4/13

Mathematics for Computer Science
MIT 6.042J/18.062J

Bookkeeper Rule
Pigeonhole Principle

bookkeeper rule

\# permutations of the word **bookkeeper** ? $\dfrac{10!}{2!2!3!}$

- \# perms $\mathbf{bo_1o_2k_1k_2e_1e_2pe_3r} = 10!$
- map perm $\mathbf{o_1be_1o_2k_1rk_2e_2pe_3}$ to **obeokrkepe**
- 2 o's, 2 k's, 3 e's: map is $2!\cdot2!\cdot3!$-to-1

bookkeeper rule

\# permutations of length-n word with n_1 **a**'s, n_2 **b**'s, ..., n_k **z**'s:

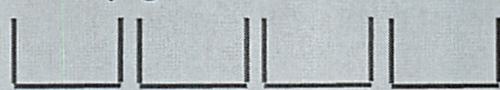
$$\binom{n}{n_1, n_2, \cdots, n_k} ::= \frac{n!}{n_1! n_2! \cdots n_k!}$$

multinomial coefficient

Pigeonhole Principle

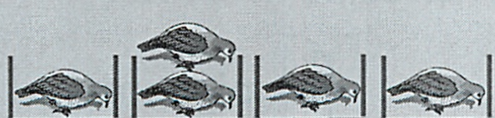
If more pigeons

than pigeonholes,



Pigeonhole Principle

then some hole must have $\geq$ two pigeons!

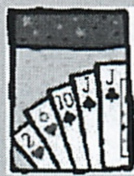


Pigeonhole Principle

Mapping Rule: [$\geq$1 out]/[$\leq$1 in] from A to B implies $|A| \leq |B|$.

If $|A| > |B|$, then no [$\geq$1 out]/[$\leq$1 in] from A to B.

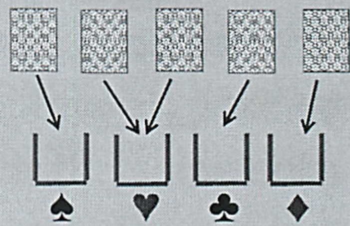
example: 5 Card Draw



set of 5 cards:
must have ≥ 2
with the same suit.

5 Card Draw

5 cards (pigeons)

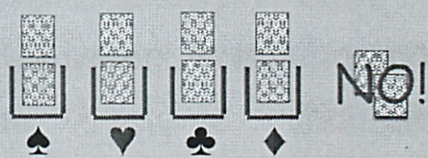


4 suits (holes)

10 Card Draw

10 cards: how many have the same suit?

10 Card Draw



<3 cards in every hole?

10 Card Draw

\# cards with same suit

$$\geq \left\lceil \frac{10}{4} \right\rceil = 3$$

"ceiling," means round up

Generalized Pigeonhole Principle

If n pigeons and h holes, then some hole has $\geq$

$$\left\lceil \frac{n}{h} \right\rceil \text{ pigeons.}$$

Team Problems

Problems

1—4

Bookkeeper Rule

Permutations of "b" "o" "o" etc okkeeper

If could tell difference b/w letters

$$10! = b\, o_1 o_2 k_1 k_2 e_1 e_2 p\, e_3 r$$

Now map to un-subscripted letters

Os can be in either order
ks " " "
e " " "

Can mix them and still same

So its a $2!\,2!\,3!$ to 1 mapping

So division rule

$$\frac{10!}{2!\,2!\,3!}$$

So basically n_1 a's, n_2 b's, ... n_{26} z's
of length n

$$\frac{n!}{n_1!\,n_2!\cdots n_{26}!}$$ = multinomial coefficent

②

Short hand is $\binom{n}{n_1, n_2, \ldots n_{26}}$

Called "counting permutations w/ indistinguisible elements"

<u>Pigonhole Principle</u>

If pigons > pidgon holes

Then some holes have ≥ 1 pidgon

Generalization of <u>mapping rule</u>

$\frac{\geq 1 \text{ out}}{\leq 1 \text{ in}}$ from A to B implies $|A| \leq |B|$

If $|A| > |B|$ then <u>no</u> $\frac{\geq 1 \text{ out}}{\leq 1 \text{ in}}$ from A to B

(3)

5-card-draw

If have 5 cards, at least 1 needs > 1 suit

Hint know exactly what you are doing

- what are pigons
- " " holes
- what is mapping rules ← difficult part

Put a pidgon in its suit hole

Must be 2 cards w/ same suit

10 card draw

How many have same suit?

