In-Class Problems Week 10, Mon.

Problem 1.
Answer the following quesions using the Generalized Product Rule.

(a) Next week, I'm going to get really fit! On day 1, I'll exercise for 5 minutes. On each subsequent day, I'll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 9, 11, 12. How many such sequences are possible?

(b) An *r-permutation* of a set is a sequence of r distinct elements of that set. For example, here are all the 2-permutations of $\{a, b, c, d\}$:

$$
\begin{array}{llll}
(a,b) & (a,c) & (a,d) \\
(b,a) & (b,c) & (b,d) \\
(c,a) & (c,b) & (c,d) \\
(d,a) & (d,b) & (d,c)
\end{array}
$$

How many r-permutations of an n-element set are there? Express your answer using factorial notation.

(c) How many $n \times n$ matrices are there with *distinct* entries drawn from $\{1, \ldots, p\}$, where $p \geq n^2$?

Problem 2. (a) There are 30 books arranged in a row on a shelf. In how many ways can eight of these books be selected so that there are at least two unselected books between any two selected books?

(b) How many nonnegative integer solutions are there for the following equality?

$$x_1 + x_2 + \cdots + x_m = k. \tag{1}$$

(c) How many nonnegative integer solutions are there for the following inequality?

$$x_1 + x_2 + \cdots + x_m \leq k. \tag{2}$$

(d) How many length-m weakly increasing sequences of nonnegative integers $\leq k$ are there?

Problem 3.
Your class tutorial has 12 students, who are supposed to break up into 4 groups of 3 students each. Your Teaching Assistant (TA) has observed that the students waste too much time trying to form balanced groups, so they decided to pre-assign students to groups and email the group assignments to his students.

(a) Your TA has a list of the 12 students in front of him, so they divides the list into consecutive groups of 3. For example, if the list is ABCDEFGHIJKL, the TA would define a sequence of four groups to be $(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\})$. This way of forming groups defines a mapping from a list of twelve students to a sequence of four groups. This is a k-to-1 mapping for what k?

(b) A group assignment specifies which students are in the same group, but not any order in which the groups should be listed. If we map a sequence of 4 groups,

$$(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}),$$

into a group assignment

$$\{\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}\},$$

this mapping is j-to-1 for what j?

(c) How many group assignments are possible?

(d) In how many ways can $3n$ students be broken up into n groups of 3?

Problem 4.

A pizza house is having a promotional sale. Their commercial reads:

> We offer 9 different toppings for your pizza! Buy 3 large pizzas at the regular price, and you can get each one with as many different toppings as you wish, absolutely free. That's $22, 369, 621$ different ways to choose your pizzas!

The ad writer was a former Harvard student who had evaluated the formula $(2^9)^3/3!$ on his calculator and gotten close to $22, 369, 621$. Unfortunately, $(2^9)^3/3!$ is obviously not an integer, so clearly something is wrong. What mistaken reasoning might have led the ad writer to this formula? Explain how to fix the mistake and get a correct formula.

1. Answer

a) 5 min

Then next day $^+0, 1, 2, 3$ min

· <u>What are all the permutations</u>

They are not asking sum which would be harder!

$1 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

4^6 ✓

b) r-permutation of set is seq of r distinct elements

So a 2-permutation of set of size 4

· How many r-perm for set of size n

$4 \cdot (4-1) \cdot$
   ↑   ↑ # terms
   n

$$\prod_{i=0}^{r} \cancel{all} (n-i)$$

$$\prod_{i=1}^{r} (n-i+1)$$

② Then
$$\frac{n!}{(n-r)!}$$

Since $(n) \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$ ⟵ these terms

Then ∘ by $\dfrac{(n-r)!}{(n-r)!}$

This cancles out the

$-1 \quad -2 \quad \cdots \quad -r+1$

So $\dfrac{n!}{(n-r)!}$

$(n-r) (n-r-1) (n-r-2) (n-r-3) \cdots$

Divide by $(n-r)$ to cut out ⟵ these terms

that would be there from $n!$

(3)

c) How many $n \times n$ matrices are there w/ distinct entries drawn from $\{1, ..., p\}$ where $p \geq n^2$

$$\binom{p}{n^2!} n^2! = \frac{p!}{\cancel{n^2!}(p-n)!!} \cancel{n^2!} = \frac{p!}{(p-n^2)!}$$

$$\hookrightarrow = \frac{p!}{n^2! (p-n^2)!} n^2! = \frac{p}{(p-n^2)!}$$

But why?
Must be $n \times n$, right?

