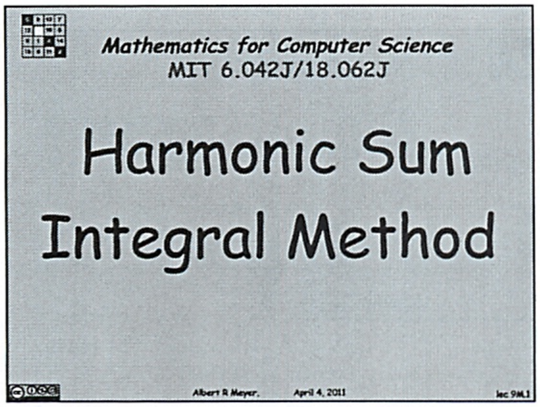
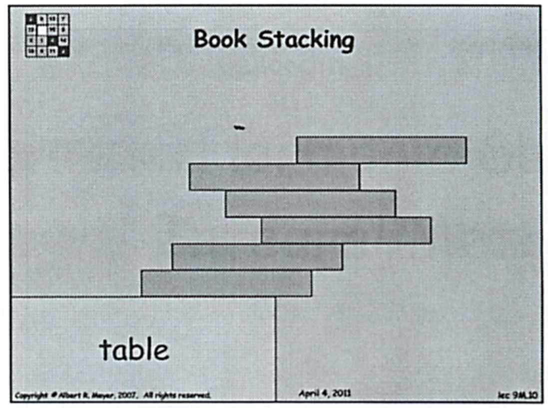
4/4


Mathematics for Computer Science
MIT 6.042J/18.062J
Harmonic Sum
Integral Method
Albert R Meyer, April 4, 2011
lec 9M.1


Book Stacking
table
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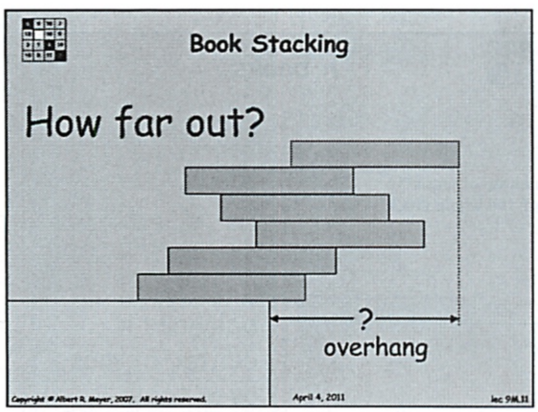
April 4, 2011
lec 9M.10


Book Stacking
How far out?
?
overhang
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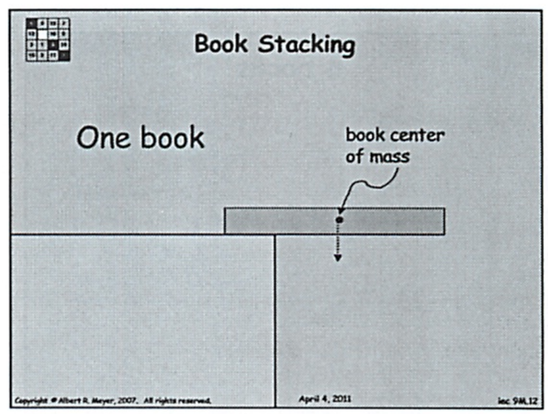
lec 9M.11


Book Stacking
One book
book center
of mass
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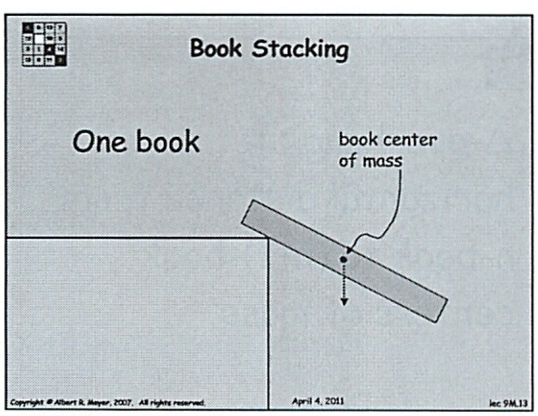
Book Stacking
One book
book center
of mass
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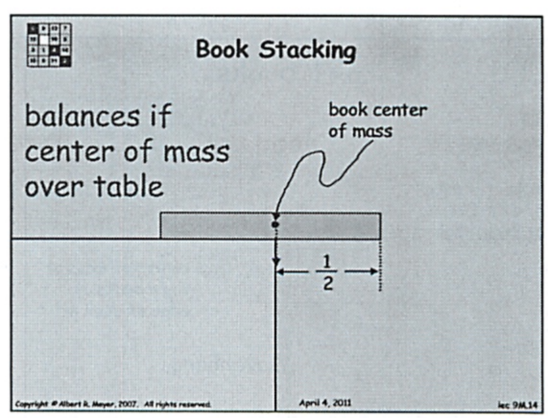
April 4, 2011
lec 9M.13


Book Stacking
balances if
center of mass
over table
book center
of mass
1/2
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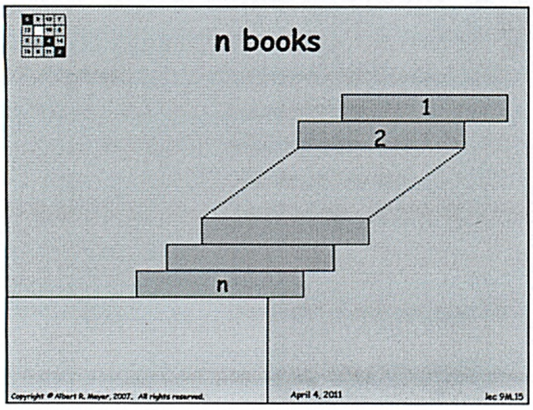
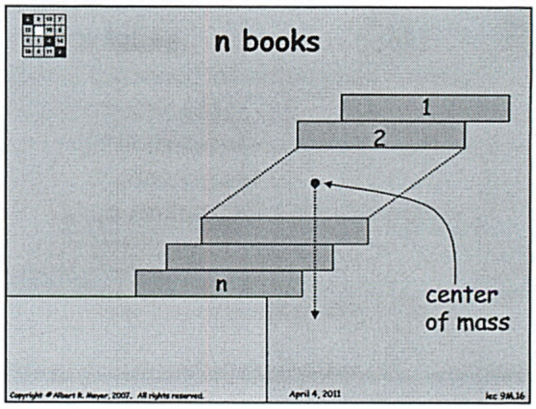
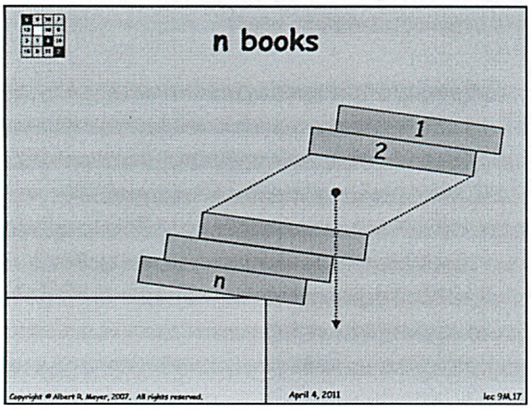
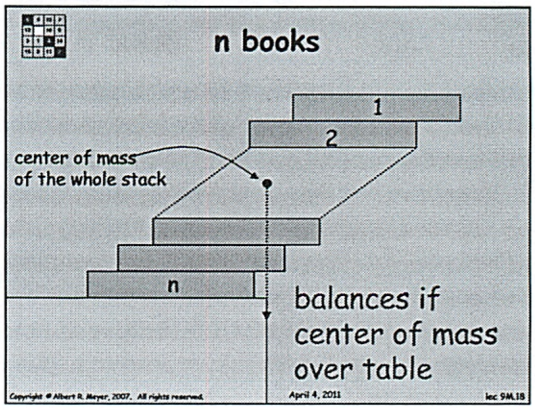
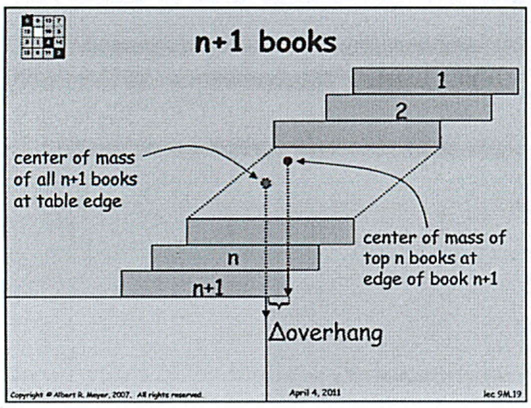
April 4, 2011
lec 9M.14







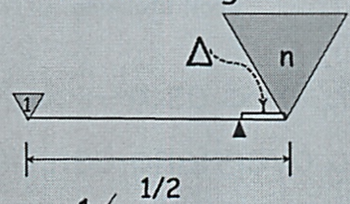




Δ-overhang ::=
horizontal distance from
n-book to (n+1)-book
centers of mass

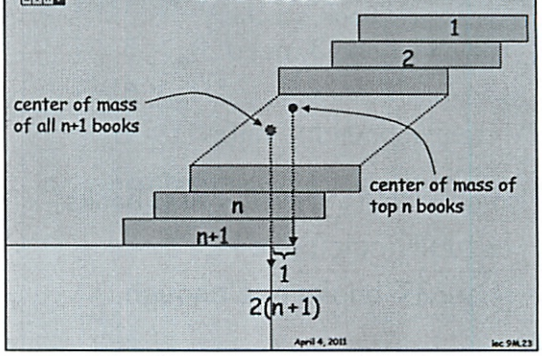
Albert R Meyer, April 4, 2011 lec 9M.20

Δ-overhang



$$\Delta = \frac{\frac{1}{2}}{n+1} = \frac{1}{2(n+1)}$$

n+1 books



Book stacking summary

B_n ::= overhang of n books

$B_1 = 1/2$

$$B_{n+1} = B_n + \frac{1}{2(n+1)}$$

$$B_n = \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$$

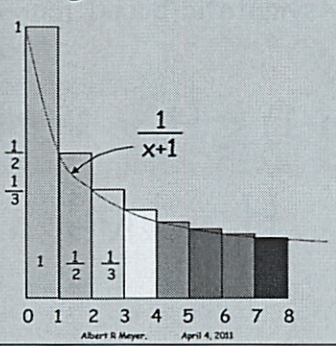
Harmonic Sums

$$H_n ::= 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

n^{th} Harmonic number

$$B_n = H_n/2$$

Integral estimate for H_n



Integral estimate for H_n

H_n = area of rectangles

> area under $1/(x+1)$ =

$$\int_0^n \frac{1}{x+1}\,dx = \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$$

Book stacking

for overhang 3, need $B_n \geq 3$

$H_n \geq 6$

integral bound: $\ln(n+1) \geq 6$

so ok with $n \geq \lceil e^6 - 1 \rceil = 403$ books

actually calculate H_n:

227 books are enough.

Book stacking

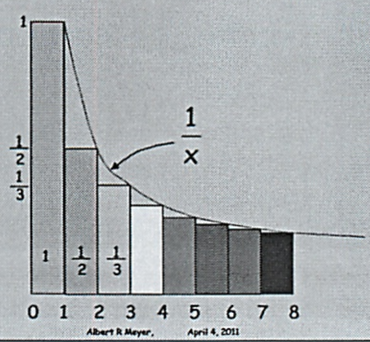
$H_n \to \infty$ as $n \to \infty$,

so overhang can be as big as desired!

CD cases over the edge

43 cases high --top 4 cases completely off the table --1.8 or 1.9 case-lengths

Upper bound for H_n



Upper bound for H_n

$$H_n < 1 + \int_1^n \frac{1}{x}\,dx$$

$$= 1 + \ln(n)$$

Asymptotic bound for H_n

$$\ln(n+1) < H_n < 1 + \ln(n)$$

$$H_n \sim \ln(n)$$

Asymptotic Equivalence

Def: $f(n) \sim g(n)$

$$\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = 1$$

Asymptotic Equivalence ~

Example: $(n^2 + n) \sim n^2$

pf:

$$\lim_{n\to\infty}\frac{n^2+n}{n^2} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$$

Team Problems

Problems

1—4

6.042

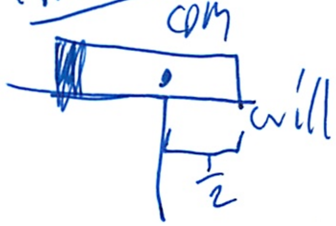
4/4

(5 min late)

Book stacking

- so does not fall off table

1 book



Max overhang = $\frac{1}{2}$

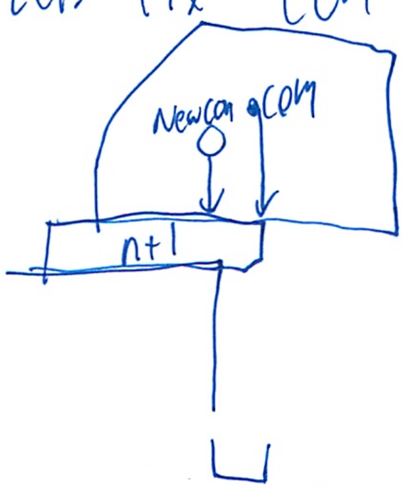
n books

take avg of all books COM

- avg COM must be on table to balance

n+1 books

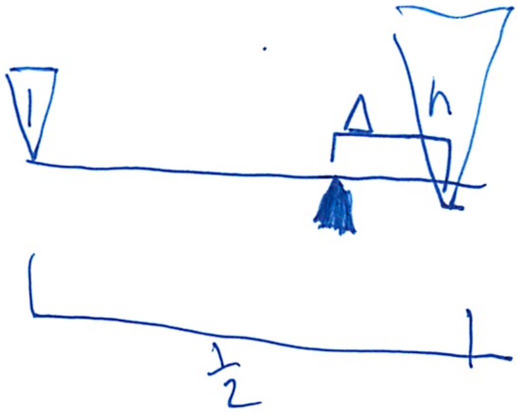
Lets fix COM of top n books at edge of book n+1



so New COM is still stable

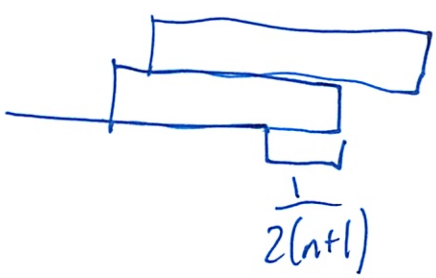
this is the new overhang Δ overhang

2) Want it to balance



$$\Delta = \frac{\frac{1}{2}}{n+1} = \frac{1}{2(n+1)}$$

That is distance



Recursive construction

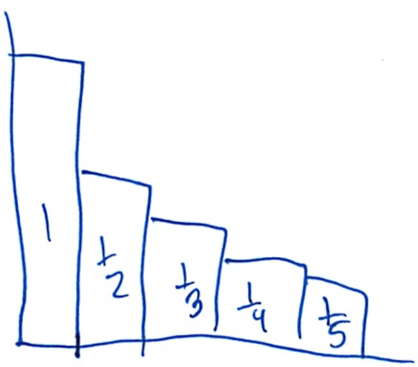
(did not copy)

(3)

Harmonic sum

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

- $\frac{1}{2}$ of the previous



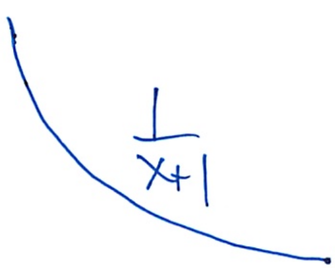
No nice closed form

Need to estimate

- use sum of size of rectangles

By turning sum into integration

(proof by picture)



$\frac{1}{x+1}$ is lower-band on our area

H_n = area rectangles > $\frac{1}{x+1}$ area rectangles

$\int_0^n \frac{1}{x+1} \, dn$ ... did not see

(9)

log grows ∞ - so can always put more books at

for overhang 3 - need ~~...~~ 403 books (estimate)

actual is 227 -

- calculate w/ sum

hard to actually do w/ books - compress

do w/ CD cases

estimates ~~...~~ upper bound

$\frac{1}{x}$ + area first rectangle (1)

$H_n < 1 + \frac{1}{x}$

So

$\ln(n+1) < H_n < 1 + \ln(n)$

estimate by integration

$H_n \sim \ln(n)$

"asymptotic to" - means ratio goes to 1 in limit

5

<u>Def</u> ... missed ...

