verticies.

Thus if the procedure terminates, no more steps are possible. Once no steps are possible, no steps will ever be possible because nothing changes.

2/2

A line graph is always a DAG

6b.

A DAG (directed acyclic graph) is a directional graph with no cycles.

A directed graph is basically a graph with arrows that are directional.

A line graph is a DAG because

- it is a directed graph by definition
- it is acyclic, because we found in b) that if there was a cycle then it would not terminate and thus not be a line graph.

(7)

d) A topological sort of a partial order $\preccurlyeq$ on a set A is the total ordering $\sqsubseteq$ such that

$$a \preccurlyeq b \rightarrow a \sqsubseteq b$$

2/2

A corellary is that $P(u,v)$ is true

$P(u,v) ::=$ there is a directed path from u to v

This is a preserved invariant for all pairs of verticies of a DAG.

We showed above that if the procedure terminates then we have a line graph (b) and that a line graph is a DAG (c).

We can "unwrap" our line graph to lie horizontally. A line graph means that all the edges line on one path. This means that if u,v are on the path then there will be a directed path between u and v.

Proof by structural induction on $P(u,v)$

<u>Base</u> u,v directly connected

true by definition

⑧

Constructor. If we have two paths

$a \to b$

$b \to c$

Which we assume to be line graphs and thus have a directed path $v \to v$

Then connecting them

$a \to b \to c$

leads to a bigger line graph that still has a directed path $v \to v$

e) Proof. In order to show that something is finite in this class we show that a procedure terminates at 0 or some value.

~~For this example.~~

~~Starts at $1a + 4b + 5 + c = k$~~

1.5/2

Step 1 ~~will lead to $0a + 4b + 5 + c = k - 2$~~

So as long as a is positive, the equation will decrease for every edge removed, as s ↓ by 1

Step 2 will lead to e decreases by 1, so the expression decreases

Step 3 causes p to increase and e to increase, As long as b is negitive and larger than one, Then the

(9)

expression will decrease each step.

C should be large and positive such that the equation starts positive and decreases to 0 when it terminates

Because it terminates with a line graph, at termination (which has $n-1$ edges)

$$s = 0$$
$$e = n-1$$
$$p = \frac{n^2-n}{2}$$

And if we define $a = 1$ $b = -2$ then

$$1 \cdot 0 + -2 \cdot \frac{n^2-n}{2} + n-1 + C = 0$$

$$C = (n-1)^2$$

So for $a = 1$, $b = -2$, $C = (n-1)^2$ the equation $as + bp + e + c$ starts positive and decreass at each step
↑ prev page

When it can no longer decrease then we are at a line graph + done

Michael Plasmeier
Oshani
Table 12

9

#3, a. $\lceil n/p \rceil$

This is the best-case amount of time it takes to do n tasks using p processors. Best case means that each task can be done independently (in any order) (no dependencies)

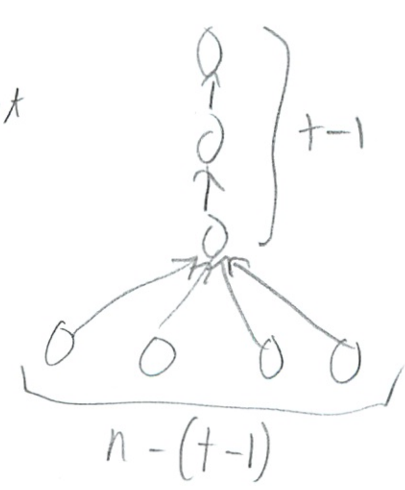
Every time period I give a job to each processor. This means I can do $n = t \cdot p$ jobs

+2

If I had n, jobs I could do them in $\left\lceil \frac{n}{p} \right\rceil$ time. I need to round up because if only have the processers are busy those half still need a full time period to complete the job. (ie two processors can't split the 1 job and do it in half the time)

b) Special partial order $D_{n,t}$

all prereqs →



2

b) $D_{n,t}$ could be done in minimal time with $n-(t-1)$ processors. All would be in use on the first step. It would take $t-1+1 = t$ seconds.

This is a unique schedule because there is no other way to do this in minimal time.