$p > n^2$ since need distinct elements

---

2. 30 books — How many ways can p books be sected so at least 2 unselected books

- was in textbook, right
- with 1_s to divide groups
- have $8 \to 1_s$ and $8 \to 0_s$

$$\binom{16}{8}$$

④

The first 7 1s ↓represent 3 books each — are a book picked and 2 unpicked book — last 1 is just a book you picked

8 0s are other books

$\binom{16}{8}$

b) How many nonneg int solutions?

$$X_1 + X_2 + \cdots X_m$$

How does this relate to previous ~~prefer~~ previous problem

So if had 5 w/ 3

$$1 + 2 + 2$$

↑don't have to be unique

Can be 0

$$0 + 0 + 5$$

Order matters — he asked

$$0 + 5 + 0$$

~~0+5~~

$$5 + 0 + 0$$

$$1 + 4 + 0$$

$$1 + 0 + 4$$

⑤ There are lot!

What is the pattern?

Allowing 0s and order matters makes this seem large

c) Is this one ∞ now?

— no its ≤ and we can't use ⊖

d) How many length -m weakly ↑ sea of ≤ th non neg ints are there

So this is previous ans, but limited to only weakly ↑

— So like half of before?

3. k-to-1 mapping for what k?

a) 4 seq of lenght 3

Each subseq has $3! = 6$ orders.

Each of 4 seq can be reordered ind.

6^4 to 1

~~there~~ But what abat

{A, k, c} {D, E, F} {G, H, I} {J, A, L}

This is disalloued in the instructions.
We've already divided into groups
__Then__ we are rearranging

b) Here assign students to group, but don't
set what order group should be in

$4!$

(7)

c) How many group assignments

 So is this the starting from scratch section?

Yes.

$$\frac{12!}{6! \, 4!}$$

d) How many ways can $3n$ students be broken up into n groups of 3?

$$(3n)!$$

6^n 1st map to some group seq

$n!$ Sequence per group assignment

$$\frac{(3n)!}{n! \, 6^n}$$

2 is most important
-b_{ij} in important
b,c were #3 from friday

4. Harvard student ad

$9!$

? also no toppings is possible

Topping 1   yes or no
        2   "
        ⋮
        9   "   "

Is actually ~ 2^9

Thats for one pizza
Now for 3 pizzas — just multiply by 3
Order does not matter
Can have 2 identical pizzas

$3 \cdot 2^9$

Well to think about

its not $2^9 + 2^9 + 2^9$

or

its $2^9 \cdot 2^9 \cdot 2^9$ so $\left(2^9\right)^3$

so + is then like if declined a pizza

$2^9 + \left(2^9\right)^2 + \left(2^9\right)^3$

Think about why he is wrong

Why does he $\overline{3!}$

I never really got this - what she said
Something about identical pizzas

$$\underline{\left(2^9\right)\left(2^9-1\right)\left(2^9-2\right)}$$

k - to 1 mapping

eliminate that many choices

if 2 are same, 1 different

Solutions to In-Class Problems Week 10, Mon.

Problem 1.
Answer the following quesions using the Generalized Product Rule.

(a) Next week, I'm going to get really fit! On day 1, I'll exercise for 5 minutes. On each subsequent day, I'll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 9, 11, 12. How many such sequences are possible?