Used to see which parts are dominating the growth

In-Class Problems Week 9, Mon.

Problem 1.
You've seen this neat trick for evaluating a geometric sum:

$$\begin{aligned} S &= 1 + z + z^2 + \ldots + z^n \\ zS &= z + z^2 + \ldots + z^n + z^{n+1} \\ S - zS &= 1 - z^{n+1} \\ S &= \frac{1 - z^{n+1}}{1 - z} \end{aligned}$$

Use the same approach to find a closed-form expression for this sum:

$$T = 1z + 2z^2 + 3z^3 + \ldots + nz^n$$

Problem 2.
An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine d days walk from the nearest oasis. In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to make multiple trips carrying up to a gallon each time to create water caches out in the desert.

For example, if the shrine were $2/3$ of a day's walk into the desert, then she could recover the Holy Grail after two days using the following strategy. She leaves the oasis with 1 gallon of water, travels $1/3$ day into the desert, caches $1/3$ gallon, and then walks back to the oasis—arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks $1/3$ day into the desert, tops off her water supply by taking the $1/3$ gallon in her cache, walks the remaining $1/3$ day to the shrine, grabs the Holy Grail, and then walks for $2/3$ of a day back to the oasis—again arriving with no water to spare.

But what if the shrine were located farther away?

(a) What is the most distant point that the explorer can reach and then return to the oasis if she takes a total of only 1 gallon from the oasis?

(b) What is the most distant point the explorer can reach and still return to the oasis if she takes a total of only 2 gallons from the oasis? No proof is required; just do the best you can.

(c) The explorer will travel using a recursive strategy to go far into the desert and back drawing a total of n gallons of water from the oasis. Her strategy is to build up a cache of $n-1$ gallons, plus enough to get home, a certain fraction of a day's distance into the desert. On the last delivery to the cache, instead of returning home, she proceeds recursively with her $n-1$ gallon strategy to go farther into the desert and return to the cache. At this point, the cache has just enough water left to get her home.

Prove that with n gallons of water, this strategy will get her $H_n/2$ days into the desert and back, where H_n is the nth Harmonic number:

$$H_n ::= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Conclude that she can reach the shrine, however far it is from the oasis.

(d) Suppose that the shrine is $d = 10$ days walk into the desert. Use the asymptotic approximation $H_n \sim \ln n$ to show that it will take more than a million years for the explorer to recover the Holy Grail.

Problem 3.
There is a number a such that $\sum_{i=1}^{\infty} i^p$ converges iff $p < a$. What is the value of a? Prove it.

Problem 4.
Suppose $f, g : \mathbb{N}^+ \to \mathbb{N}^+$ and $f \sim g$.

(a) Prove that $2f \sim 2g$.

(b) Prove that $f^2 \sim g^2$.

(c) Give examples of f and g such that $2^f \not\sim 2^g$.

In Class 9 Mon 4/4

1. We've seen trick for evaluating geo. sum
Do it for

$$T = 1z + 2z^2 + 3z^3 + \dots + nz^n$$

Try z2

$$zT = z^2 + 2z^3 + 3z^4 + \dots + nz^{n+1}$$

~~$T - zT =$ no thats not very nice~~

~~$T + 1$?~~

Our group did

$$T - zT = 1z + 1z^2 + 1z^3 + \dots - nz^{n-1}$$

$$= \frac{z^{n+1}}{1-z}$$

$$- nz^{n+1}$$

2a. $\frac{1}{2}$ day

b $\frac{3}{4}$ day

c)

Oh I see it can be represented w/ Geometric seq
Find the closed form to know how far she can go

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

She can get $\frac{H_n}{2}$ days into the dessert

d) Suppose shrine is $d = 10$ days into desert
Use asymptotic approx $H_n \sim \ln(n)$ to show
it will take more than a million years.

Well makes sense - b/c additions get smaller
+ smaller each day - as go further out
need more + more water

Treat cache as oasis
Build up to $n-1$ gallon

③

3. Tutor problem! Except now you actually have to prove it

$\sum_{i=1}^{\infty} i^p$ converges if $p < a$

What is a?

-1

just copy ans

Sum is $\Theta\left(\int_1^{\infty} x^p \, dx\right)$

for $p \neq -1$ ind. integral is $\frac{x^{p+1}}{p+1}$

- If $p < -1$ then $p+1 < 0$ so $\lim_{x \to \infty} x^{p+1} = 0$
  definite integral from $1 \to \infty$ evals to $-\frac{1}{(p+1)}$
  Here sum bounded from above - since increasing, finite limit, so it converges
- If $p > -1$ then $p+1 > 0$ so $\lim_{x \to \infty} x^{p+1} = \infty$
  so diverges

(4)

$p=-1$ indef. int. is $\log x$, which also approaches ∞ as $x \to \infty$, so diverges

4. Suppose f, g $N^+ \to N^+$ and $f \sim g$

a. Prove $2f \sim 2g$

so this is ratios

$f \sim g$ means $\frac{f}{g} \approx 1$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$$

So
$$\lim_{x \to \infty} \frac{2f}{2g} = 1$$

$\frac{2}{2}$ cancles

b) $\lim \frac{f^2}{g^2}$ take $\sqrt{\ }$ of ca $\frac{f}{g} = 1$

? legal move on its own

(5)

c) Give 2 counter examples s.t.

$$2^f \not< 2^g$$

That is not allowed mathwise

But what could you say? for a counter-example?

---

2. Our board

a) $\frac{1}{2}$

b) $\frac{3}{4}$

c) For the explorer to deposit $n-1$ gallons at a position, using n gallons, he needs to make n trips drinking a total of 1 gallon ~~on each trip~~. Hence, each trip will have to be $\frac{1}{n}$ days long round trip or $\frac{1}{2n}$ days long 1-way.

Doing this recursively, the first cache always has $n-1$ at dist $\frac{1}{2n}$. 2nd has $n-2$ at $\frac{1}{2n-1}$

⑥

So total distance is

$$\sum_{i=0}^{n-1} \frac{1}{2n-i} = \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{2n-4} + \dots + \frac{1}{6} + \frac{1}{4} + \frac{1}{2}$$

$$= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)$$

$$= \frac{1}{2} H_n$$

d) $d = \frac{1}{2} H_n$

$d = 10 \approx \frac{1}{2} \ln n$

$n = e^{20} = 4.8 E 8$ days $= 1.7 E 6$ years

= Very big!

---

3. Prove $\sum_{i=1}^{\infty} i^p$ converges if $p < -1$

Case 1 $p > -1$

Then $\lim_{p \to \infty}$

... forget it

(7)

Case $p=-1$

$\sum_{i=1}^{\infty} i^p$ is the harmonic series

---

4. a) Like I had it

b) $\lim_{n\to\infty} \frac{f(n)^2}{g(n)^2} = \lim \frac{f(n)}{g(n)} \cdot \frac{f(n)}{g(n)}$

$f \sim g$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$

See how that was written! pay attention to detail

as long as $g(n) \neq 0$

So if $f \sim g$ $\lim_{n\to\infty} \frac{f(n)}{g(n)} \cdot \frac{f(n)}{g(n)} = 1 \cdot 1 = 1$

So $f \sim g \rightarrow f^2 \sim g^2$

c) $n \sim n+1$, but $\lim_{n\to\infty} \frac{2^{n+1}}{2^n} = \frac{2 \cdot 2^n}{2^n} = 2$

Solutions to In-Class Problems Week 9, Mon.

Problem 1.
An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine d days walk from the nearest oasis. In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to make multiple trips carrying up to a gallon each time to create water caches out in the desert.

For example, if the shrine were $2/3$ of a day's walk into the desert, then she could recover the Holy Grail after two days using the following strategy. She leaves the oasis with 1 gallon of water, travels $1/3$ day into the desert, caches $1/3$ gallon, and then walks back to the oasis—arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks $1/3$ day into the desert, tops off her water supply by taking the $1/3$ gallon in her cache, walks the remaining $1/3$ day to the shrine, grabs the Holy Grail, and then walks for $2/3$ of a day back to the oasis—again arriving with no water to spare.

But what if the shrine were located farther away?

(a) What is the most distant point that the explorer can reach and then return to the oasis if she takes a total of only 1 gallon from the oasis?

Solution. At best she can walk $1/2$ day into the desert and then walk back. ∎

(b) What is the most distant point the explorer can reach and still return to the oasis if she takes a total of only 2 gallons from the oasis? No proof is required; just do the best you can.

Solution. The explorer walks $1/4$ day into the desert, drops $1/2$ gallon, then walks home. Next, she walks $1/4$ day into the desert, picks up $1/4$ gallon from her cache, walks an additional $1/2$ day out and back, then picks up another $1/4$ gallon from her cache and walks home. Thus, her maximum distance from the oasis is $3/4$ of a day's walk. ∎

(c) The explorer will travel using a recursive strategy to go far into the desert and back drawing a total of n gallons of water from the oasis. Her strategy is to build up a cache of $n-1$ gallons, plus enough to get home, a certain fraction of a day's distance into the desert. On the last delivery to the cache, instead of returning home, she proceeds recursively with her $n-1$ gallon strategy to go farther into the desert and return to the cache. At this point, the cache has just enough water left to get her home.

Prove that with n gallons of water, this strategy will get her $H_n/2$ days into the desert and back, where H_n is the nth Harmonic number:

$$H_n ::= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Conclude that she can reach the shrine, however far it is from the oasis.

Solution. To build up the first cache of $n-1$ gallons, she should make n trips $1/(2n)$ days into the desert, dropping off $(n-1)/n$ gallons each time. Before she leaves the cache for the last time, she has $n-1$ gallons plus enough for the walk home. Then she applies her $(n-1)$-day strategy. So letting D_n be her maximum distance into the desert and back, we have

$$D_n = \frac{1}{2n} + D_{n-1}.$$

So

$$\begin{aligned} D_n &= \frac{1}{2n} + \frac{1}{2(n-1)} + \frac{1}{2(n-2)} + \cdots + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 1} \\ &= \frac{1}{2}\left(\frac{1}{n} + \frac{1}{(n-1)} + \frac{1}{(n-2)} + \cdots + \frac{1}{\cdot 2} + \frac{1}{\cdot 1}\right) \\ &= \frac{H_n}{2}. \end{aligned}$$

■

(d) Suppose that the shrine is $d = 10$ days walk into the desert. Use the asymptotic approximation $H_n \sim \ln n$ to show that it will take more than a million years for the explorer to recover the Holy Grail.

Solution. She obtains the Grail when:

$$\frac{H_n}{2} \approx \frac{\ln n}{2} \geq 10.$$

This requires $n \geq e^{20} = 4.8 \cdot 10^8$ days $> 1.329M$ years.

■

Problem 2.
There is a number a such that $\sum_{i=1}^{\infty} i^p$ converges iff $p < a$. What is the value of a? Prove it.

Solution. $a = -1$.

For $p = -1$, the sum is the harmonic series which we know does not converge. Since the term i^p is increasing in p for $i > 1$, the sum will be larger, and hence also diverge for $p > -1$.

For $p < -1$ there exists an $\epsilon > 0$ such that $p = -(1 + \epsilon)$. By the integral method,

$$\begin{aligned} \sum_{i=1}^{\infty} i^{-(1+\epsilon)} &\leq 1 + \int_1^{\infty} x^{-(1+\epsilon)}\, dx \\ &= 1 + \epsilon^{-1} - \epsilon^{-1} \lim_{\alpha \to \infty} \alpha^{-\epsilon} \\ &= 1 + \epsilon^{-1} \\ &< \infty \end{aligned}$$

Hence the sum is bounded above, and since it is increasing, it has a finite limit, that is, it converges. ■

Problem 3.
Suppose $f, g : \mathbb{N}^+ \to \mathbb{N}^+$ and $f \sim g$.

(a) Prove that $2f \sim 2g$.

Solution.

$$\frac{2f}{2g} = \frac{f}{g},$$

silly

so they have the same limit as $n \to \infty$.

■

(b) Prove that $f^2 \sim g^2$.

Solution.

$$\lim_{n\to\infty} \frac{f(n)^2}{g(n)^2} = \lim_{n\to\infty} \frac{f(n)}{g(n)} \cdot \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f(n)}{g(n)} \cdot \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1 \cdot 1 = 1.$$

■

(c) Give examples of f and g such that $2^f \not\sim 2^g$.