One would want to complete the horizontal $n-(t-1)$ tasks first in 1 time step, because it makes no sense to only do a fraction of them in the first time step if one has ∞ processors, since in the second step one would have to do the remainder of the $n-(t-1)$ tasks. Then one can only do one of the vertical $t-1$ tasks each turn since they are each prereqs of one another. It makes no sense to take a round off. +2

c) $$\underbrace{\left\lceil \frac{n-(t-1)}{p} \right\rceil}_{\text{? the horizontal tasks based off a)}} + \underbrace{t-1}_{\text{the vertical tasks}}$$ +2

(3)

d) Now prove.

Proof by induction on t

$P(t)$ = Any partial order
n vertices
t max chain size
p processors
runs in $\left\lceil \frac{n-(t-1)}{p} \right\rceil + t - 1$

+3 see solutions

Base $t=0$

Can't really have a 0 chain size

Base $t=1$

So this is an all flat schedule - do anything any time just like a)

$$\left\lceil \frac{n-(1-1)}{p} \right\rceil + 1 - 1$$

$$\left\lceil \frac{n}{p} \right\rceil$$

Example
$n=5$

Base $t=2$

(Not needed, but demonstrate)

$$\left\lceil \frac{n-(2-1)}{p} \right\rceil + 2 - 1 = \left\lceil \frac{n-1}{p} \right\rceil + 1$$

Example
$n=5$

(4)

Inductive

Assume $P(t)$

Will show for $P(t+1)$

$$\left\lceil \frac{n-((t+1)-1)}{p} \right\rceil + (t+1)-1$$

$$\left\lceil \frac{n-t}{p} \right\rceil + t$$

Which makes sense. With fixed n and p, when you increase t by one, one task is removed from the horizontal row and is added to the vertical column. The numerator of the first part of the equation decreases by 1 and the entire number increases by 1

Thus $M(n,t,p)$ holds for every t.

Michael Plasmeier

Oshani

Table 12

#4 a) Just guess + check ¡

Increasing 1 2 5 8 max lenght = 4

1 2 3 8

10

Decreasing ~~647~~

953

753 max lenght = 3

653

643

641, 642

b) So I am assuming that I can only draw a partial order arrow if it is earlier in order and less then numerically.

So 6 $\prec$ 7, 9, 8

4 $\prec$ 7, 9, 5, 8

7 $\prec$ 9, 8

9 $\prec$ $\emptyset$

1 $\prec$ 2, 5, 3, 8

2 $\prec$ 5, 3, 8

5 $\prec$ 8

3 $\prec$ 8

8 $\prec$ $\emptyset$

7

Minimal elements = elements that only start arrows
no arrows in

6, 4, 1

maximal elements = elements that only end arrows
no arrows out

9, 8

DAG

③

c) "Connection":

Well follow the arrows down the diagraph to get the "increasing" subsequences $1 \to 2 \to 5 \to 8$

~~If you take the inverse of the relation $\prec^{-1}$ (flip the direction of the arrows) then you could~~

If you instead defined

$$a \preceq' a' ::= a > a' \text{ and } a <_s a'$$

then you would have the arrows for "decreasing" subsequences

Q4: * Also this is the number of <u>anti-chains</u> in the original diagraph *

So the chains are increasing subsequences

d) Proof. Start by constructing a diagraph by defining the partial order

$$a \preceq a' ::= a < a' \text{ and } a <_s a'$$

Now, from c we know that increasing subsequences are the chains in the diagraph and decreasing subsequences are the antichains in the diagraph.

4

Then by Lemma 9.10.10 (Dilworth's) Lemma we know that for all $t > 0$ every partially ordered set with n elements must either have a chain greater than t or an antichain of size at least n/t.

So we have a set with n elements. We set: $t = \sqrt{n}$

Now either a) we have a chain greater than t - which means that we have an increasing subsequence of lenght greater than $\sqrt{n}$ which is stronger than the requirement that the increasing subsequence be at least of lenght $\sqrt{n}$.

b) we have an antichain of "size" $\frac{n}{t}$ or $\frac{n}{\sqrt{n}}$,

Well $\frac{n}{\sqrt{n}}$ simplifies to $\sqrt{n}$ so this satisfys the requirement that we have a decreasing subsequence of at least "size" $\sqrt{n}$ $\blacksquare$

Solutions to Problem Set 5

Reading: Chapter 9–9.10.1, Parallel Task Scheduling.
Skip Chapter 9.11, Equivalence Relations and Chapter 10, Communication Nets, which will not be covered this term.