Solution. The number of minutes on the first day can be selected in 1 way. The number of minutes on each subsequent day can be selected in 4 ways. Therefore, the number of exercise sequences is $1 \cdot 4^6$ by the generalized product rule. ∎

(b) An *r-permutation* of a set is a sequence of r distinct elements of that set. For example, here are all the 2-permutations of $\{a, b, c, d\}$:

$$
\begin{array}{llll}
(a,b) & (a,c) & (a,d) \\
(b,a) & (b,c) & (b,d) \\
(c,a) & (c,b) & (c,d) \\
(d,a) & (d,b) & (d,c)
\end{array}
$$

How many r-permutations of an n-element set are there? Express your answer using factorial notation.

Solution. There are n ways to choose the first element, $n-1$ ways to choose the second, $n-2$ ways to choose the third, $\ldots$, and $n-r+1$ ways to choose the r-th element. Thus, there are:

$$
n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!}
$$

r-permutations of an n-element set. ∎

(c) How many $n \times n$ matrices are there with *distinct* entries drawn from $\{1, \ldots, p\}$, where $p \geq n^2$?

Solution. There are p ways to choose the first entry, $p-1$ ways to choose the second for each way of choosing the first, $p-2$ ways of choosing the third, and so forth. In all there are

$$
p(p-1)(p-2)\cdots(p-n^2+1) = \frac{p!}{(p-n^2)!}
$$

such matrices. Alternatively, this is the number of n^2-permutations of a p element set, which is $p!/(p-n^2)!$. ∎

Problem 2. (a) There are 30 books arranged in a row on a shelf. In how many ways can eight of these books be selected so that there are at least two unselected books between any two selected books?

Solution. The answer is the number of length-16 bitstrings with eight 1's, namely

$$\binom{16}{8}.$$

As in Problem 15.3, there is a obvious bijection between length 30 bit-strings with eight 1's and selections of eight among 30 books on a shelf. And there is another obvious bijection between length 30 bit-strings with eight 1's that are at least two apart and length 16 bit-strings with eight 1's, namely, insert two 0's after the first seven 1's in a 16 bit-string with eight 1's, to obtain a 30 bit-string with 8 occurrences of 1's that are at least two apart. ∎

(b) How many nonnegative integer solutions are there for the following equality?

$$x_1 + x_2 + \cdots + x_m = k. \tag{1}$$

Solution. There are

$$\binom{m+k-1}{k}$$

nonnegative integer solutions to (1).

As in Problem 15.4, mapping $(x_1, x_2, \ldots, x_m) \in \mathbb{N}^m$ to $0^{x_1}10^{x_2}1\ldots0^{x_m}$ defines a bijection between solutions to (1) and length-$k + m - 1$ bit strings with k 0's. ∎

(c) How many nonnegative integer solutions are there for the following inequality?

$$x_1 + x_2 + \cdots + x_m \le k. \tag{2}$$

Solution. There are

$$\binom{m+k}{k}$$

nonnegative integer solutions to (2).

There is an obvious bijection between integer solutions to the equality

$$x_1 + x_2 + \cdots + x_m + x_{m+1} = k,$$

and solutions to (2), and the answer follows as in part (b). ∎

(d) How many length-m weakly increasing sequences of nonnegative integers $\le k$ are there?

Solution. There are

$$\binom{m+k}{k}$$

length-m weakly increasing sequences of nonnegative integers $\le k$.

As in Problem 15.4, there is a bijection between these weakly increasing sequences and solutions to (2), namely, map a solution $(x_1, x_2, \ldots, x_m)$ to a sequence $(x_1, \ x_1 + x_2, \ x_1 + x_2 + x_3, \ \ldots, \ \sum_{i=1}^{m} x_i)$. So the answer follows from part (b). ∎

Problem 3.
Your class tutorial has 12 students, who are supposed to break up into 4 groups of 3 students each. Your Teaching Assistant (TA) has observed that the students waste too much time trying to form balanced groups, so they decided to pre-assign students to groups and email the group assignments to his students.

(a) Your TA has a list of the 12 students in front of him, so they divides the list into consecutive groups of 3. For example, if the list is ABCDEFGHIJKL, the TA would define a sequence of four groups to be $(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\})$. This way of forming groups defines a mapping from a list of twelve students to a sequence of four groups. This is a k-to-1 mapping for what k?