2 cards - no! Could only fit 8 then

Some hole must have ≥ 3

Generally

$$\geq \left\lceil \frac{\overset{n \text{ card draw}}{\downarrow}\\10}{4} \right\rceil$$

= 3 here

(9)

Generalized Pigon hole

n pidgons

h holes

Some hole has

$\geq \left\lceil \frac{n}{h} \right\rceil$ pidgeons

In-Class Problems Week 10, Wed.

Problem 1.
The Tao of BOOKKEEPER: we seek enlightenment through contemplation of the word $BOOKKEEPER$.

(a) In how many ways can you arrange the letters in the word $POKE$?

(b) In how many ways can you arrange the letters in the word BO_1O_2K? Observe that we have subscripted the O's to make them distinct symbols.

(c) Suppose we map arrangements of the letters in BO_1O_2K to arrangements of the letters in $BOOK$ by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

O_2BO_1K	
KO_2BO_1	$BOOK$
O_1BO_2K	$OBOK$
KO_1BO_2	$KOBO$
BO_1O_2K	...
BO_2O_1K	
...	

(d) What kind of mapping is this, young grasshopper?

(e) In light of the Division Rule, how many arrangements are there of $BOOK$?

(f) Very good, young master! How many arrangements are there of the letters in $KE_1E_2PE_3R$?

(g) Suppose we map each arrangement of $KE_1E_2PE_3R$ to an arrangement of $KEEPER$ by erasing subscripts. List all the different arrangements of $KE_1E_2PE_3R$ that are mapped to $REPEEK$ in this way.

(h) What kind of mapping is this?

(i) So how many arrangements are there of the letters in $KEEPER$?

(j) *Now you are ready to face the BOOKKEEPER!*
How many arrangements of $BO_1O_2K_1K_2E_1E_2PE_3R$ are there?

(k) How many arrangements of $BOOK_1K_2E_1E_2PE_3R$ are there?

(l) How many arrangements of $BOOKKE_1E_2PE_3R$ are there?

(m) How many arrangements of $BOOKKEEPER$ are there?

Remember well what you have learned: subscripts on, subscripts off.
This is the Tao of Bookkeeper.

(n) How many arrangements of $VOODOODOLL$ are there?

(o) How many length 52 sequences of digits contain exactly 17 two's, 23 fives, and 12 nines?

Problem 2.
Solve the following problems using the pigeonhole principle. For each problem, try to identify the *pigeons*, the *pigeonholes*, and a *rule* assigning each pigeon to a pigeonhole.

(a) In a certain Institute of Technology, Every ID number starts with a 9. Suppose that each of the 75 students in a class sums the nine digits of their ID number. Explain why two people must arrive at the same sum.

(b) In every set of 100 integers, there exist two whose difference is a multiple of 37.

(c) For any five points inside a unit square (not on the boundary), there are two points at distance *less than* $1/\sqrt{2}$.

(d) Show that if $n+1$ numbers are selected from $\{1, 2, 3, \ldots, 2n\}$, two must be consecutive, that is, equal to k and $k+1$ for some k.

Problem 3.
Here are the solutions to the next 10 problem parts, in no particular order.

$$n^m \qquad m^n \qquad \frac{n!}{(n-m)!} \qquad \binom{n+m}{m} \qquad \binom{n-1+m}{m} \qquad \binom{n-1+m}{n} \qquad 2^{mn}$$

(a) How many solutions over the natural numbers are there to the inequality $x_1 + x_2 + \cdots + x_n \leq m$? ________

(b) How many length m words can be formed from an n-letter alphabet, if no letter is used more than once? ________

(c) How many length m words can be formed from an n-letter alphabet, if letters can be reused? ________

(d) How many binary relations are there from set A to set B when $|A| = m$ and $|B| = n$? ________

(e) How many injections are there from set A to set B, where $|A| = m$ and $|B| = n \geq m$? ________

(f) How many ways are there to place a total of m distinguishable balls into n distinguishable urns, with some urns possibly empty or with several balls? ________

(g) How many ways are there to place a total of m indistinguishable balls into n distinguishable urns, with some urns possibly empty or with several balls? ________

(h) How many ways are there to put a total of m distinguishable balls into n distinguishable urns with at most one ball in each urn? ________

Problem 4.
Solve the following counting problems. Define an appropriate mapping (bijective or k-to-1) between a set whose size you know and the set in question.

(a) An independent living group is hosting nine new candidates for membership. Each candidate must be assigned a task: 1 must wash pots, 2 must clean the kitchen, 3 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. Write a multinomial coefficient for the number of ways this can be done.