Problem 1.
How many binary relations are there on the set $\{0, 1\}$?

How many are there that are transitive?, …asymmetric?, …reflexive?, …irreflexive?, …strict partial orders?, …weak partial orders?

Hint: There are easier ways to find these numbers than listing all the relations and checking which properties each one has.

Solution. There are $2^4 = 16$ such relations, since in any such relation there are four possible arrows between $\{0, 1\}$ and itself, each of which may or may not be there.

There are 3 **in**transitive transitive relations, because the only way transitivity can fail in a relation on two elements is when there is an arrow in both directions between the elements, but one or the other or both the elements are missing a *self-loop*, that is, an arrow that starts and ends at the element. So there are $13 = 16 - 3$ transitive relations.

There are 3 asymmetric relations. Asymmetry implies no self-loops, and at most one of the two possible arrows between 0 and 1. So the only 3 possibilities are no arrows, arrow from 0 to 1, arrow from 1 to 0.

There are 4 reflexive relations, because two of the four possible arrows (the self-loops) must be present, the remaining two arrows can be either present or not present, which yields 2^2 relations. There are 4 irreflexive relations for the same reason.

There are 3 strict partial orders, because the 3 asymmetric relations are all transitive.

There are 3 weak partial orders, because the 3 strict partial orders remain distinct after adding self-loops to both elements.

■

Problem 2.
The following procedure can be applied to any digraph, G:

1. Delete an edge that is in a cycle.
2. Delete edge $\langle u \to v \rangle$ if there is a path from vertex u to vertex v that does not include $\langle u \to v \rangle$.
3. Add edge $\langle u \to v \rangle$ if there is no path in either direction between vertex u and vertex v.

Repeat these operations until none of them are applicable.

This procedure can be modeled as a state machine. The start state is G, and the states are all possible digraphs with the same vertices as G.

(a) Let G be the graph with vertices $\{1, 2, 3, 4\}$ and edges

$$\{\langle 1 \to 2 \rangle, \langle 2 \to 3 \rangle, \langle 3 \to 4 \rangle, \langle 3 \to 2 \rangle, \langle 1 \to 4 \rangle\}$$

What are the possible final states reachable from G?

Solution. There are six::

$$\{\langle 1 \to 2\rangle, \langle 2 \to 3\rangle, \langle 3 \to 4\rangle\}$$
$$\{\langle 1 \to 3\rangle, \langle 3 \to 2\rangle, \langle 2 \to 4\rangle\}$$
$$\{\langle 3 \to 1\rangle, \langle 1 \to 2\rangle, \langle 2 \to 4\rangle\}$$
$$\{\langle 1 \to 3\rangle, \langle 3 \to 4\rangle, \langle 4 \to 2\rangle\}$$
$$\{\langle 3 \to 1\rangle, \langle 1 \to 4\rangle, \langle 4 \to 2\rangle\}$$
$$\{\langle 3 \to 4\rangle, \langle 4 \to 1\rangle, \langle 1 \to 2\rangle\}$$

The last five can all be reached by deleting first $\langle 1 \to 4\rangle$ and then $\langle 2 \to 3\rangle$.

■

A *line graph* is a graph whose edges are all on one path. All the final graphs in part (a) are line graphs.

(b) Prove that if the procedure terminates with a digraph, H, then H is a line graph with the same vertices as G.

Hint: Show that if H is *not* a line graph, then some operation must be applicable.

Solution. Since vertices are not changed in any transition, H will have the same vertices as G. So we need only show that if H is *not* a line graph, then an operation is applicable.

Now if H has a directed cycle, then operation 1. applies. So H must be a DAG. Further, if there are two incomparable elements, $u \neq v$ in the partial order defined by this DAG, then operation 3. would be applicable to add either $\langle u \to v\rangle$ or $\langle u \to v\rangle$. So the DAG must define a path-total order.

All that remains is to prove that no vertex has in-degree or out-degree greater than one. The proof for in-degree and out-degree is virtually the same, and we'll just prove that out-degree is at most one.