Solution.

$$\begin{aligned} f(n) &::= n + 1 \\ g(n) &::= n. \end{aligned}$$

Then $f \sim g$ since $\lim(n+1)/n = 1$, but $2^f = 2^{n+1} = 2 \cdot 2^n = 2 \cdot 2^g$ so

$$\lim \frac{2^f}{2^g} = 2 \neq 1.$$

■

Problem 4. was #1

You've seen this neat trick for evaluating a geometric sum:

$$\begin{aligned} S &= 1 + z + z^2 + \ldots + z^n \\ zS &= z + z^2 + \ldots + z^n + z^{n+1} \\ S - zS &= 1 - z^{n+1} \\ S &= \frac{1 - z^{n+1}}{1 - z} \end{aligned}$$

Use the same approach to find a closed-form expression for this sum:

$$T = 1z + 2z^2 + 3z^3 + \ldots + nz^n$$

Solution.

$$\begin{aligned} zT &= 1z^2 + 2z^3 + 3z^4 + \ldots + nz^{n+1} \\ T - zT &= z + z^2 + z^3 + \ldots + z^n - nz^{n+1} \\ &= \frac{1 - z^{n+1}}{1 - z} - 1 - nz^{n+1} \\ T &= \frac{1 - z^{n+1}}{(1-z)^2} - \frac{1 + nz^{n+1}}{1 - z} \end{aligned}$$

■

4/6

Mathematics for Computer Science
MIT 6.042J/18.062J

Asymptotic Notation

Asymptotic Equivalence

Def: $f(n) \sim g(n)$

$$\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = 1$$

Stirling's Formula

A precise approximation:

$$\boxed{n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$$

Little Oh: $o(\cdot)$

Asymptotically smaller :

Def: $f(n) = o(g(n))$

iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

Little Oh: $o(\cdot)$

$$n^2 = o(n^3)$$

because

$$\lim_{n\to\infty} \frac{n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} = 0$$

Big Oh: $O(\cdot)$

Asymptotic Order of Growth:

$f(n) = O(g(n))$

$$\lim\cancel{\sup}_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) < \infty$$

a technicality -- ignore now

Big Oh: $O(\cdot)$

$$3n^2 = O(n^2)$$

because

$$\lim_{n\to\infty} \frac{3n^2}{n^2} = 3 < \infty$$

Theta: $\Theta(\cdot)$

Same Order of Growth:

$$f(n) = \Theta(g(n))$$

Def: $f(n)=O(g(n))$

and

$g(n)=O(f(n))$

Asymptotics: Intuitive Summary

$f \sim g$:	f & g nearly equal
$f = o(g)$:	f much less than g
$f = O(g)$:	f roughly $\le$ g
$f = \Theta(g)$:	f & g roughly equal

The Oh's

lemma:

If $f = o(g)$ or $f \sim g$, then $f = O(g)$

lim = 0 or lim = 1 IMPLIES lim $< \infty$

The Oh's

If $f = o(g)$, then $g \ne O(f)$

$\lim \frac{f}{g} = 0$ IMPLIES $\lim \frac{g}{f} = \infty$

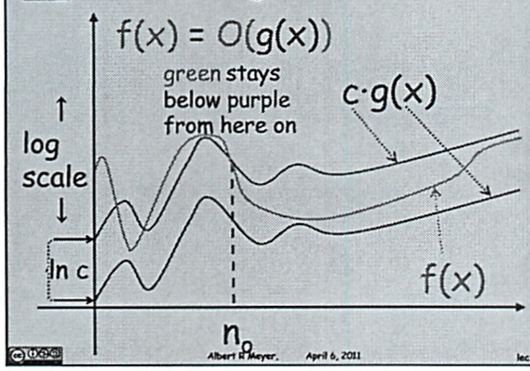
Big Oh: $O(\cdot)$

Equivalent definition:

$$f(n) = O(g(n))$$

$$\exists c, n_0\ \forall n \ge n_0.\ f(n) \le c \cdot g(n)$$

Big Oh: $O(\cdot)$



Little Oh: $o(\cdot)$

Lemma: $\boxed{x^a = o(x^b) \text{ for } a < b}$

Proof: $\dfrac{x^a}{x^b} = \dfrac{1}{x^{b-a}}$ and $b - a > 0$

so as $x \to \infty$, $\dfrac{1}{x^{b-a}} \to 0$

Little Oh: $o(\cdot)$

Lemma:

$$\boxed{\ln x = o(x^\varepsilon)}$$

for $\varepsilon > 0$.

Little Oh: $o(\cdot)$

Lemma:

$$\boxed{x^n = o(a^x)}$$

for $a > 1$.

Little Oh: $o(\cdot)$

proofs:
L'Hopital's Rule,
McLaurin Series
(see a Calculus text)

Big Oh Mistakes

"$\cdot = O(\cdot)$" defines a relation
Don't write $O(g) = f$.
Otherwise: $x = O(x)$, so $O(x) = x$.
But $2x = O(x)$, so
$2x = O(x) = x$,
therefore $2x = x$.
Nonsense!

Team Problems

Problems

1–3

Miniquiz 4

6.042 Asymptotic Notation

4/6

8 vertices

Need 7 edges

Only 12

So can't have 2 sep sets of disconnected edges

Last #3 - never true - so invarient

urg

missed this

Asymptotic Notation

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

A precise approx, can do w/ log(n) squarings

No exact closed form

Asym equal

% diff b/w goes to 0 as $n \to \infty$

Called Stirling's Formula

Need to know rate limit is approached

- w/o can't really use

(2)

Messy approx for non limiting states
Takes 1 page to prove

Little Oh notation

Def $f(n) = o(g(n))$

iff $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$

for $n^2 = o(n^3)$

$\lim_{n \to \infty} \frac{n^2}{n^3} \boxed{= \frac{1}{n}}$ and that goes to 0

Big Oh

- asymptotic order of growth
- interested in time or ~~energy~~ memory
- precicese value is uncertain
- $f(n) = O(g(n))$

$\lim \sup_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) < \infty$

is finite

technicality

③

$3n^2 = O(n^2)$

Since $\lim_{n \to \infty} \frac{3n^2}{n^2} = 3 < \infty$

finite

so n^2 is O of $3n^2$

---

④ Theta within _ _ .

- both of big Oh of each other
- equivilance relations

$f \sim g \to f, g$ nearly $=$

$f = o(g) \Rightarrow f$ much less than g

$f = O(g) \Rightarrow f$ roughly $\leq g$

may need to amplify w/ constant factor

$f = \Theta(g) \to f, g$ roughly $=$

If $f = o(g)$ or $f \sim g$ then $f = O(g)$

$\lim = 0$ or $\lim = 1$ $\to$ $\lim < \infty$

If $f = o(g)$ then $g \neq O(f)$

$\lim \frac{f}{g} = 0 \to \lim \frac{g}{f} = \infty$

4

Big Oh ~~O~~ O()

- messy way in lit

$f(n) = O(g(n))$

$\exists c, n_0 \; \forall n \geq n_0. \quad f(n) \leq c \cdot g(n)$

~~small~~ amp factor

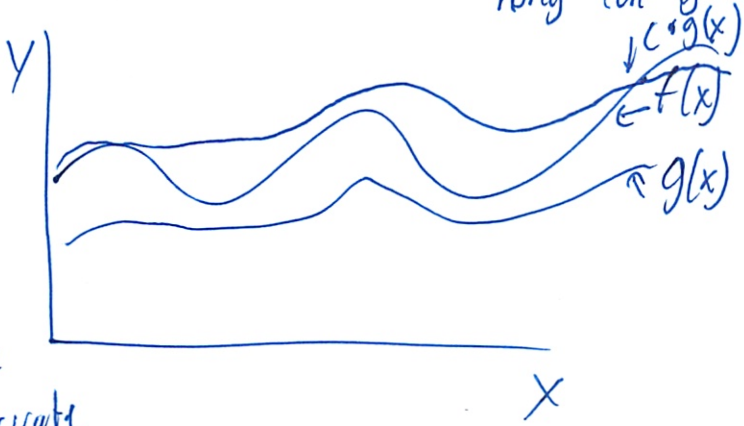
small

from a certain point on

- long run growth rate

log

so that way not that accurate



not much bigger so C. makes it almost =

amplify

always < f(x)

Little Oh o()

$x^a = o(x^b)$ for $a < b$

$\frac{x^a}{x^b} = \frac{1}{x^{b-a}}$ and $b - a > 0$

so as $\lim_{x \to \infty}$ $\frac{1}{x^{b-a}} \to 0$

(5)

$\ln x = o(x^{\varepsilon})$ for $\varepsilon > 0$

log x grows smaller than any root of x

$x^n = o(a^x)$ for $a > 1$

any polynomial grows slower than exponential as long as $a > 1$

CS

polynomial ~ feasible
exponential ~ not feasable

Mistakes

- don't seperate equality from $O()$
  $= O()$ defines the relation
- don't write $O(g) = f$ - historical bad notation
- otherwise $x = O(x)$ so $O(x) = x$
  true / not true
- $\boxed{2x = O(x)}$ so $2x = O(x) = x$
  so $2x = x$ NO

In-Class Problems Week 9, Wed.

Problem 1.
Recall that for functions f, g on $\mathbb{N}$, $f = O(g)$ iff

$$\exists c \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)|. \tag{1}$$

For each pair of functions below, determine whether $f = O(g)$ and whether $g = O(f)$. In cases where one function is O() of the other, indicate the *smallest nonegative integer*, c, and for that smallest c, the *smallest corresponding nonegative integer* n_0 ensuring that condition (1) applies.

(a) $f(n) = n^2, g(n) = 3n$.

$f = O(g)$	YES	NO	If YES, $c =$ ______, $n_0 =$ ______
$g = O(f)$	YES	NO	If YES, $c =$ ______, $n_0 =$ ______

(b) $f(n) = (3n - 7)/(n + 4), g(n) = 4$

$f = O(g)$	YES	NO	If YES, $c =$ ______, $n_0 =$ ______
$g = O(f)$	YES	NO	If YES, $c =$ ______, $n_0 =$ ______

(c) $f(n) = 1 + (n \sin(n\pi/2))^2, g(n) = 3n$

$f = O(g)$	YES	NO	If yes, $c =$ ______ $n_0 =$ ______
$g = O(f)$	YES	NO	If yes, $c =$ ______ $n_0 =$ ______

Problem 2.

False Claim.

$$2^n = O(1). \tag{2}$$

Explain why the claim is false. Then identify and explain the mistake in the following bogus proof.

Bogus proof. The proof by induction on n where the induction hypothesis, $P(n)$, is the assertion (2).

base case: $P(0)$ holds trivially.

inductive step: We may assume $P(n)$, so there is a constant $c > 0$ such that $2^n \leq c \cdot 1$. Therefore,

$$2^{n+1} = 2 \cdot 2^n \leq (2c) \cdot 1,$$

which implies that $2^{n+1} = O(1)$. That is, $P(n + 1)$ holds, which completes the proof of the inductive step.

We conclude by induction that $2^n = O(1)$ for all n. That is, the exponential function is bounded by a constant.

■

Problem 3.

(a) Define a function $f(n)$ such that $f = \Theta(n^2)$ and $\text{NOT}(f \sim n^2)$.

(b) Define a function $g(n)$ such that $g = O(n^2)$, $g \neq \Theta(n^2)$ and $g \neq o(n^2)$.

Asymptotic Notations

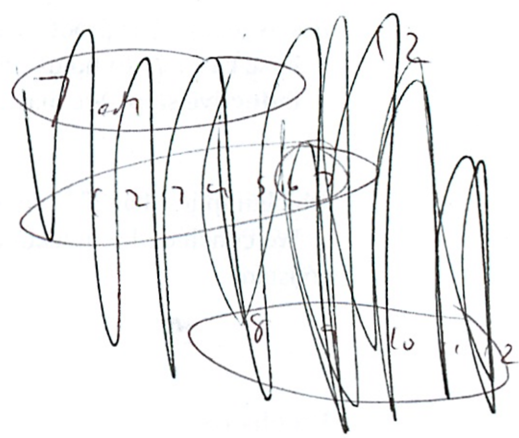
Let f, g be functions from $\mathbb{R}$ to $\mathbb{R}$.

- f is *asymptotically equal* to g: $f(x) \sim g(x)$ iff $\lim_{x\to\infty} f(x)/g(x) = 1$.
- f is *asymptotically smaller* than g: $f(x) = o(g(x))$ iff $\lim_{x\to\infty} f(x)/g(x) = 0$.
- for f, g nonnegative, $f = O(g)$ iff $\limsup_{x\to\infty} f(x)/g(x) < \infty$, where $\limsup_{x\to\infty} h(x) ::= \lim_{x\to\infty} \text{lub}_{y\geq x} h(y)$.

  An alternative, equivalent, definition is

  $$f = O(g) \quad \text{iff} \quad \exists c, x_0 \in \mathbb{R}^+ \, \forall x \geq x_0.\ f(x) \leq c g(x).$$

- Finally, $f = \Theta(g)$ iff $f = O(g)$ AND $g = O(f)$.



1. For f, g on $\mathbb{N}$ $f = O(g)$ iff

$$\exists c \in \mathbb{N}\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0$$
$$c \cdot g(n) \geq |f(n)|$$

For each pair determine whether $f = O(g)$
$g = O(f)$

In cases where one fn is $O()$ of other indicate smallest non-neg int c and smallest corresponding nonneg int n_0 for that c

a) $f(n) = n^2$ $g(n) = 3n$

$f = O(g)$

$f = O(g)$ means f is roughly $\leq g$

$f = o(g)$ means f is much less than g

So $\frac{n^2}{3n}$ is $\frac{n}{3}$ is infinite so ~~Yes~~ NO

$\frac{\infty}{3} = \infty$

c is scaling factor

~~3 must be int~~ Can't do it no

②

$\frac{3n}{n^2} \to \frac{3}{n}$ which is 0 as $n \to \infty$

So is true

~~[illegible]~~

want ~~[illegible]~~ find $C \cdot 3n \le n^2$

~~n [illegible] smallest is 1~~

~~- smallest possible~~

want smallest c where this is true

Must be non neg int so $c = 1$

$1 \cdot 3n \le n^2$ since above was for $f = O(g)$

when $g = O(f)$ need to reverse

n where this is true

$n = 3$

Ok - think I finally get

---

b) $\lim \frac{\frac{3n-7}{n+4}}{4}$ as $n \to \infty$ look highest term $\frac{3\frac{\infty}{\infty}}{4} = \frac{3}{4}$

Yes true

(3)

lim $c > 4 \geq \left| \frac{3n-7}{n+4} \right|$

If $c > 1$ n would be

$$4 = \frac{3n-7}{n+4}$$

$$4(n+4) = 3n-7$$

$$4n + 16 = 3n - 7$$

$$-16 \quad -3n$$

$$n = -23$$

?but must be non neg

; how solve c, n sim

Want a c - where for any n it will be bigger. Then find n

$c = 1$
$n = 1$) he got

Doesn't look possible

Oh I forgot abs val at first

Solutions to In-Class Problems Week 9, Wed.