(b) How many nonnegative integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 17?

10 Wed InClass

4/13

1. ~~Tom~~ Tao

a) $4!$ - trival

b) $\frac{4!}{\cancel{2!}}$ Os are distinct - read!

c) Wow

$\frac{4!}{2!}$

d) Since 2 to 1

e) $\frac{4!}{2!}$ or $\frac{4!}{2} = \frac{4 \cdot 3 \cdot 2}{2} = 4 \cdot 3 = 12$

f) $6!$

g) 3 to 1

$3!$ to 1

6

h) So 6 to 1

i) $\frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 6 \cdot 5 \cdot 4$

or $5!$

I see a diff way of simplifying

ℓ)

j) $10!$

k) $\frac{10!}{2!}$

l) $\frac{10!}{2! \cdot 2!}$

m) $\frac{10!}{2! \cdot 2! \cdot 3!}$

n) 5 os

2 ds

2 ls

10 total

$$\frac{10!}{5! \cdot 2! \cdot 2!}$$

o) $\frac{52!}{17! \cdot 23! \cdot 12!}$

everything is either 2, 5, 9

③

2. Try to find pigon
pigonhole
rule

a) 9 digits of ID #
First digit = 9
Students sum digits
Why must two people arrive at same sum?

Σ all 0s = 9
beside the 9

Σ all nines = 81

So 81 − 9 = 72 possible sums
pigon holes

With 75 students
pigon

By pigon rule, students must have sum

(3)

b) In every set of 100 intigers - there exist two whose difference is multiple of 37.

take mod 37 of a # - means its the same if = / congruent

will get 0 → 36 places
↑ pidgon holes

100 pidgons

38 - 1 = 37

So 38, 1 are congruent (mod 37) = 1

So that is why diff = 37

c) Map pts in Quadrants

If > 1 in quad cant be $1/\sqrt{2}$ apart

(4)

d) Show that if $n+1$ are selected from $\{1, 2, 3, \ldots 2n\}$ two must be consecutive - that is = to k and $k+1$ for some k

Put int into even and odds ?

3. Sort soltions

a) How many solutions are there

$$x_1 + x_2 + \ldots + x_n \leq m$$

(Asymptotes stuff I don't get)

(5)

4. Counting problem

- mapping (bij or k to 1)

a) ILG 9 canidates

Oh 2 people not person #2

$1! \cdot 2! \cdot 3! \cdot 1! \cdot 2!$

$1 \cdot 2 \cdot 3 \cdot 2 \cdot 1 \cdot 2 = 24$

Not a mapping - need to map like w/ binary seq

~~[illegible]~~ Need mapping approach

$$\frac{9!}{1! \cdot 2! \cdot 3! \cdot 1! \cdot 2!}$$ ← remember this!

{(wash pots), (clean kitchens 1, clean kitchens 2), (bath 1, bath 2, bath 3) etc...

(6)

b) How many - int < 1,000,000 have exactly 1 digit = 9 and have sum = 17

- thought I remember doing this before....

5! combos

- 9 other digits

$6 \cdot \binom{12}{4}$

6 places to put 9

so 5 digits that sum to 8

8 0s and 4 dividers

7

3. (our board)

a) $\binom{n+m}{n}$

b) $\frac{n!}{(n-m)!}$

c) n^m

d) 2^{mn}

e) $\cancel{(m+1)^n}$ $\frac{n!}{(n-m)!}$

t is supposed to be total → ≥1 arrow coming at
injective ≥1 arrow going in

TA: Question is badly written

f) n^m

g) $\binom{n-1+m}{m}$

h) Assuming $n \geq m$, $\frac{n!}{(n-m)!}$, Else 0

Solutions to In-Class Problems Week 10, Wed.

Problem 1.
The Tao of BOOKKEEPER: we seek enlightenment through contemplation of the word $BOOKKEEPER$.

(a) In how many ways can you arrange the letters in the word $POKE$?

Solution. There are 4! arrangements corresponding to the 4! permutations of the set $\{P, O, K, E\}$. ■

(b) In how many ways can you arrange the letters in the word BO_1O_2K? Observe that we have subscripted the O's to make them distinct symbols.

Solution. There are 4! arrangements corresponding to the 4! permutations of the set $\{B, O_1, O_2, K\}$. ■

(c) Suppose we map arrangements of the letters in BO_1O_2K to arrangements of the letters in $BOOK$ by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

O_2BO_1K	
KO_2BO_1	$BOOK$
O_1BO_2K	$OBOK$
KO_1BO_2	$KOBO$
BO_1O_2K	...
BO_2O_1K	
...	