So suppose to the contrary that in H, a vertex u has out-degree of 2 or more. So there are vertices $v \neq w$ and edges $\langle u \to v\rangle$ and $\langle u \to w\rangle$ in H. Now since H defines a path-total order, there must be a directed path, π, in one direction or the other between v and w; moreover π does not go through u (if it did, there would be a positive length cycle). Hence, the path u, π goes from u to w without including $\langle u \to w\rangle$, which means that $\langle u \to w\rangle$ could be deleted by applying operation 2. ■

(c) Prove that being a DAG is a preserved invariant of the procedure.

Solution. Deleting an edge cannot create a cycle, and neither can adding an edge between unconnected vertices. So if there was no positive length cycle in a graph, there wouldn't be any after one state transition.

■

(d) Prove that if G is a DAG and the procedure terminates, then the path relation of the final line graph is a topological sort of G.

Hint: Verify that the predicate

$$P(u, v) ::= \text{there is a directed path from } u \text{ to } v$$

is a preserved invariant of the procedure, for any two vertices u, v of a DAG.

Solution. *Proof.* To prove $P(u, v)$ is an invariant, suppose $P(u, v)$ holds in some DAG H. Then operation 1. won't be applicable since there are no cycles. Also, since adding an edge preserves all existing paths, operation 3. will preserve $P(u, v)$. This leaves only operation 2, to consider.

So suppose operation 2. is applied to delete an edge $\langle x \to y \rangle$ of H. By definition of the operation, this would only be possible if there remains a directed path, π, from x to y. Hence for any directed path that traversed $\langle x \to y \rangle$, there remains a directed path between the same endpoints obtained by replacing edge $\langle x \to y \rangle$ by π. So $P(u, v)$ will still hold.

Since G is a DAG, its path relation is a partial order, $\preceq_G$. By part (b), the procedure terminates with a DAG, H, that defines a path-total order, $\preceq_H$. So to show that $\preceq_H$ is a topological sort of $\preceq_G$, we need only check that

$$u \preceq_G v \text{ IMPLIES } u \preceq_H v.$$

But $u \preceq_G v$ is equivalent to $P(u, v)$ holding in G, and since P is preserved, $P(u, v)$ still holds in H, and this is equivalent to $u \preceq_H v$.

■

(e) Prove that if G is finite, then the procedure terminates.

Hint: Let s be the number of cycles, e be the number of edges, and p be the number of pairs of vertices with a directed path (in either direction) between them. Note that $p \leq n^2$ where n is the number of vertices of G. Find coefficients a, b, c such that $as + bp + e + c$ is nonnegative integer valued and decreases at each transition.

Solution. Since $s, e \in \mathbb{N}$ and $0 \leq p \leq n^2$, where n is the number of vertices of G, the value

$$2n^2 s - 2p + e + 2n^2$$

is always nonnegative. We claim it is strictly decreasing. To prove this, we consider the effect of the three kinds of operations.

Adding edge $\langle u \to v \rangle$ by operation 3. adds one to e and leaves s unchanged. Also, pairs of vertices connected by a directed path remain connected after adding an edge, and adding $\langle u \to v \rangle$ creates the new connected pair, (u, v), so p increases by at least one. Therefore $2n^2 s - 2p + e$ decreases by at least one.

Deleting an edge by operation 1. decreases e and s by at least one. It could also decrease p, but not by more than the total number, n^2, of pairs of vertices, so $2n^2 s - 2p + e$ decreases by at least one.

Finally, deleting an edge decreases e by one and never increases s. Further, deleting an edge by operation 2. does not change the path relation, as explained in the solution to part (c), so p does not change. so $2n^2 s - 2p + e$ decreases by at least one.

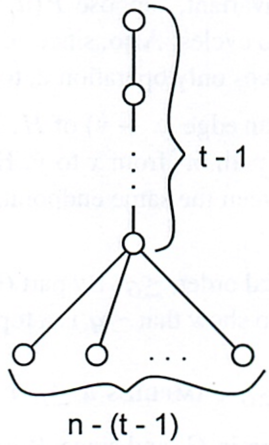
■

Problem 3.

We want to schedule n partially ordered tasks.

(a) Explain why any schedule that requires only p processors must take time at least $\lceil n/p \rceil$.