Problem 1.
Recall that for functions f, g on $\mathbb{N}$, $f = O(g)$ iff

$$\exists c \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)|. \tag{1}$$

For each pair of functions below, determine whether $f = O(g)$ and whether $g = O(f)$. In cases where one function is O() of the other, indicate the *smallest nonegative integer*, c, and for that smallest c, the *smallest corresponding nonegative integer* n_0 ensuring that condition (1) applies.

(a) $f(n) = n^2, g(n) = 3n$.

$f = O(g)$ YES NO If YES, $c =$ ______, $n_0 =$ ______

Solution. NO. ■

$g = O(f)$ YES NO If YES, $c =$ ______, $n_0 =$ ______

Solution. YES, with $c = 1, n_0 = 3$, which works because $3^2 = 9, 3 \cdot 3 = 9$. ■

(b) $f(n) = (3n - 7)/(n + 4), g(n) = 4$

$f = O(g)$ YES NO If YES, $c =$ ______, $n_0 =$ ______

Solution. YES, with $c = 1, n_0 = 0$ (because $|f(n)| < 3$). ■

$g = O(f)$ YES NO If YES, $c =$ ______, $n_0 =$ ______

Solution. YES, with $c = 2, n_0 = 15$.
Since $\lim_{n\to\infty} f(n) = 3$, the smallest possible c is 2. For $c = 2$, the smallest possible $n_0 = 15$ which follows from the requirement that $2f(n_0) \geq 4$. ■

(c) $f(n) = 1 + (n \sin(n\pi/2))^2, g(n) = 3n$

$f = O(g)$ YES NO If yes, $c =$ ______ $n_0 =$ ______

Solution. NO, because $f(2n) = 1$, which rules out $g = O(f)$ since $g = \Theta(n)$. ■

$g = O(f)$ YES NO If yes, $c =$ ______ $n_0 =$ ______

Solution. NO, because $f(2n + 1) = n^2 + 1 \neq O(n)$ which rules out $f = O(g)$. ■

Problem 2.

False Claim.

$$2^n = O(1). \tag{2}$$

Explain why the claim is false. Then identify and explain the mistake in the following bogus proof.

Bogus proof. The proof by induction on n where the induction hypothesis, $P(n)$, is the assertion (2).
base case: $P(0)$ holds trivially.
inductive step: We may assume $P(n)$, so there is a constant $c > 0$ such that $2^n \leq c \cdot 1$. Therefore,

$$2^{n+1} = 2 \cdot 2^n \leq (2c) \cdot 1,$$

which implies that $2^{n+1} = O(1)$. That is, $P(n+1)$ holds, which completes the proof of the inductive step.
We conclude by induction that $2^n = O(1)$ for all n. That is, the exponential function is bounded by a constant.

■

Solution. A function is $O(1)$ iff it is bounded by a constant, and since the function 2^n grows unboundedly with n, it is not $O(1)$.

The mistake in the bogus proof is in its misinterpretation of the expression 2^n in assertion (2). The intended interpretation of (2) is

$$\text{Let } f \text{ be the function defined by the rule } f(n) ::= 2^n. \text{ Then } f = O(1). \tag{3}$$

But the bogus proof treats (2) as an assertion, $P(n)$, about n. Namely, it misinterprets (2) as meaning:

Let f_n be the constant function equal to 2^n. That is, $f_n(k) ::= 2^n$ for all $k \in \mathbb{N}$. Then

$$f_n = O(1). \tag{4}$$

Now (4) is true since every constant function is $O(1)$, and the bogus proof is an unnecessarily complicated, but *correct*, proof that that for each n, the constant function f_n is $O(1)$. But in the last line, the bogus proof switches from the misinterpretation (4) and claims to have proved (3).

So you could say that the exact place where the proof goes wrong is in its first line, where it defines $P(n)$ based on misinterpretation (4). Alternatively, you could say that the proof was a correct proof (of the misinterpretation), and its first mistake was in its last line, when it switches from the misinterpretation to the proper interpretation (3). ■

Problem 3.

(a) Define a function $f(n)$ such that $f = \Theta(n^2)$ and NOT$(f \sim n^2)$.

Solution. Let $f(n) ::= 2n^2$. ■

(b) Define a function $g(n)$ such that $g = O(n^2)$, $g \neq \Theta(n^2)$ and $g \neq o(n^2)$.

Solution. Let $g(n) ::= (n \sin(n\pi/2))^2 + n\,(\cos(n\pi/2))^2$.
For odd n, we have $g(n) = n^2$, which implies that $g \neq o(n^2)$. For even n, we have $g(n) = n$, which implies $n^2 \neq O(g)$ and hence $g \neq \Theta(n^2)$.

■

Problem Set 7

Due: April 8

Reading: Chapter 11.7–11.11.3, Coloring, Connectedness, & Trees; Chapter 12, Planar Graphs; Chapter 14, Sums and Asymptotics.
Skip the following sections which will not be covered this term: Chapter 11.11.4, Minimum Weight Spanning Trees, Chapter 13, State Machines, Chapter 14.6, Double Sums, & Chapter 14.7.5, Omega notation.

Problem 1. **(a)** Give an example of a simple graph that has two vertices $u \neq v$ and two distinct paths between u and v, but no cycle including either u or v.

Hint: There is an example with 5 vertices.

(b) Prove that if there are different paths between two vertices in a simple graph, then the graph has a cycle.

Problem 2.
The entire field of graph theory began when Euler asked whether the seven bridges of Königsberg could all be crossed exactly once. Abstractly, we can represent the parts of the city separated by rivers as vertices and the bridges as edges between the vertices. Then Euler's question asks whether there is a closed walk through the graph that includes every edge in a graph exactly once. In his honor, such a walk is called an *Euler tour*.

So how do you tell in general whether a graph has an Euler tour? At first glance this may seem like a daunting problem. The similar sounding problem of finding a cycle that touches every vertex exactly once is one of those Millenium Prize NP-complete problems known as the *Traveling Salesman Problem*). But it turns out to be easy to characterize which graphs have Euler tours.

Theorem. *A connected graph has an Euler tour if and only if every vertex has even degree.*

(a) Show that if a graph has an Euler tour, then the degree of each of its vertices is even.

In the remaining parts, we'll work out the converse: if the degree of every vertex of a connected finite graph is even, then it has an Euler tour. To do this, let's define an Euler *walk* to be a walk that includes each edge *at most* once.

(b) Suppose that an Euler walk in a connected graph does not include every edge. Explain why there must be an unincluded edge that is incident to a vertex on the walk.

In the remaining parts, let **w** be the *longest* Euler walk in some finite, connected graph.

(c) Show that if **w** is a closed walk, then it must be an Euler tour.

Hint: part (b)

(d) Explain why all the edges incident to the end of **w** must already be in **w**.

(e) Show that if the end of **w** was not equal to the start of **w**, then the degree of the end would be odd.

Hint: part (d)

(f) Conclude that if every vertex of a finite, connected graph has even degree, then it has an Euler tour.

Problem 3.

False Claim. *Let G be a graph whose vertex degrees are all $\leq k$. If G has a vertex of degree strictly less than k, then G is k-colorable.*

(a) Give a counterexample to the False Claim when $k = 2$.

(b) Underline the exact sentence or part of a sentence that is the first unjustified step in the following bogus proof of the False Claim.

Bogus proof. Proof by induction on the number n of vertices:

Induction hypothesis:

$P(n)$::= "Let G be an n-vertex graph whose vertex degrees are all $\leq k$. If G also has a vertex of degree strictly less than k, then G is k-colorable."

Base case: ($n = 1$) G has one vertex, the degree of which is 0. Since G is 1-colorable, $P(1)$ holds.

Inductive step:

We may assume $P(n)$. To prove $P(n + 1)$, let G_{n+1} be a graph with $n + 1$ vertices whose vertex degrees are all k or less. Also, suppose G_{n+1} has a vertex, v, of degree strictly less than k. Now we only need to prove that G_{n+1} is k-colorable.

To do this, first remove the vertex v to produce a graph, G_n, with n vertices. Let u be a vertex that is adjacent to v in G_{n+1}. Removing v reduces the degree of u by 1. So in G_n, vertex u has degree strictly less than k. Since no edges were added, the vertex degrees of G_n remain $\leq k$. So G_n satisfies the conditions of the induction hypothesis, $P(n)$, and so we conclude that G_n is k-colorable.

Now a k-coloring of G_n gives a coloring of all the vertices of G_{n+1}, except for v. Since v has degree less than k, there will be fewer than k colors assigned to the nodes adjacent to v. So among the k possible colors, there will be a color not used to color these adjacent nodes, and this color can be assigned to v to form a k-coloring of G_{n+1}. ■

(c) With a slightly strengthened condition, the preceding proof of the False Claim could be revised into a sound proof of the following Claim:

Claim. *Let G be a graph whose vertex degrees are all $\leq k$. If* ⟨statement inserted from below⟩ *has a vertex of degree strictly less than k, then G is k-colorable.*

Circle each of the statements below that could be inserted to make the Claim true.

- G is connected and
- G has no vertex of degree zero and
- G does not contain a complete graph on k vertices and
- every connected component of G
- some connected component of G

Problem 4.

Use integration to find upper and lower bounds that differ by at most 0.1 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^2}$$

Problem 5.
Determine which of these choices

$$\Theta(n), \quad \Theta(n^2 \log n), \quad \Theta(n^2), \quad \Theta(1), \quad \Theta(2^n), \quad \Theta(2^{n \ln n}), \quad \text{none of these}$$

describes each function's asymptotic behavior. Full proofs are not required, but briefly explain your answers.

(a)

$$n + \ln n + (\ln n)^2$$

(b)

$$\frac{n^2 + 2n - 3}{n^2 - 7}$$

(c)

$$\sum_{i=0}^{n} 2^{2i+1}$$

(d)

$$\ln(n^2!)$$

(e)

$$\sum_{k=1}^{n} k\left(1 - \frac{1}{2^k}\right)$$

Doing P-set 7

4/6

Felt like I just did this

1. Simple graph $U \neq V$

two distinct paths

but no cycle

w/ 5 verticles

- ?how does this work

path - all must be unique

Hmm

Jwang

Look closely what a cycle is

- Never looked closely at defn.
- And said cycle involving U, V

(2)

b) Prove that if diff paths in simple graph then is a cycle

- somehow it must connect
- but how to prove?
  - splitting, merging?

Jwang's bad

Hope that works
Actually that is kinda what jwang had
but in split/merge terms

#2 7 bridges crossed once

sections = dots

bridges = edges

Euler tour

if only if every vertex has even degree

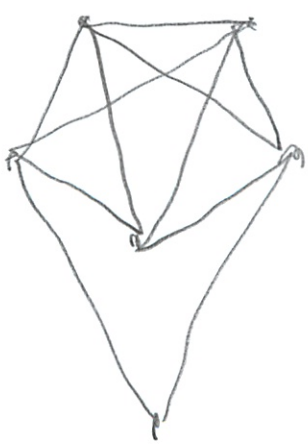
How many points?

WP: starts + ends on same pts

Bridges problem can't be solved?

wp has example

So complete graph?



even b/c in + out on each

does it work both ways?

(2)

a) if Euler every vertex even

Induction

Base One dot • one walk

× degree not even 0

two no •—• odd

3

induction on # of vertices

Add just to make ring

Inductive is weird - works for this case

But I'm not showing any Euler tour

- isomorphism?
- No

Juang had basically what I was thinking at 1st

---

b) incident = edges are incident to endpoints

*its a connected graph

Some sum formula?

③

Jwang makes no sense

Oh connected ≠ complete

Jwang makes more sense now

But there can be Euler walks in connected graphs that don't have every edge?

Well no - just suppose

Confused...

c) Show if w is walk, must be tour

Confused here too

d) Explain why all edges incident to end of w must be in w

e) Wrong approach I think - theorm must be true

Or is it?

Why can't I concentrate on this complex logic?

f) Now go through + check the multiple steps

(4)

Euler $\xleftrightarrow{\text{Theorm}}$ even

$\xrightarrow{a}$

b) Euler walk = every edge at most once

b on to prove Euler $\leftarrow$ even

tour = exactly once

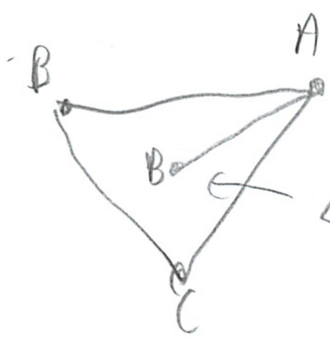
So walk - edge might not be included

#3 False claim

G is graph w/ vertex degrees all $\leq k$
If G has vertex of degree strictly less than k so k - colorable

No - contradicts

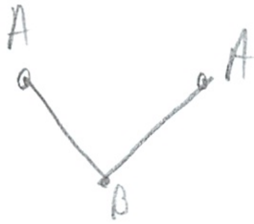
$k=2$



↙ all ≤ 2 degrees

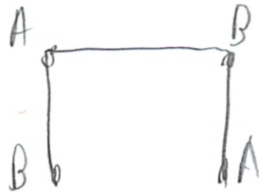
$< k$ deg

but still 3 colorable



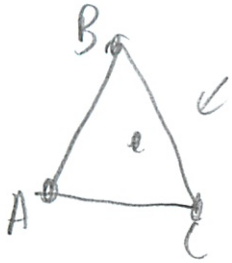
$k=2$

2 colorable



2

Jwang



So this works

$k=2$ - not 2 colorable

But wait - mine was like that

oh deg $\leq k$ so 2

So my original was correct

b) So where does it go wrong.

Base case does not hold ??

Why ind on # vertices - not k

- actually that makes sense

Base case - needs to have strictly less - not true

Jwang is unsure

Also it proves wrong way

c) Condition streanghtened to sound proof

- should have looked at 1st

Claim G is graph vertex degrees $\leq k$.

If $\overline{}$ has a vertex $< k$, then

G is k-colorable

(3)

Oh that would not have worked to help b

But what must it be

Could be multiple

Connected - would work?

- well no connected = every pair of vertices are connected

Opps connected - means a path

Complete - every pt to every pt

Opps thought everyone would be false

Rem conditions could never be true - statement = True

Oh 2 need a diff counterexample!

- No counter example did work

Every connected component?