(d) What kind of mapping is this, young grasshopper?

Solution. 2-to-1 ■

(e) In light of the Division Rule, how many arrangements are there of $BOOK$?

Solution. $4!/2$ ■

(f) Very good, young master! How many arrangements are there of the letters in $KE_1E_2PE_3R$?

Solution. 6! ■

(g) Suppose we map each arrangement of $KE_1E_2PE_3R$ to an arrangement of $KEEPER$ by erasing subscripts. List all the different arrangements of $KE_1E_2PE_3R$ that are mapped to $REPEEK$ in this way.

Solution. $RE_1PE_2E_3K$, $RE_1PE_3E_2K$, $RE_2PE_1E_3K$, $RE_2PE_3E_1K$, $RE_3PE_1E_2K$, $RE_3PE_2E_1K$ ■

(h) What kind of mapping is this?

Solution. 3!-to-1 ■

(i) So how many arrangements are there of the letters in $KEEPER$?

Solution. 6!/3! ■

(j) *Now you are ready to face the BOOKKEEPER!*

How many arrangements of $BO_1O_2K_1K_2E_1E_2PE_3R$ are there?

Solution. 10! ■

(k) How many arrangements of $BOOK_1K_2E_1E_2PE_3R$ are there?

Solution. 10!/2! ■

(l) How many arrangements of $BOOKKE_1E_2PE_3R$ are there?

Solution. $10!/(2! \cdot 2!)$ ■

(m) How many arrangements of $BOOKKEEPER$ are there?

Solution.

$$\binom{10}{1,2,2,3,1,1} ::= \frac{10!}{1!\,2!\,2!\,3!\,1!\,1!} = \frac{10!}{(2!)^2\,3!}$$

■

Remember well what you have learned: subscripts on, subscripts off.
This is the Tao of Bookkeeper.

(n) How many arrangements of $VOODOODOLL$ are there?

Solution.

$$\binom{10}{1,2,5,2} ::= \frac{10!}{1!\,2!\,5!\,2!}$$

■

(o) How many length 52 sequences of digits contain exactly 17 two's, 23 fives, and 12 nines?

Solution.

$$\binom{52}{17,23,12} ::= \frac{52!}{17!\,23!\,12!}$$

■

Problem 2.
Solve the following problems using the pigeonhole principle. For each problem, try to identify the *pigeons*, the *pigeonholes*, and a *rule* assigning each pigeon to a pigeonhole.

(a) In a certain Institute of Technology, Every ID number starts with a 9. Suppose that each of the 75 students in a class sums the nine digits of their ID number. Explain why two people must arrive at the same sum.

Solution. The students are the pigeons, the possible sums are the pigeonholes, and we map each student to the sum of the digits in his or her MIT ID number. Every sum is in the range from $9 + 8 \cdot 0 = 9$ to $9 + 8 \cdot 9 = 81$, which means that there are 73 pigeonholes. Since there are more pigeons than pigeonholes, there must be two pigeons in the same pigeonhole; in other words, there must be two students with the same sum. ■

(b) In every set of 100 integers, there exist two whose difference is a multiple of 37.

Solution. The pigeons are the 100 integers. The pigeonholes are the numbers 0 to 36. Map integer k to $\text{rem}(k, 37)$. Since there are 100 pigeons and only 37 pigeonholes, two pigeons must go in the same pigeonhole. This means $\text{rem}(k_1, 37) = \text{rem}(k_2, 37,)$, which implies that $k_1 - k_2$ is a multiple of 37. ■

(c) For any five points inside a unit square (not on the boundary), there are two points at distance *less than* $1/\sqrt{2}$.

Solution. The pigeons are the points. The pigeonholes are the four subsquares of the unit square, each of side length 1/2.

Pigeons are assigned to the subsquare that contains them, except that if the pigeon is on a boundary, it gets assigned to the leftmost and then lowest possible subsquare that includes it (so the point at $(1/2, 1/2)$ is assigned to the lower left subsquare).

There are five pigeons and four pigeonholes, so more than one point must be in the same subsquare. The diagonal of a subsquare is $1/\sqrt{2}$, so two pigeons in the same hole are at most this distance. But pigeons must be inside the unit square, so two pigeons cannot be at the opposite ends of the same subsquare diagonal. So at least one of them must be inside the subsquare, so their distance is less than the length of the diagonal. ■

(d) Show that if $n + 1$ numbers are selected from $\{1, 2, 3, \ldots, 2n\}$, two must be consecutive, that is, equal to k and $k + 1$ for some k.