Solution. At most p tasks can be completed at any step in a schedule, so the total number of tasks that can be completedin t parallel steps is tp. So to complete all the tasks requires $tp \geq n$, which implies $t \geq \lceil n/p \rceil$. ■



(b) Let $D_{n,t}$ be the strict partial order with n elements that consists of a chain of $t-1$ elements, with the bottom element in the chain being a prerequisite of all the remaining elements as in the following figure:

What is the minimum time schedule for $D_{n,t}$? Explain why it is unique. How many processors does it require?

Solution. There's no choice but to schedule each of the $t-1$ vertices on the path one at a time in order. A minimum time schedule then does all the remaining $n-(t-1)$ vertices at the tth time interval. The number of processors required is therefore $n-t+1$. ■

(c) Write a simple formula, $M(n,t,p)$, for the minimum time of a p-processor schedule to complete $D_{n,t}$.

Solution. As in part (b), there's no choice but to schedule each of the $t-1$ vertices on the path one at a time in order. A minimum time schedule then does all the remaining $n-(t-1)$ vertices p at a time, for a total time of

$$M(n,t,p) ::= (t-1) + \left\lceil \frac{n-(t-1)}{p} \right\rceil. \qquad (1)$$

■

(d) Show that *every* partial order with n vertices and maximum chain size, t, has a p-processor schedule that runs in time $M(n,t,p)$.

Hint: Induction on t.

Solution. *Proof.* Induction on t with induction hypothesis that the statement of this problem part (d) holds for all positive integers, n, p.

Base case ($t = 1$). In this case there are n vertices and no edges between them. So any partition of the vertices into $\lceil n/p \rceil$ blocks of size at most p will be a p-processor schedule taking time $\lceil n/p \rceil = 0 + \lceil (n-0)/p \rceil = M(n,1,p)$.

Inductive step: Assume $P(t)$ and conclude $P(t+1)$ where $t \geq 1$.

Let G be any partial order with n elements and maximum chain size $t+1$. Suppose k elements are endpoints of maximum-size chains in G. These elements must be maximal, for otherwise there would be a chain of length one more than the maximum chain size. Let H be the subset of G obtained by removing these k vertices.

Now H is a partial order with $n-k$ elements and maximum chain size t, so by Induction Hypothesis, there is a p-processor schedule for H taking time $M(n-k,t,p)$.

This p-processor schedule for H can be extended to one for G by adding $\lceil k/p \rceil$ disjoint blocks of the endpoints, all of size $\leq p$. So the time for this schedule for G is

$$\begin{aligned}
&M(n-k,t,p)+\left\lceil \frac{k}{p} \right\rceil \\
&=(t-1)+\left\lceil \frac{n-k-(t-1)}{p} \right\rceil+\left\lceil \frac{k}{p} \right\rceil && \text{(def of } M\text{)} \\
&=(t-1)+\left\lceil \frac{n-t}{p}-\frac{k-1}{p} \right\rceil+\left\lceil \frac{k}{p} \right\rceil && (2)
\end{aligned}$$

We complete the proof by showing that the expression (2) is $\leq M(n,t+1,p)$. To do this, we consider two cases:

- **Case 1:** ($k-1$ is not a multiple of p): We have

$$\left\lceil \frac{k-1}{p} \right\rceil=\left\lceil \frac{k}{p} \right\rceil, \qquad (3)$$

so

$$\begin{aligned}
(2) &\leq (t-1)+\left(1+\left\lceil \frac{n-t}{p} \right\rceil-\left\lceil \frac{k-1}{p} \right\rceil\right)+\left\lceil \frac{k}{p} \right\rceil && \text{(by (4))} \\
&=(t-1)+\left(1+\left\lceil \frac{n-t}{p} \right\rceil-\left\lceil \frac{k}{p} \right\rceil\right)+\left\lceil \frac{k}{p} \right\rceil && \text{(by (3))} \\
&=t+\left\lceil \frac{n-t}{p} \right\rceil \\
&=M(n,t+1,p) && \text{(def of } M\text{)}.
\end{aligned}$$