 Not works  Not

Some  ~~Some~~ No since  no 2 colorable  No

Stand my

(4)

Ohhhhh

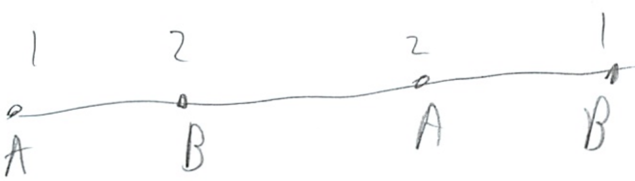
$\triangle$ is $k \neq 2$ since degree 3

Oppps

— redo whole pset!

— well just pg 2

#1



But how prove? Do you need to?

(2)

c) Which would make statement true?

#1) [G is connected] - No my counter example in a) was Connected, and it was proven false

(A path exists between every 2 points)

#2) No, Again my counterexample has no deg 0

#3) This statement itself is true, A complete graph can not have a vertex with degree less than k - so the statement itself will disqualify,

However, it does not prove the converse - the lack of it being a complete graph does not mean that it works. See my favorite Counter-example where G is not Complete. False.

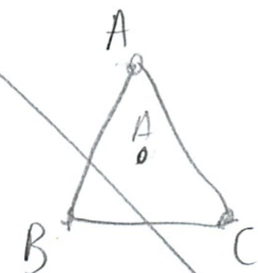
#4) False. My counter example had only 1 connected Component, but the claim would still be false

#5) False. Same as #4

None could make it tre?

3

#2 Still false, but my canter example does not work. But this canter example does



It is odd cycle so $\chi = 3$, but degree ≤ 2 and one < 2 and one vertex degree 0

#4 Have not read this yet

Use $\int$ to find upper + lower bands that differ by .1

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^2}$$

May need to add some terms

Try WA.

∞ sum

$$= \frac{1}{8}(\pi^2 - 8) = .233701$$

Appox sum chap

- strictly $\downarrow$?

$$I + f(n) \le S \le I + f(1)$$

Better than jwang's suggestion

Sum $\neq$ $\int$ integral

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Oshani

Table 12

P-Set 7

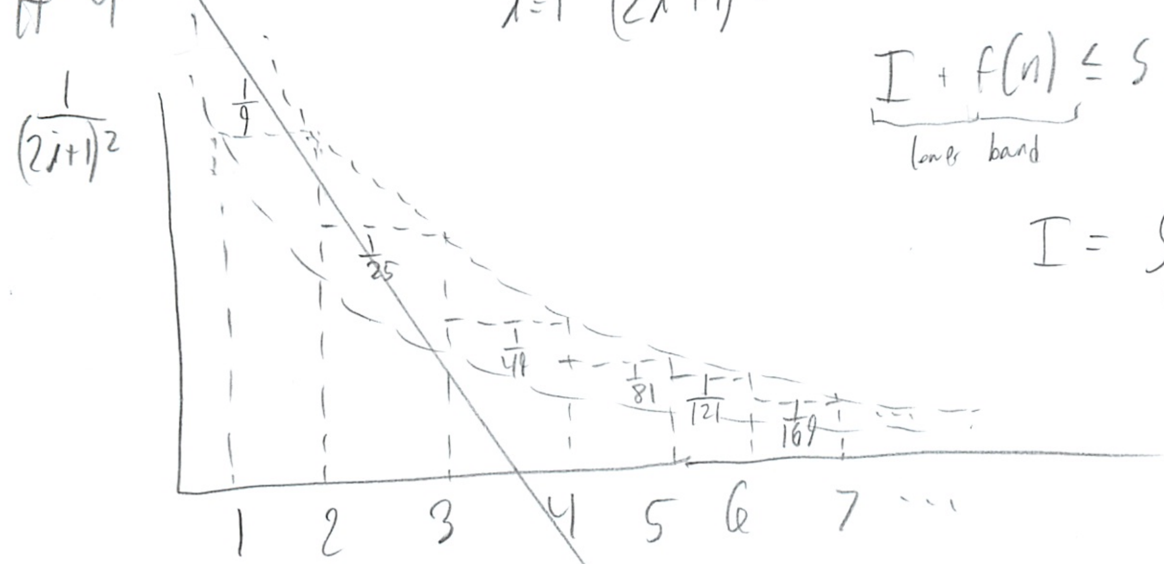
4

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^2}$$

Strictly decreasing

$$\underbrace{I + f(n)}_{\text{lower band}} \le S \le \underbrace{I + f(1)}_{\text{upper band}}$$

$$I = \int_1^n f(x)\,dx$$



Upper

$$\frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \int_5^{\infty} \frac{1}{2x-1}\,dx$$

$$= \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \frac{1}{18}$$

$$= \frac{47513}{198450} = .2394$$

#5 Finally something to work through

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} < \infty$$

finite

$f = O(g)$

But Θ is equal both ways $O(\,)$

$f = O(g) \quad g = O(f)$

¿ Use Wolfram G + V ¿

Ne gives answers for both

Student's Solutions to Problem Set 7

Your name: Michael Plasmeier

Due date: April 8

Submission date: 4/8

Circle your TA/LA: Ali Nick Oscar Oshani **Table number** 12

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with: Wolfram Alpha

got help from:[1] jwang7

and referred to:[2] Wikipedia ; Eulerian path
Seven Bridges of Königsberg

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	10
2	
3	
4	
5	
Total	

[1] People other than course staff.

[2] Give citations to texts and material other than the Spring '11 course materials.

#1

a) 

cycle = closed walk whose vertices are distinct except for starting and ending vertices

path = a walk where all points are distinct

b) If there exists two different paths (two paths were there is not a bijection between vertices and edges) then the simple graph has a cycle.

This is because the two paths must be connected at at least the start and end points (u and v in our example) and they must diverge somewhere in order to be different (at minimum what is shown in a). A cycle will exist where the two paths differ up to the point where the two paths connect (which can be at minimum, at the start + end points) Thus if there are diff paths b/w 2 vertices on a simple graph – there must be a cycle.

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Oshani

Table 12

P-Set 7

9

#2 Proof by induction over n = # verticies

$P(n)$ ::= that a graph with n verticies has even degree

Base $n=3$

can also have

The only possible Euler tour (Closed walk that gets every vertice once) is a triangle. Here the degree of every vertex $=2$. $P(3)$ = true

Inductive Add a vertex by placing a vertex on an existing line. This will make the degree of the new vertex $=2$ - one for each end of the line. The graph remains a Euler tour and every vertex has degree 2. 1

does not create all graph with Euler Tour!

(2)

a revisited) However this does not prove for a generic Euler tour - that is not a loop. Basically for each vertex you must first enter and then leave it to add degree 2 to the degree count of a vertex, You may do this multiple times from a vertex to/from different vertexes each time, but you will always have a degree \| 2. The starting vertex will be "matched" at the end So it is like going in and then leaving, because the starting point can be anywhere on a closed walk

---

Defn Euler walk = every edge at most once
Euler tour = " " exactly "

(3).

b) There can exist a connected graph for which a Euler walk does not include every edge. We are supposing this is the case, However the unincluded edges must be incident to a vertex on the walk. This is because there must be a vertex connecting the Euler walk to the untouched edge in order for the graph to be connected.

This edge must be added for the Euler walk to become an Euler tour

W is the logest Euler walk possible

- which would be a tour
- unless it could be lenghtened ← this is c!

(4).

c) If w is a closed walk, then it must be a Euler tour. This is because a closed walk will visit every vertex (except the start = end vertex) only once. ~~This will include every edge once because, if not, b) will happen. An unincluded edge will be incident to a vertex on~~ the ~~walk.~~

Every edge that is part of the walk will be included once.

If w is the longest possible walk, it is a Euler tour because it includes every edge. If it did not include every edge, then it would not be the longest possible walk – it could be lenghtened. This is through b) –. the walk could be lenghened if there was an edge incident to a vertex included on the walk

(5).

d) If there was an edge that was incident to the end of w, then w would be extended to include it. If there are no edges, then these edges must already be in w. Thus all edges incident to the end of w must already be in w.

(6).

e) If the end of w was not equal to the start of w, then the degree would be odd. If the degree would be odd, then the theorm would no longer be true, This is a condiction to a) where we proved that a Euler tour has degree = 2. Since this can not happen, the end of w must be equal to the start of w, for w to be a Euler walk or a Euler tour.

7.

f) So if every vertex of a finite, connected graph has an even degree, then it is a Euler tour.

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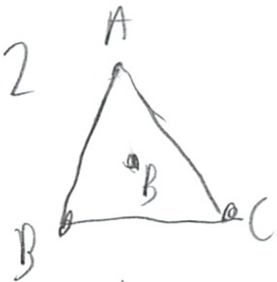
Table 12

6

P-Set 7

#3 a) $k=2$



All verticies have degree $\leq k$
2, 2, 2, 1

There is also a vertex of degree $<$ 2 - which is degree 1 (the middle)
strictly less than

However, this graph is not k-colorable.
The outer ring is a ring of 3 vertices, which is odd. This means since $\chi(C_{odd}) = 3$ the graph is 3-colorable, not 2-colorable

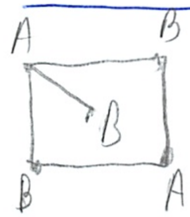
b) The base case is wrong. When you only have 1 vertex, k must $= 0$ since there can be no edges.
No it doesn't
G also requires a vertex of degree strictly less than k. However this is -1 which is not possible. The proof glosses over this and talks about the converse instead - that if G is 1-colorable P(1) holds. You can not conclude this.

②

#1) This would cause my counterexample to fail.
But would there be another counterexample?

Cond. →

#2) Would also cause counterexample to fail
But what about



$k=3$

This is enough...

True?

#3) If graph was complete it would fail -
Since could be no vertex with deg $< k$
So this statement does nothing. False.

#4). This would also cause my counterexample
to fail - this is the same as #1, True.

#5). My counterexample in a) meets this test, but
the statement is not true, so False.

(3)

#1 cond) Since every graph has at least $|V(G)| - |E(G)|$ Connected components, And we have by def. 1 connected components

$$1 \leq |V(G)| - |E(G)|$$

Also every connected graph w/ n vertices has at least $n-1$ edges

$$|V(G)| \geq |E(G)| - 1$$

So only last part holds

$\therefore$

True

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Oshani

Table 12

P-Set 7

#4 2

$$\sum \frac{1}{(2i+1)^2}$$

is strictly decreasing, so

$$\underbrace{I + f(n)}_{\text{lower band}} \leq S \leq \underbrace{I + f(1)}_{\text{Upper pound}}$$

$$S = \sum_{i=1}^{n} f(i)$$

$$I = \int_1^n f(x)\, dx$$

<u>lower band</u>

$$\int_1^n \frac{1}{(2x+1)^2}\, dx + \frac{1}{(2\infty+1)^2}$$

$$\frac{1}{6} + \frac{1}{\infty}$$

$$\frac{1}{6}$$

(2)

Upper

$I + f(1)$

$\frac{1}{6} + \frac{1}{(2\cdot 1+1)^2}$

$\frac{1}{6} + \frac{1}{9}$

$\frac{5}{18}$

~~Check if acceptable~~

$\cancel{\frac{11}{18} - \frac{1}{2} = \frac{1}{9}}$

$\cancel{\frac{1}{9} > .1}$

Check vs the actual value

Wolfram Alpha

$$\sum_{x=1}^{\infty} \frac{1}{(2x+1)^2} = \frac{1}{8}(\pi^2 - 8) \approx .233701$$

So $\left|\frac{1}{6} - .233701\right| = .067$ ✓

$\left|\frac{5}{18} - .233701\right| = .044$ ✓

bounds must be at most 0.1 from each other. Not actual value.

$\frac{1}{9} > 0.1$

4

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Oshani
Table 12
P-Set 7

6

#5 a. $n + \ln n + (\ln n)^2$

So Θ means $f = O(g)$ and $g = O(f)$

$f = O(g)$ means $\lim_{x \to \infty} \frac{g(x)}{f(x)} < \infty$
is finite

So basically asks what is asyptomic limit of the function?

Look at the biggest term

$(\ln n)^2 = \ln^2 n$

So $\Theta(\ln^2 n)$ but that is not a choice

None of the above

-2

$\Theta(n)$

b) $\frac{n^2+2n-3}{n^2-7}$

Look at largest terms $\frac{n^2}{n^2} \approx 1$

So $\Theta(1)$.

Test

$$\lim_{n\to\infty} \frac{\frac{n^2+2n-3}{n^2-7}}{1} = \frac{\frac{\infty}{\infty}}{1} = \frac{1}{1} = 1 \checkmark$$

$$\lim_{n\to\infty} \frac{1}{\frac{n^2+2n-3}{n^2-7}} = \frac{1}{\frac{\infty}{\infty}} = \frac{1}{1} = 1 \checkmark$$

c) $\sum_{i=0}^{n} 2^{(2i+1)}$

Try to convert to closed form

$$= \frac{2}{3}\left(2^{2n+2} - 1\right)$$

Which means $\frac{4}{3}^{2n+2}$ dominates

So $\Theta(2^n)$ is closest

But it would really be $\frac{4}{3}^{2n}$

So none of above

d) $\ln(n^2!)$

$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ Stirling's approx

So I am guessing

I prob messed up here

$$n^2! \sim \sqrt{2\pi n^2}\left(\frac{n^2}{e}\right)^{n^2}$$

Look at term that is leading

$(\ )^{n^2}$ is dominating

$$\ln\left(\frac{(n^2)^{n^2}}{e^{n^2}}\right) \xrightarrow{\text{roughly}} \ln {n^2}^{n^2}$$ This must be wrong

$n^2 \log n$ looks closest

e) $\sum_{k=1}^{n} k\left(1-\frac{1}{2^k}\right)$

Try to convert to closed form

$1\left(1-\frac{1}{2^1}\right) + 2\left(1-\frac{1}{2^2}\right) + \cdots$ so $\sim n(1-0)$

$\sim n$

gets larger

gets smaller

to 0

Roughly linear

$\Theta(n)$ $-2\ \Theta(n^2$

Check

$$\frac{\sum_{k=1}^{n} k\left(1-\frac{1}{2^k}\right)}{n} = \frac{n}{n} = 1$$

$$\frac{n}{\sum_{k=1}^{n} k\left(1-\frac{1}{2^k}\right)} = \frac{n}{n} = 1$$

Solutions to Problem Set 7

Reading: Chapter ??–??, Coloring, Connectedness, & Trees; Chapter ??, Planar Graphs; Chapter ??, Sums and Asymptotics.
Skip the following sections which will not be covered this term: Chapter ??, Minimum Weight Spanning Trees, Chapter ??, State Machines, Chapter ??, Double Sums, & Chapter ??, Omega notation.

Problem 1. **(a)** Give an example of a simple graph that has two vertices $u \neq v$ and two distinct paths between u and v, but no cycle including either u or v.