Solution. Two lists map to the same sequence of groups iff the first 3 students are the same on both lists, and likewise for the second, third, and fourth consecutive sublists of 3 students. So for a given sequence of 4 groups, the number of lists which map to it is

$$(3!)^4$$

because there are 3! ways to order the students in each of the 4 consecutive sublists. ∎

(b) A group assignment specifies which students are in the same group, but not any order in which the groups should be listed. If we map a sequence of 4 groups,

$$(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}),$$

into a group assignment

$$\{\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}\},$$

this mapping is j-to-1 for what j?

Solution. 4!.

Each of the 4! sequences of a particular set of four groups maps to that set of groups. ∎

(c) How many group assignments are possible?

Solution.

$$\frac{12!}{4! \cdot (3!)^4} = 15400$$

different assignments.

There are 12! possible lists of students, and we can map each list to an assignment by first mapping the list to a sequence of four groups, and then mapping the sequence to the assignment. Since the first map is $(3!)^4$-to-1 and and the second is 4!-to-1, the composite map is $(3!)^4 \cdot 4!$-to-1. So by the Division Rule, $12! = ((3!)^4 \cdot 4!) A$ where A is the number of assignments. ∎

(d) In how many ways can $3n$ students be broken up into n groups of 3?

Solution.
$$\frac{(3n)!}{(3!)^n n!}.$$

This follows simply by replacing "12" by "$3n$" and "4" by "n" in the solution to the previous problem parts. ∎

Problem 4.

A pizza house is having a promotional sale. Their commercial reads:

> We offer 9 different toppings for your pizza! Buy 3 large pizzas at the regular price, and you can get each one with as many different toppings as you wish, absolutely free. That's $22,369,621$ different ways to choose your pizzas!

The ad writer was a former Harvard student who had evaluated the formula $(2^9)^3/3!$ on his calculator and gotten close to $22,369,621$. Unfortunately, $(2^9)^3/3!$ is obviously not an integer, so clearly something is wrong. What mistaken reasoning might have led the ad writer to this formula? Explain how to fix the mistake and get a correct formula.

Solution. The number of ways to choose toppings for one pizza is the number of the possible subsets of the set of 9 toppings, namely, 2^9. The ad writer presumably then used the Product Rule to conclude that there were $(2^9)^3$ *sequences* of three topping choices. Then he probably reasoned that each way of making three topping choices arises from 3! sequences, so the Division Rule would imply that the number of ways to choose three pizzas is $(2^9)^3/3!$.   *↳ why division rule ↴*

   It's true that every set of three *different* topping choices arises from 3! different length-3 sequences of choices. The mistake is that if some of the three choices are the same, then the set of three choices arises from *fewer* than 3! sequences. For example, if all three pizzas have the same toppings, there is only one sequence of topping choices for them.

   One fix is to consider ways to choose toppings with 1, 2 and 3 different topping choices. There are $2^9(2^9-1)(2^9-2)/3!$ ways to choose a set of 3 different choices, $2^9(2^9-1)$ ways to choose one topping choice to be used on two pizzas and a second choice for the third pizza, and 2^9 ways to choose one topping for all three pizzas, giving

↓ why not $\frac{1}{2!}$?   *assumes can't have 2 of topping*

$$\frac{2^9(2^9-1)(2^9-2)}{3!} + 2^9(2^9-1) + 2^9 = 22,500,864.$$

↑ can do it anyway

ways to choose three pizzas.

   Alternatively, we can observe that this is exactly the problem of selecting a dozen donuts of five possible different kinds—except now there are 3 donuts and 2^9 kinds. Hence, there is a bijection to the number of (2^9+2)-bit strings with exactly 2^9-1 ones and 3 zeros:

↗   *← isn't this flipped?*

$$\binom{2^9+2}{3} = 22,500,864.$$

diff way of choosing 2^9-1 *zeros*   3 *ones*

length of strings

∎