Solution. The pigeonholes will be the n sets $\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{2n - 1, 2n\}$. The pigeons will be the $n + 1$ selected numbers. A pigeon is assigned to the unique pigeon hole of which it is a member. By the Pigeonhole Principle, two pigeons must assigned to some hole, and these are the two consecutive numbers required. Notice that we've actually shown a bit more: there will be two consecutive numbers with the smaller being odd. ■

Problem 3.
Here are the solutions to the next 10 problem parts, in no particular order.

$$n^m \qquad m^n \qquad \frac{n!}{(n-m)!} \qquad \binom{n+m}{m} \qquad \binom{n-1+m}{m} \qquad \binom{n-1+m}{n} \qquad 2^{mn}$$

(a) How many solutions over the natural numbers are there to the inequality $x_1 + x_2 + \cdots + x_n \le m$?

Solution.

$$\binom{n+m}{m}$$

This is the same as the number of solutions to the equation the equality $x_1 + x_2 + \cdots + x_n + y = m$, and which has a bijection to sequences with m stars and n bars. ■

(previous problem!

(b) How many length m words can be formed from an n-letter alphabet, if no letter is used more than once?

Solution. (in sep alphas ! ← but why division rule

$$\frac{n!}{(n-m)!}$$

There are n choices for the first letter, $n-1$ choices for the second letter, $\ldots n-m+1$ choices for the mth letter, so by the Generalized Product rule, the number of words is

$$n \cdot (n-1) \cdots (n-m+1).$$

Pay attention to this one ■

(c) How many length m words can be formed from an n-letter alphabet, if letters can be reused?

Solution. n^m by the Product Rule. ■

p-set

(d) How many binary relations are there from set A to set B when $|A| = m$ and $|B| = n$?

Solution.

$$2^{mn}$$

The graph of a binary relations from A to B is a subset of $A \times B$. There are on 2^{mn} such subsets because $|A \times B| = mn$. ■

(e) How many injections are there from set A to set B, where $|A| = m$ and $|B| = n \geq m$?

Solution.

injective

≤1 arrow in

$$\frac{n!}{(n-m)!}$$

There is a bijection between the injections and the length m sequences of distinct elements of B. By the Generalized Product rule, the number of such sequences is

$$n \cdot (n-1) \cdots (n-m+1).$$

■

I should have done in class

(f) How many ways are there to place a total of m distinguishable balls into n distinguishable urns, with some urns possibly empty or with several balls?

Solution.

$$n^m$$

There is a bijection between a placement of the balls and length m sequence whose ith element is the urn where the ith ball is placed. So the number of placements is the same as the number of length m sequences of elements from a size-n set. ■

(g) How many ways are there to place a total of m indistinguishable balls into n distinguishable urns, with some urns possibly empty or with several balls?

flipped?

Solution.

$$\binom{n-1+m}{m}$$

This is the same as the number of selections of m donuts with n possible flavors, which is the number of sequences with m stars and $n-1$ bars. ■

(h) How many ways are there to put a total of m distinguishable balls into n distinguishable urns with at most one ball in each urn?

Solution.

Slow down + think about

$$\frac{n!}{(n-m)!}$$

There is a bijection between a placement of balls and a length m sequence whose ith element is the urn containing the ith ball. So the number of ball placements is the same as number of length m sequences of distinct elements from a set of n elements. ■

Problem 4.

Solve the following counting problems. Define an appropriate mapping (bijective or k-to-1) between a set whose size you know and the set in question.

(a) An independent living group is hosting nine new candidates for membership. Each candidate must be assigned a task: 1 must wash pots, 2 must clean the kitchen, 3 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. Write a multinomial coefficient for the number of ways this can be done.

Solution. There is a bijection from sequences containing one P, two K's, three B's, a C, and two D's. In any such sequence, the letter in the ith position specifies the task assigned to the ith candidate. Therefore, the number of possible assignments is:

$$\binom{9}{1,2,3,1,2} ::= \frac{9!}{1!\,2!\,3!\,1!\,2!}$$

■

(b) How many nonnegative integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 17?