Hint: There is an example with 5 vertices.

Solution. Define

$$V ::= \{u, v, a, b, c\},$$
$$E ::= \{\langle u\text{—}a\rangle, \langle a\text{—}b\rangle, \langle b\text{—}c\rangle, \langle c\text{—}a\rangle, \langle c\text{—}v\rangle\}.$$

Two paths from from u to v are

$$u \langle u\text{—}a\rangle\, a \langle a\text{—}c\rangle\, c \langle c\text{—}v\rangle\, v$$

and

$$u \langle u\text{—}a\rangle\, a \langle a\text{—}b\rangle\, b \langle b\text{—}c\rangle\, c \langle c\text{—}v\rangle\, v.$$

■

(b) Prove that if there are different paths between two vertices in a simple graph, then the graph has a cycle.

Solution. *Proof.* Call a two vertices $u \neq v$ *different-path-pair* (dpp) if there are different paths paths between them. Suppose u, v is a dpp whose distance is minimum among all dpp's, and let $\mathbf{p}$ be a shortest path between u and v. By definition of dpp, there must be another path $\mathbf{q} \neq \mathbf{p}$ between u and v.

We claim that, other than u and v, there cannot be a vertex that appears in both paths $\mathbf{p}$ and $\mathbf{q}$. This implies that $\mathbf{q}\,\widehat{}\,\mathit{reverse}(\mathbf{p})$ is a cycle.

So we just have to prove the claim: suppose to the contrary there was such a vertex, w, appearing in both $\mathbf{p}$ and $\mathbf{q}$. This means that

$$\mathbf{p} = \mathbf{p_1}\, \widehat{w}\, \mathbf{p_2}$$

and

$$\mathbf{q} = \mathbf{q_1}\, \widehat{w}\, \mathbf{q_2}$$

for some walks $\mathbf{p_1}$, $\mathbf{q_1}$ that start at u and end at w, and walks $\mathbf{p_2}$, $\mathbf{q_2}$ that start at w and end at v. But since $\mathbf{p} \neq \mathbf{q}$, either $\mathbf{p_1} \neq \mathbf{q_1}$ or $\mathbf{p_2} \neq \mathbf{q_2}$, which implies that either u, w is a dpp or w, v is a dpp, and this dpp will be have a shorter path between them than u, v. This contradicts the fact that among all dpp's, u, v have a shortest length path between them. So the claim must be true.

■

Another proof can be given that is very similar to the proof of Theorem ??.??.

■

Problem 2.
The entire field of graph theory began when Euler asked whether the seven bridges of Königsberg could all be crossed exactly once. Abstractly, we can represent the parts of the city separated by rivers as vertices and the bridges as edges between the vertices. Then Euler's question asks whether there is a closed walk through the graph that includes every edge in a graph exactly once. In his honor, such a walk is called an *Euler tour*.

So how do you tell in general whether a graph has an Euler tour? At first glance this may seem like a daunting problem. The similar sounding problem of finding a cycle that touches every vertex exactly once is one of those Millenium Prize NP-complete problems known as the *Traveling Salesman Problem*). But it turns out to be easy to characterize which graphs have Euler tours.

Theorem. *A connected graph has an Euler tour if and only if every vertex has even degree.*

(a) Show that if a graph has an Euler tour, then the degree of each of its vertices is even.

Solution. Let tour $C ::= v_1, v_2, \dots, v_r, v_1$ be an Euler tour. Consider any vertex v. Then every time v occurs in C, there is a vertex a which comes immediately before v and a vertex b which comes immediately after v. Note that this holds for $v = v_1$ as well since C is a tour. Moreover, (a, v) and (v, b) must be distinct edges of G since C is an Euler tour. It follows that if v occurs s times in C, then it has degree $2s$ since every edge incident to v occurs in C exactly once. Thus, v has even degree. ■

In the remaining parts, we'll work out the converse: if the degree of every vertex of a connected finite graph is even, then it has an Euler tour. To do this, let's define an Euler *walk* to be a walk that includes each edge *at most* once.

(b) Suppose that an Euler walk in a connected graph does not include every edge. Explain why there must be an unincluded edge that is incident to a vertex on the walk.

Solution. If either end of the unincluded edge is on the Euler walk, that already is the desired edge. So suppose there's an unincluded edge, e, both of whose endpoints are not on the Euler walk. Since the graph is connected, there must be a shortest walk, $\mathbf{p}$, from an endpoint of e to a vertex on the Euler walk. Then none of the edges on $\mathbf{p}$ can be on $\mathbf{p}$ or $\mathbf{p}$ could be shortened. So the last edge on $\mathbf{p}$ will be the desired edge. ■

In the remaining parts, let $\mathbf{w}$ be the *longest* Euler walk in some finite, connected graph.

(c) Show that if $\mathbf{w}$ is a closed walk, then it must be an Euler tour.

Hint: part (b)

Solution. Suppose an edge was in $\mathbf{w}$. By part (b), there must be a vertex on $\mathbf{w}$ incident to an edge not in $\mathbf{w}$. Starting at this vertex, go around $\mathbf{w}$ back to that vertex, and then the follow the edge. This makes a longer Euler walk, contradicting the maximality of $\mathbf{w}$. So no edge can be missing from $\mathbf{w}$. ■

(d) Explain why all the edges incident to the end of $\mathbf{w}$ must already be in $\mathbf{w}$.

Solution. Otherwise we could extend $\mathbf{w}$ to a longer Euler walk with any edge from the end not already in $\mathbf{w}$. ■

(e) Show that if the end of $\mathbf{w}$ was not equal to the start of $\mathbf{w}$, then the degree of the end would be odd.

Hint: part (d)

Solution. Let v be the end vertex of $\mathbf{w}$. Given that v is not the start of $\mathbf{w}$, it follows that at any occurrence of v in $\mathbf{w}$ other than at the end, $\mathbf{w}$ would enter and leave that occurrence of v with a pair of edges. Since $\mathbf{w}$ is an Euler walk, all the edges in all these pairs are distinct. In addition, the final edge in $\mathbf{w}$ as it ends at v is distinct from all the paired edges. Altogether, this imples that there are an odd number of edges in $\mathbf{w}$ that are incident to v. But by part (d), these are all the edges incident to v, proving that v has odd degree. ■

(f) Conclude that if every vertex of a finite, connected graph has even degree, then it has an Euler tour.

Solution. If all vertices in G have even degree, then by part (e), the only possibility is that the end of $\mathbf{w}$ equals the start, that is, $\mathbf{w}$ is closed. So by part (c), $\mathbf{w}$ is an Euler tour. ■

Problem 3.

False Claim. *Let G be a graph whose vertex degrees are all $\leq k$. If G has a vertex of degree strictly less than k, then G is k-colorable.*

(a) Give a counterexample to the False Claim when $k = 2$.

Solution. One node by itself, and a separate triangle (K_3). The graph has max degree 2, and a node of degree zero, but is not 2-colorable. ■

(b) Underline the exact sentence or part of a sentence that is the first unjustified step in the following bogus proof of the False Claim.

Bogus proof. Proof by induction on the number n of vertices:

Induction hypothesis:

$P(n)$::= "Let G be an n-vertex graph whose vertex degrees are all $\leq k$. If G also has a vertex of degree strictly less than k, then G is k-colorable."

Base case: ($n = 1$) G has one vertex, the degree of which is 0. Since G is 1-colorable, $P(1)$ holds.

Inductive step:

We may assume $P(n)$. To prove $P(n+1)$, let G_{n+1} be a graph with $n+1$ vertices whose vertex degrees are all k or less. Also, suppose G_{n+1} has a vertex, v, of degree strictly less than k. Now we only need to prove that G_{n+1} is k-colorable.

To do this, first remove the vertex v to produce a graph, G_n, with n vertices. Let u be a vertex that is adjacent to v in G_{n+1}. Removing v reduces the degree of u by 1. So in G_n, vertex u has degree strictly less than k. Since no edges were added, the vertex degrees of G_n remain $\leq k$. So G_n satisfies the conditions of the induction hypothesis, $P(n)$, and so we conclude that G_n is k-colorable.

Now a k-coloring of G_n gives a coloring of all the vertices of G_{n+1}, except for v. Since v has degree less than k, there will be fewer than k colors assigned to the nodes adjacent to v. So among the k possible colors, there will be a color not used to color these adjacent nodes, and this color can be assigned to v to form a k-coloring of G_{n+1}. ■

Solution. The flaw is that if v has degree 0, then no such u exists. In such a case, removing v will not reduce the degree of any vertex, and so there may not be any vertex of degree less than k in G_n, as in the counterexample of part (a).

So the mistaken sentence is "Let u be a vertex that is adjacent to v in G_{n+1}."

Alternatively, you could say that it's OK to reason about a nonexistent u, and the only mistake is the claim that u exists. This claim is hidden in the phrase "So G_n satisfies the conditions of the induction hypothesis, $P(n)$". ■

(c) With a slightly strengthened condition, the preceding proof of the False Claim could be revised into a sound proof of the following Claim:

Claim. *Let G be a graph whose vertex degrees are all* $\leq k$. *If* ⟨statement inserted from below⟩ *has a vertex of degree strictly less than k, then G is k-colorable.*

Circle each of the statements below that could be inserted to make the Claim true.

- G is connected and
- G has no vertex of degree zero and
- G does not contain a complete graph on k vertices and
- every connected component of G
- some connected component of G

Solution. Either the first statement "G is connected and" or the fourth statement "every connected component of G" will work. ■

Problem 4.

Use integration to find upper and lower bounds that differ by at most 0.1 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^2}$$

Solution. Let's first try standard bounds:

$$\int_0^{\infty} \frac{1}{(2x+3)^2}\,dx \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \int_0^{\infty} \frac{1}{(2x+1)^2}\,dx$$

Evaluating the integrals gives:

$$-\frac{1}{2(2x+3)}\Bigg|_0^{\infty} \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad -\frac{1}{2(2x+1)}\Bigg|_0^{\infty}$$

$$\frac{1}{6} \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \frac{1}{2}$$

These bounds are too far apart, so let's sum the first couple terms explicitly and bound the rest with integrals.

$$\frac{1}{3^2} + \frac{1}{5^2} + \int_2^{\infty} \frac{1}{(2x+3)^2}\,dx \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \frac{1}{3^2} + \frac{1}{5^2} + \int_2^{\infty} \frac{1}{(2x+1)^2}\,dx$$

Integration now gives:

$$\frac{1}{3^2} + \frac{1}{5^2} + \left(-\frac{1}{2(2x+3)}\Bigg|_2^{\infty}\right) \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \frac{1}{3^2} + \frac{1}{5^2} + \left(-\frac{1}{2(2x+1)}\Bigg|_2^{\infty}\right)$$

$$\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{14} \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{10}$$

Now we have bounds that differ by $1/10 - 1/14 < 1/10 = 0.1$. ■

Problem 5.
Determine which of these choices

$$\Theta(n), \quad \Theta(n^2 \log n), \quad \Theta(n^2), \quad \Theta(1), \quad \Theta(2^n), \quad \Theta(2^{n \ln n}), \quad \text{none of these}$$

describes each function's asymptotic behavior. Full proofs are not required, but briefly explain your answers.

(a)

$$n + \ln n + (\ln n)^2$$

Solution. Both $n > \ln n$ and $n > (\ln n)^2$ hold for all sufficiently large n. Thus, for all sufficiently large n:

$$n < n + \ln n + (\ln n)^2 < n + n + n$$

So $n + \ln n + (\ln n)^2 = \Theta(n)$. ■

(b)

$$\frac{n^2 + 2n - 3}{n^2 - 7}$$

Solution. Observe that:

$$\lim_{n \to \infty} \frac{n^2 + 2n - 3}{n^2 - 7} = 1$$

This means that, for all sufficiently large n, the fraction lies, for example, between 0.99 and 1.01 and is therefore $\Theta(1)$. ■

(c)

$$\sum_{i=0}^{n} 2^{2i+1}$$

Solution. Geometric sums are dominated by their largest term, which is $2^{2n+1} = 2 \cdot 4^n$. This is $\Theta(4^n)$, which does not appear in the list provided. ■

(d)

$$\ln(n^2!)$$

Solution. By Stirling's formula:

$$n^2! \sim \sqrt{2\pi n^2} \left(\frac{n^2}{e}\right)^{n^2}$$

Taking logarithms gives:

$$\begin{aligned}
\ln(n^2!) &\sim \ln(\sqrt{2\pi n^2} \left(\frac{n^2}{e}\right)^{n^2}) \\
&= \ln(\sqrt{2\pi n^2}) + \ln \left(\frac{n^2}{e}\right)^{n^2} \\
&= \frac{1}{2} \ln 2\pi + \ln n + n^2 \ln(\frac{n^2}{e}) \\
&= \frac{1}{2} \ln 2\pi + \ln n + n^2(2 \ln n - 1)
\end{aligned}$$

It is then easy to see that this expression and $n^2 \ln n$ are big-O of each other by looking at limits as n goes to ∞, so we conclude that $\ln(n^2!) = \Theta(n^2 \ln n)$. ■

(e)

$$\sum_{k=1}^{n} k\left(1 - \frac{1}{2^k}\right)$$

Solution. The expression in parentheses is always at least $1/2$ and at most 1. Thus, we have the bounds:

$$\frac{1}{2}\sum_{k=1}^{n} k \le \sum_{k=1}^{n} k\left(1 - \frac{1}{2^k}\right) \le \sum_{k=1}^{n} k$$

Since the first expression and the last are both $\Theta(n^2)$, so is the one in the middle. ■

TP 8 cont

4/7

TP 8.6

$\int$ Integral method

4/8

$$S_{57} = \sum_{n=1}^{57} (n+7)^{-1/3}$$

$$I = \int_1^n (n+7)^{-1/3}$$

lower $I + f(1)$ use Wolfram Alpha

upper $I + f(n)$

57 here

$I = 18$

$f(1) = 8^{-1/3} = .5$

$f(57) = (57+7)^{-1/3} = .25$

diff $= .25$

②

TP 8.7 Big O practice

$$\lim \frac{f(x)}{g(x)} < \infty \qquad f = O(g)$$

find least n so $f(x)$ is $O(x^n)$

a.) Just the largest term

$2x^{3} + (\log x)\, x^2$

how does this fit in?

3 ✓

b) $2x^2 + (\log x)\, x^3$

4? ✓

c) $(1.1)^x$

- is this polynomial or exponential?

log $O(\log n)$

poly n^x n is what?

exp x^n ← so this

(3)

But ~~the~~ what is ~~other part~~ ans then?