  Here the first inequality used the fact that

$$\lceil a-b \rceil \leq 1+\lceil a \rceil-\lceil b \rceil \qquad (4)$$

  for all real numbers a, b.
- **Case 2:** ($k-1$ is a multiple of p): Now we have

$$\left\lceil \frac{k}{p} \right\rceil=1+\frac{k-1}{p}, \qquad (5)$$

so

$$\begin{aligned}
(2) &=(t-1)+\left(\left\lceil \frac{n-t}{p} \right\rceil-\frac{k-1}{p}\right)+\left\lceil \frac{k}{p} \right\rceil && \text{(since } (k-1)/p \in \mathbb{Z}\text{)} \\
&=(t-1)+\left\lceil \frac{n-t}{p} \right\rceil-\frac{k-1}{p}+\left(1+\frac{k-1}{p}\right) && \text{(by (5))} \\
&=t+\left\lceil \frac{n-t}{p} \right\rceil \\
&=M(n,t+1,p). && \text{(def of } M\text{)}
\end{aligned}$$

■

Problem 4.
Let S be a sequence of n different numbers. A *subsequence* of S is a sequence that can be obtained by deleting elements of S.

For example, if

$$S = (6, 4, 7, 9, 1, 2, 5, 3, 8)$$

Then 647 and 7253 are both subsequences of S (for readability, we have dropped the parentheses and commas in sequences, so 647 abbreviates $(6, 4, 7)$, for example).

An *increasing subsequence* of S is a subsequence of whose successive elements get larger. For example, 1238 is an increasing subsequence of S. Decreasing subsequences are defined similarly; 641 is a decreasing subsequence of S.

(a) List all the maximum length increasing subsequences of S, and all the maximum length decreasing subsequences.

Solution. The maximum length increasing subsequences are 1238 and 1258. The maximum length decreasing subsequences are

$$641, 642, 643, 653, 753, 953$$

■

Now let A be the *set* of numbers in S. (So $A = \{1, 2, 3, \ldots, 9\}$ for the example above.) There are two straightforward ways to path-total order A. The first is to order its elements numerically, that is, to order A with the $<$ relation. The second is to order the elements by which comes first in S; call this order $<_S$. So for the example above, we would have

$$6 <_S 4 <_S 7 <_S 9 <_S 1 <_S 2 <_S 5 <_S 3 <_S 8$$

Next, define the partial order $\prec$ on A defined by the rule

$$a \prec a' \quad ::= \quad a < a' \text{ and } a <_S a'.$$

(It's not hard to prove that $\prec$ is strict partial order, but you may assume it.)

(b) Draw a diagram of the partial order, $\prec$, on A. What are the maximal elements,... the minimal elements?

Solution. The maximal elements are 8 and 9; the minimal are 1, 4,and 6: ■

(c) Explain the connection between increasing and decreasing subsequences of S, and chains and antichains under $\prec$.

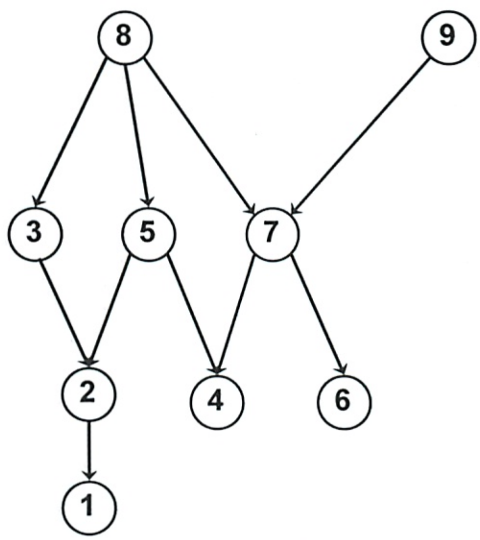
Solution. A *chain*, with its elements listed in numerically increasing order, is an *increasing* subsequence and an *antichain*, with its elements listed in numerically decreasing order, is a *decreasing* subsequence. ■

(d) Prove that every sequence, S, of length n has an increasing subsequence of length greater than $\sqrt{n}$ or a decreasing subsequence of length at least $\sqrt{n}$.

Solution. By Dilworth's Lemma, either a chain or an antichain must have size at least $\sqrt{n}$, which, by the previous problem part, means there is either an increasing or a decreasing subsequence of this size. ■

(e) (Optional, tricky) Devise an efficient procedure for finding the longest increasing and the longest decreasing subsequence in any given sequence of integers. (There is a nice one.)

Solution. $\boxed{\text{TBA - reference to Floyd algorithm}}$ ■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