$1.1^1 \quad 1.1^2 \quad 1.1^3 \quad 1.1^4 \quad \ldots \quad 1.1^{100}$

$1.1 \quad 1.21 \quad 1.331 \quad 1.46 \quad \ldots \quad 13780$

$\sim n^2$? (X)

n^1 (X)

Oh can try none (✓)

d) $.1^n$

none (X)

1 (X)

2 (X)

0 (✓) ← I had feeling it has this but did not want to put

e) $\frac{x^4+x^2+1}{x^3+1}$

4 (X) just look at largest term

~~just look at largest term~~

(4)

Or does it cancle

$$\frac{x^4}{x^3} = x^1$$ ✓

e) $$\frac{x^4 + 5\log x}{x^4 + 1} = x^0$$

If it was $\frac{5x^4}{x^4}$ would be $5x^0$

f) $2^{3\log_2 x^2}$

none? ⊗

$\log x^2$ – only natural #s allowed

2 ⊗

3 ⊗

6 ✓

TP9

4/8

TP9.1

Oh could have ans before reading - since did in 6.041

4 TF

$2 \cdot 2 \cdot 2 \cdot 2 \quad + 4 \quad + \quad 6$

$2^4 \quad + 4 \quad + 6$ ✗

Or is it times

- yeah x is and
- + would be or

$2^4 \cdot 4 \cdot 6$ ✓

TP9.2

Won't load, emailed in

TP9.3

$X ::= \{1, 2, 3, 4, 5, 6\}$

How many subsets of X contain 1?

② ~~Did~~ ~~this~~

Hmm

1 | 1
1, 2 |
1, ... ~~5~~6 | ~~1~~ 5

~~1, 5~~

1, 2, 3 ..., 6

1, 3, 2, 4, ... 6 $\quad 5 \cdot 4 \quad$ add since diff lenghts

1, 2, 3, 4 ... 6 $\quad 5 \cdot 4 \cdot 3$

1, 2, 3

len 1 $\quad$ 2 $\quad$ 3 $\quad$ 4 $\quad$ 5 $\quad$ 6 $\quad$ 7

$1 + 5 + 5 \cdot 4 + 5 \cdot 4 \cdot 3 + 5 \cdot 4 \cdot 3 \cdot 2 + 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 + 1$

x

$1 + 5 + 5 \cdot 4 + 5 \cdot 4 \cdot 3 + 5 \cdot 4 \cdot 3 \cdot 2 + 1$ x

That should be it!

207

try B) How many subsets 2, 3 but not 6

– must be eaiser way to do this

3

len 2   2, 3   1

len 3   2, 3, something $1 \cdot 3$
1, 4, 5

len 4   2, 3   2 of something $1 \cdot 3 \cdot 2$
1, 4, 5

len 5 $2, 3, \underbrace{1, 4, 5,}_{\text{all } 3}$ 1

$1 + 3 + 6 + 1$ X

Subset - order does not matter

Give up

A) $2^5 = 32$

Bij · X subsets ↔ binary strings len 6

- first position fixed must be 1

2 choices for oter positions

But how does deal w/ smaller subsets?

B) 2^3 3 pos fixed so 2^3

So 1 means in subset?

(4)

TP9.4 Mississippi

How many permutations?

- So any order

- do later

4/8

Mathematics for Computer Science
MIT 6.042J/18.062J

Counting

Counting in Gambling

What *fraction* of poker hands are "a pair of Jacks?"

(probability of a pair of Jacks)

Counting in Algorithms

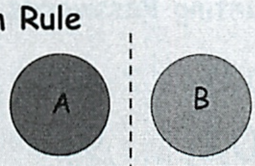
- # ops to update a data structure (# comparisons needed to sort n items)
- # steps in a computation (# multiplies to compute d^n)

Counting in Cryptography

possible passwords

possible keys

Sum Rule



If sets A and B are disjoint, then

|A B| = |A| + |B|

The Sum Rule

- Class has 43 women, 54 men so total enrollment = 43 + 54 = 97
- 26 lower case letters, 26 upper case letters, and 10 digits, so # characters = 26+26+10 = 62

The Product Rule

If there are 4 boys and 3 girls, there are

$$4 \cdot 3 = 12$$

different boy/girl couples

Product Rule

If $|A| = m$ and $|B| = n$, then
$$|A \times B| = m \cdot n$$

$A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$

$A \times B = \{(a,1),(a,2),(a,3),$
$(b,1),(b,2),(b,3),$
$(c,1),(c,2),(c,3),$
$(d,1),(d,2),(d,3)\ \}$

Product Rule: Counting Strings

\# length-4 binary strings

$= |B \times B \times B \times B|$

$= |B^4|$ where $B ::= \{0,1\}$

$= 2 \cdot 2 \cdot 2 \cdot 2 = 2^4$

Product Rule: Counting Strings

\# length n strings from an alphabet of size m is

$$m^n$$

Example: Counting Passwords

Password conditions:

- characters are digits & letters
- between 6 & 8 characters long
- starts with a letter
- case sensitive

Counting Passwords

$L ::= \{a,b,\ldots,z,A,B,\ldots,Z\}$

$D ::= \{0,1,\ldots\ldots,9\}$

P_n ::= length n words starting w/letter

$= L \times (L \quad D)^{n-1}$

Counting Passwords

$|L \times (L \quad D)^{n-1}|$

$= |L| \cdot |(L \quad D)|^{n-1}$

$= |L| \cdot (|L| + |D|)^{n-1}$

$= 52 \cdot (52+10)^{n-1}$

Albert R Meyer, April 8, 2011 lec 9F.16

Counting Passwords

set of passwords:

$P ::= P_6 \quad P_7 \quad P_8$

$|P| = |P_6| + |P_7| + |P_8|$

$= 52 \cdot (62^5 + 62^6 + 62^7)$

$\approx 19 \cdot 10^{14}$

Albert R Meyer, April 8, 2011 lec 9F.17

4-digit nums w/ $\geq$ one 7

cases by 1st occurrence of 7:

x: any digit o: any digit $\neq 7$

7xxx or o7xx or oo7x or ooo7

$10^3 + 9 \cdot 10^2 + 9^2 \cdot 10 + 9^3$

$= 3439$

Albert R Meyer, April 8, 2011 lec 9F.18

at least one 7: another way

|4-digit nums w/ $\geq$ one 7|

= |4-digit nums|

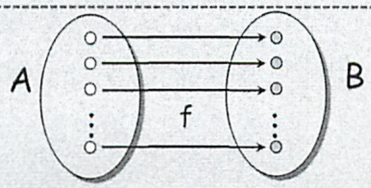
− |those w/ no 7|

$= 10^4 - 9^4 = 3439$

Albert R Meyer, April 8, 2011 lec 9F.19

Mapping Rule: Bijections

If f is a bijection from A to B, then $|A| = |B|$



Albert R Meyer, April 8, 2011 lec 9F.20

Counting Doughnut Selections

From 5 kinds of doughnuts select a dozen.

let A ::= all selections of 12 doughnuts

$\underbrace{00}_{\text{chocolate}} \quad \underbrace{\text{(none)}}_{\text{lemon}} \quad \underbrace{000000}_{\text{sugar}} \quad \underbrace{00}_{\text{glazed}} \quad \underbrace{00}_{\text{plain}}$

Albert R Meyer, April 8, 2011 lec 9F.24

Counting Doughnut Selections

B::= 16-bit words with four 1's

$\underbrace{00}_{\text{chocolate}}\ 1\ \underbrace{\text{(none)}}_{\text{lemon}}\ 1\ \underbrace{000000}_{\text{sugar}}\ 1\ \underbrace{00}_{\text{glazed}}\ 1\ \underbrace{00}_{\text{plain}}$

Counting Doughnut Selections

B::= 16-bit words with four 1's

00 1 1000000 1 00100

$\underbrace{00}_{\text{chocolate}}\ \underbrace{\text{(none)}}_{\text{lemon}}\ \underbrace{000000}_{\text{sugar}}\ \underbrace{00}_{\text{glazed}}\ \underbrace{00}_{\text{plain}}$

Counting Doughnut Selections

B::= 16-bit words with four 1's

0011000000100100

a bijection: $|A| = |B|$

$\underbrace{00}_{\text{chocolate}}\ \underbrace{\text{(none)}}_{\text{lemon}}\ \underbrace{000000}_{\text{sugar}}\ \underbrace{00}_{\text{glazed}}\ \underbrace{00}_{\text{plain}}$

Team Problems

Problems

1—4

6.042

4/8

Counting comes from gambling

Used to multiply # in computers

For example how many possible passwords w/
certain rules

- to check Brute Force possibity

If sets are disjoint   <u>Sum Rule</u>

A   B

$|A| + |B| = |A \cup B|$

54 men + 43 women = 97 people

26 lower + 26 upper + 10 = 62 alphanumeric chars

<u>Product Rule</u>

4 boys   3 girls

$4 \cdot 3$ possible pairs

$|A \times B| = |A| \cdot |B|$

(2)

$(a,1)$ $(a,2)$ $(a,3)$
$(b,1)$ $(b,2)$ $(b,3)$
$(c,1)$ $(c,2)$ $(c,3)$ 12 possible pairs
$(d,1)$ $(d,2)$ $(d,3)$

Counting strings

$= |B \times B \times B \times B|$

$= |B^4|$ where $B: \{0,1\}$

$= 2^4$

2^n # of bit strings of lenght n

B: # sets ~~and~~ bitstrings

If size m alphabet, # of n-lenght strings =
m^n

Use to find # of passwords for password condiditons

③

$L = \{a, b, \dots, z, A, B, \dots Z\}$

$D = \{0, 1, \dots, 9\}$

P_n = lenght n words starting w/ Letter.

$\cancel{\#} \left| L \times (L \cup D)^{n-1} \right|$

?but this is must be n lenght

$= |L| \times |(L \cup D)^{n-1}|$

$= |L| \times (|L| + |D|)^{n-1}$

$= 52 \cdot (52 + 10)^{n-1}$

Could be 6, 7, 8 letters

$P = P_6 \cup P_7 + P_8$

$= |P_6| + |P_7| + |P_8|$

$\approx$ over a trillion

(4)

No tools to know that you are right

Can almost always reduce to counting sequences

- with bij

\# 4 digit nums w/ $\geq$ one 7

- can start w/ 0
- have at least one 7

Can do by cases

1st occurance of 7

7xxx or o7xx or oo7x or ooo7

x = any digit ~~except~~

o = " " except 7

Add them up for ans

$10^3 + 9 \cdot 10^2 + 9^2 \cdot 10 + 9^3$

$= 3439$

5

At least one 7 another way

- count the # of strings w/ no 7s

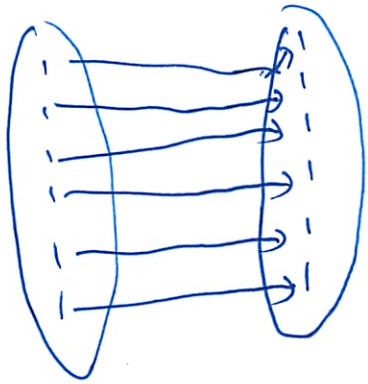
$= |4 \text{ digit nums}| - |\text{those w/ no 7s}|$

$= 10^4 - 9^4$

$= 3439$

Again Bij

$|A| = |B|$

A  B

How many ways to select a dozen from 5 kinds of donuts

Find bij to something easy to count

⑥

~~B:~~ B:: = 16 bit words w/ 4 1s

1s to delimit the gaps

- seperate boundries
- that is a bij $|A| = |B|$
- sequence counting problem

Week 9 Friday InClass

4/8

1. License plate

   3 letters → 3 dig

   5 letters

   2 chars

a) L = all possible plates

$A =$ alpha $\quad |A| = 26$

$D =$ digits $\quad |D| = 10$

$L = A^3 \cdot D^3 + A^5 + (A \cup D)^2$

b) $= 26^3 \cdot 10^3 + 26^5 + (26+10)^2$

Actually $(A^3 \times D^3) \cup (A^5) \cup (A \cup D)^2$

notation

b) 29458672

2. n-vertex #ed tree $\{1, 2, \ldots n\}$ $n \geq 2$

   Code

   If > 2 vertices - write father - delete

②

Till 2 left

a) Proc reconstruct

Start w/ 1 — 2

~~ARM~~ At end

~~If 2 # Before~~

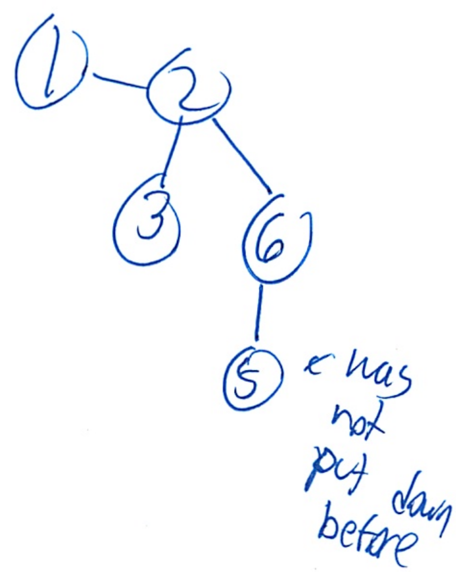
Look the next # on that   1 - 2

already 2

one before

If already put it down, its max(have) + 1

revision



Are 1 2 always connected?

Opps

So start w/ 1 Then do rest of proc.

(2)

b) We found way to reconstruct tree from code

Code - any seq $\{1, \dots n\}$

to the $n-2$

3. How many billion

$10^9 - 9^8 - 9^7 - 9^6 - 9^5 - 9^4 - 9^3 - 9^2 - 9^1$

But what about 0s?

$\cancel{10^{10} - 9^9}$

$10^{10} - 9^9$

9612579511

Only up to 10^9

$10^9 - 9^8$ instead

956953279

b) 20 books on shelf

bij to choose 6 so no 2 adj

15-bit string w/ exactly 6 1s

(4)

Is this like the donut one? - Use 1s as seperator

But how do you do adjacency?

2a) Prof: Backwards way we did was not what he thought about - but willing to accept if we write it up

3a) $S = \{2, \ldots, 9\}$

8

Are we going to bother w/ 0s?

No

$$\overset{\downarrow \{2,\ldots,9\}}{(2\text{-}9)} \times (0, 2\text{-}9)^{n+}$$

$$\sum_{1}^{9} (2\ldots9) + \{0, 2\ldots9)^{n}$$

Solutions to In-Class Problems Week 9, Fri.

Problem 1.
A license plate consists of either:

- 3 letters followed by 3 digits (standard plate)
- 5 letters (vanity plate)
- 2 characters—letters or numbers (big shot plate)

Let L be the set of all possible license plates.

(a) Express L in terms of

$$\mathcal{A} = \{A, B, C, \ldots, Z\}$$
$$\mathcal{D} = \{0, 1, 2, \ldots, 9\}$$

using unions ($\cup$) and set products ($\times$).

Solution.

$$L = (A^3 \times D^3) \cup A^5 \cup (A \cup D)^2$$

■

(b) Compute $|L|$, the number of different license plates, using the sum and product rules.

Solution.

$$\begin{aligned}
|L| &= |(A^3 \times D^3) \cup A^5 \cup (A \cup D)^2| \\
&= |(A^3 \times D^3)| + |A^5| + |(A \cup D)^2| && \text{Sum Rule} \\
&= |A|^3 \cdot |D|^3 + |A|^5 + |A \cup D|^2 && \text{Product Rule} \\
&= |A|^3 \cdot |D|^3 + |A|^5 + (|A| + |D|)^2 && \text{Sum Rule} \\
&= 26^3 \cdot 10^3 + 26^5 + 36^2 = 29458672
\end{aligned}$$

■

Problem 2.
An n-vertex *numbered tree* is a tree whose vertex set is $\{1, 2, \ldots, n\}$ for some $n > 2$. We define the *code* of the numbered tree to be a sequence of $n - 2$ integers from 1 to n obtained by the following recursive process:[1]

[1]The necessarily unique node adjacent to a leaf is called its *father*.

If there are more than two vertices left, write down the *father* of the largest leaf, delete this *leaf*, and continue this process on the resulting smaller tree.

If there are only two vertices left, then stop—the code is complete.

For example, the codes of a couple of numbered trees are shown in the Figure 1.

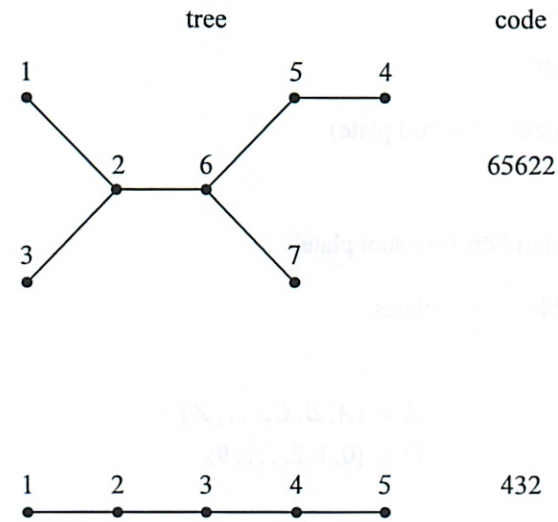


Figure 1

(a) Describe a procedure for reconstructing a numbered tree from its code.

Solution. The key observation is that, given a code of length $n-2$, the numbers between 1 and n which *do not appear* in the code are precisely the leaves of the tree. This follows because the vertices left at the end of the process are both leaves. So the procedure must have changed all the nonleaf vertices into leaves, and this implies that all the nonleaf vertices appear in the code.

Hence, the largest missing number is a leaf attached to the first number of the code. The rest of the tree can now be reconstructed by deleting the first number in the code, henceforth ignoring the largest leaf, and proceeding recursively on the rest of the code. (We're using the obvious fact that what's left after deleting a leaf from a tree is another tree.)

More precisely, the reconstruction procedure applies to any finite tree whose vertex set is totally ordered. The procedure takes *two* parameters: the vertex set, V, and a length $|V|-2$ "code" sequence, S, of elements in V. If l is the largest element in V which does not appear in S, and f is the first element of S, then the reconstructed tree is obtained by adding edge (l, f) to the tree reconstructed by calling the procedure recursively with first argument $V-\{l\}$ and second argument equal to the code obtained by erasing the initial f from S. The procedure terminates when $|V| = 2$, returning the edge between the two numbers in V.

■

(b) Conclude there is a bijection between the n-vertex numbered trees and $\{1, \ldots, n\}^{n-2}$, and state how many n-vertex numbered trees there are.

Solution. There are exactly as many n-vertex numbered trees as the number of possible code words, that is, the number of length $n-2$ sequences of integers between 1 and n. So there are n^{n-2} numbered trees.

The reason is that the map from trees to codes is a bijection. To see this, note that the tree reconstruction procedure finds *the only possible tree* with that code. So there can't be two trees with the same code, that is, the map from a tree to its code is an injection. But since the reconstruction procedure finds a tree for every possible codeword, the map from trees to codes is also a surjection. ■

Problem 3. (a) How many of the billion numbers in the range from 1 to 10^9 contain the digit 1? (*Hint:* How many don't?)

Solution. We can count up how many *do not* contain the digit 1 and subtract. So (total number) - (number without 1's) $= 10^9 - (9^9 - 1) = 612,579,512$ (the -1 is for 0 which is not in our range). ■

(b) There are 20 books arranged in a row on a shelf. Describe a bijection between ways of choosing 6 of these books so that no two adjacent books are selected and 15-bit strings with exactly 6 ones.

Solution. A selection of six among twenty books on a shelf corresponds in an obvious way to a 20-bit string with exactly six 1's. For example, the 20-bit string with 1's in exactly the 3rd, 4th, 5th, 10th, 19th and 20th positions corresponds to selecting 3rd, 4th, 5th, 10th, 19th and 20th books on the shelf.

So the problem reduces to finding a bijection between 20-bit strings with six *nonadjacent* 1's and 15-bit strings with six 1's.

But in a string, s, with six nonadjacent 1's, all but the last 1 must have a 0 to its right. So we can map s to a string with six 1's and five fewer 0's by erasing the 0's immediately to the right of each of the first five 1's. For example, erasing the underlined 0's in the 20-bit string $\mathtt{0001\underline{0}1\underline{0}01\underline{0}1\underline{0}00001\underline{0}10}$ yields the 15-bit string 000110110000110.

This map is a bijection because given any 15-bit string with six 1's, there is a unique 20-bit string with nonadjacent 1's that maps to it, namely, the string obtained by replacing each of the first five 1's in the 15-bit string by a 10. ■

Problem 4.

(a) Let $\mathcal{S}_{n,k}$ be the possible nonnegative integer solutions to the inequality

$$x_1 + x_2 + \cdots + x_k \leq n. \tag{1}$$

That is

$$\mathcal{S}_{n,k} ::= \{(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k \mid (1) \text{ is true}\}.$$

Describe a bijection between $\mathcal{S}_{n,k}$ and the set of binary strings with n zeroes and k ones.

Solution. The notation 0^x indicates a length x string of 0's.

$$(x_1, x_2, \ldots, x_k) \longleftrightarrow 0^{x_1}10^{x_2}1\ldots0^{x_k}10^{n-s},$$

where $s ::= \sum_{i=1}^{k} x_i$. ■

(b) Let $\mathcal{L}_{n,k}$ be the length k weakly increasing sequences of nonnegative integers $\leq n$. That is

$$\mathcal{L}_{n,k} ::= \{(y_1, y_2, \ldots, y_k) \in \mathbb{N}^k \mid y_1 \leq y_2 \leq \cdots \leq y_k \leq n\}.$$

Describe a bijection between $\mathcal{L}_{n,k}$ and $\mathcal{S}_{n,k}$.

Solution. $(y_1, y_2, \ldots, y_k) \longleftrightarrow (y_1,\ y_2 - y_1,\ y_3 - y_2,\ \ldots,\ y_k - y_{k-1})$.

In the other direction,

$$(x_1, x_2, \ldots, x_k) \longleftrightarrow (x_1,\ x_1 + x_2,\ x_1 + x_2 + x_3,\ \ldots,\ \sum_{i=1}^{k} x_i).$$

■

4/11

Mathematics for Computer Science

MIT 6.042J/18.062J

Generalized Counting Rules

Generalized Product Rule

lineups of 5 students in 6.042? let S::= 6.042 students

|S| = 91 so

~~|lineups of 5 students|~~ NO!

lineups have no repeats:

|seqs in S^5 with no repeats| ?

Generalized Product Rule

|seqs in S^5 with no repeats|

91 choices for 1st student,

90 choices for 2nd student,

89 choices for 3rd student,

88 choices for 4th student,

87 choices for 5th student

$$= 91 \cdot 90 \cdot 89 \cdot 88 \cdot 87 = \frac{91!}{86!}$$

Generalized Product Rule

Q a set of length-k sequences

if n_1 possible 1st elements,

n_2 possible 2nd elements (for each first entry),

n_3 possible 3rd elements (for each 1st & 2nd entry,...)

then, $\boxed{|Q| = n_1 \cdot n_2 \cdots n_k}$

Division Rule

$$\#6.042 \text{ students} = \frac{\#6.042 \text{ students' fingers}}{10}$$

Division Rule

if function from A to B is k-to-1, then

$$|A| = k|B|$$

(generalized Bijection Rule)

Counting Subsets

How many size 4 subsets of $\{1,2,\ldots,13\}$?

Let $A ::=$ permutations of $\{1,2,\ldots,13\}$

$B ::=$ size 4 subsets

map $\boxed{a_1 a_2 a_3 a_4} a_5 \ldots a_{12} a_{13} \in A$

to $\{a_1, a_2, a_3, a_4\} \in B$

Counting Subsets

$\boxed{a_1 a_3 a_2 a_4} a_5 \ldots a_{12} a_{13}$ also maps to $\{a_1, a_2, a_3, a_4\}$

so does $\underbrace{a_1 a_3 a_2 a_4}_{4!\ \text{perms}} \underbrace{a_{13} \ldots a_{12} a_5}_{9!\ \text{perms}}$

all map to same set

$4! \cdot 9!$-to-1

those can be in any permutation

these can be in any other permutation

Counting Subsets

$$13! = |A| = (4! \cdot 9!)|B|$$

so # of size 4 subsets is

$$\binom{13}{4} ::= \frac{13!}{4!9!}$$

Counting Subsets

\# m element subsets of an n element set is

$$\binom{n}{m} ::= \frac{n!}{m!(n-m)!}$$

counting 2-pair poker hands

a 2-pair hand has

- 2 cards of some rank
- 2 cards of a second rank
- 1 card of still a third rank

counting 2-pair poker hands

a 2-pair hand:

K♦, K♥, A♦, A♠, 3♣

counting 2-pair poker hands

to count, choose:

- 1^{st} pair rank (13 ranks)
- 2^{nd} pair rank (12 ranks left)
- last card rank (11 ranks left)

counting 2-pair poker hands

then choose:

- 1^{st} pair suits $\binom{4}{2}$ sets of 2 suits
- 2^{nd} pair suits $\binom{4}{2}$ sets of 2 suits
- last card suit (4 suits)

counting 2-pair poker hands

successively choosing:

K, A, 3, {♥, ♦}, {♦, ♠}, ♣

specifies 2-pair hand:

K♦, K♥, A♦, A♠, 3♣

counting 2-pair poker hands

so # 2-pair hands is

$$13 \cdot 12 \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 4$$

NO!

counting 2-pair poker hands

this method counts 6-tuples

[1^{st} card ranks]×[2^{nd} card ranks] × [last card rank] ×

[1^{st} card suits] × [2^{nd} card suits] × [last card suit]

correctly

counting 2-pair poker hands

but the correspondence to 2-pair hands is not a bijection:

(K, A, 3, {♥, ♦}, {♦, ♠}, ♣)

K♦, K♥, A♦, A♠, 3♣

(A, K, 3, {♦, ♠}, {♥, ♦}, ♣)

counting 2-pair poker hands

to count, choose: the bug

- 1^{st} pair rank (13 ranks)
- 2^{nd} pair rank (12 ranks left)
- last card rank (11 ranks left)

counting 2-pair poker hands

to count, choose: the bug

- 1^{st} pair rank (13 ranks)
- 2^{nd} pair rank (12 ranks left)
- last card rank (11 ranks left)

either pair might be 1^{st}

counting 2-pair poker hands

map from 6-tuples

(K, A, 3, {♥, ♦}, {♦, ♠}, ♣)

to 2-pair hands

K♦, K♥, A♦, A♠, 3♣

is 2-to-1

counting 2-pair poker hands

so # 2-pair hands is

$$13\cdot 12\cdot 11\cdot\binom{4}{2}\cdot\binom{4}{2}\cdot 4$$

NO!

counting 2-pair poker hands

so # 2-pair hands is really

$$\frac{1}{2}\cdot 13\cdot 12\cdot 11\cdot\binom{4}{2}\cdot\binom{4}{2}\cdot 4$$

Team Problems

Problems

1–4

6.042 Generalized Counting Rules

4/11

Generalized product rule

\# of lineups of 5 students

$S = 91$ = # 6.042 students

So 91^5?

No! Students can't repeat!

So how to do it?

Chose first student $\to 91$

Then for each of those 91, have 90 remaining students

- each a diff set of 90
- but will always be 90 students

So

$$= 91 \cdot 90 \cdot 89 \cdot 88 \cdot 87$$

$$= \frac{91!}{86!}$$ ← so it cancles out all terms below 87

So

$$= n_1 \cdot n_2 \cdot \ldots \cdot n_k$$

↑ # of choices at each set

2

Divison Rule

- generalization of bij rule
- like # students = $\frac{\text{\# fingers}}{10}$
- So fn from A to B is k-to-1, then
  $|A| = k|B|$ generalized bij rule
- # of subsets of a given size
- $A ::=$ permutations of $\{1, \ldots 13\}$
  - generalized product rule
- define mapping permutations → size of 4 subsets
  map $\underbrace{a_1 a_2 \ldots}_{\text{just take first 4 elements}} a_{13} \in A$
- but if were in diff order - still same set
  $a_1 a_3 a_2 a_4$ is still $\{a_1\ a_2\ a_3\ a_4\}$
- and only between these 4 elements

③

A $4!$ perms of first els $9!$ per of remaining

$4! \cdot 9!$ - to - 1

So
$$13! = |A| = (4! \cdot 9!)|B|$$

So # size 4 subsets is

$$\frac{13!}{4!\,9!} \overset{\text{special name}}{::=} \binom{13}{4} \overset{\text{"}}{::=} 13 \text{ choose } 4$$

So (... missed

Counting 2 pair poker hand
- 2 cards of same rank
- 2 cards of a 2nd rank
- 1 card of a 3rd rule

- not 4 of a kind

Try to set up a bij

(4)

So first card

1st pair suit $\binom{4}{2}$

2nd pair suit $\binom{4}{2}$

last card $\binom{4}{1}$

Special seq notation

$\cancel{13 \cdot 12 \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 4}$ NO! -seq and hands are not in bij

Other seq can also map

But there is no 1st and 2nd pair

Are you imposing the order or is it built in?

No way to tell 1st pair from second pair