Solution. We identify the nonnegative integers less than 1,000,000 with the length 6 strings of decimal digits. Then there is a bijection with pairs:

(position of the 9, successive values of other 5 digits)

The sum of the other 5 digits is equal to 8, so the number of ways to choose their values is equal to the number of solutions over the nonnegative integers to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8, \tag{1}$$

namely, $\binom{12}{4}$. So by the product rule there are

$$6 \cdot \binom{12}{4}$$

such integers. ■

6.042 Grade Report for *Plasmeier, Michael*

Problem Sets

id ▲	adjusted score	raw score	max	statistics
PS.01	**35.15**	28.00	50.00	link
PS.02	**35.98**	33.00	50.00	link
PS.03	**22.00**	18.50	40.00	link
PS.04	**25.35**	24.00	30.00	link
PS.05	**33.96**	32.20	40.00	link
PS.06	**38.74**	33.00	50.00	link

Note: The psets' adjusted scores reflect the psets scores after being adjusted by its corresponding MQ's score. The adjusted scores will be further increased according to final exam's performance.

Mini Quizzes

id ▲	pts	max	statistics
MQ.01	**13.00**	20.00	link
MQ.02	**7.00**	20.00	link
MQ.03	**9.00**	20.00	link
MQ.04	**13.50**	20.00	link

Reading Assignments

No grades available yet.

Tutor Problems

id ▲	pts	max
T.01	**1.00**	1.00
T.02	**1.00**	1.00
T.03	**1.00**	1.00
T.04	**1.00**	1.00
T.05	**1.00**	1.00
T.06	**1.00**	1.00
T.07	**1.00**	1.00
T.08	**1.00**	1.00

Final Exam

No grades available yet.

Class Participation

id ▲	pts	max	pending makeup
CP.01	**2.00**	2.00	
CP.02	**2.00**	2.00	
CP.03	**2.00**	2.00	
CP.04	**2.00**	2.00	
CP.05	**2.00**	2.00	
CP.06	**2.00**	2.00	
CP.07	**2.00**	2.00	
CP.08	**2.00**	2.00	
CP.09	**2.00**	2.00	
CP.10	**1.00**	2.00	
CP.11	**2.00**	2.00	
CP.12	**2.00**	2.00	
CP.13	**1.00**	2.00	
CP.14	**1.00**	2.00	
CP.15	**2.00**	2.00	
CP.16	**1.00**	2.00	
CP.17	**2.00**	2.00	
CP.18	**2.00**	2.00	
CP.19	**2.00**	2.00	
CP.20	**2.00**	2.00	
CP.21	**2.00**	2.00	
CP.22	**1.00**	2.00	
CP.23	**1.00**	2.00	
CP.24	**2.00**	2.00	
CP.25	**2.00**	2.00	

Totals

id ▲	pts	max	weight	mean	median	stddev
Problem Set	194.56	250.00	0.25	215.64	227.12	34.04
Final Exam	0.00	0.00	0.30	0.00	0.00	0.00
Class participation	36.00	38.00	0.20	36.82	38.00	3.47
Miniquiz	35.50	60.00	0.17	44.07	45.00	9.21
Reading Comments	0.00	0.00	0.03	0.00	0.00	0.00
Tutorial	8.00	8.00	0.05	7.40	8.00	1.28
Grand Total	**53.46**	**67.00**	**1.00**	**58.05**	**59.86**	**6.81**

Note: The totals only reflect grades that have been completely entered for the class. A grade with gray background signifies that the grade has not been completely entered yet.

Note: A grade with red font signifies that the grade has been dropped.

Grade Quartile

Your current rank is: ***4th quartile (79th - 101th)*** out of 101 students.

Grades compiled at: 4/13/11 7:10 PM

Please contact your TA if there is any problem with the grade report.

Problem Set 8

Due: April 15

Reading: Chapter 15–15.9, Counting Rules

Problem 1.
Let X and Y be finite sets.

(a) How many binary relations from X to Y are there?

(b) Define a bijection between the set $[X \to Y]$ of all total functions from X to Y and the set $Y^{|X|}$. (Recall Y^n is the cartesian product of Y with itself n times.) Based on that, what is $|\,[X \to Y]\,|$?

(c) Using the previous part how many *functions*, not necessarily total, are there from X to Y? How does the fraction of functions vs. total functions grow as the size of X grows? Is it $O(1)$, $O(|X|)$, $O(2^{|X|})$,...?

(d) Show a bijection between the powerset, $\mathcal{P}(X)$, and the set $[X \to \{0,1\}]$ of 0-1-valued total functions on X.

(e) Let $X ::= \{1, 2, \ldots, n\}$. In this problem we count how many bijections there are from X to itself. Consider the set $B_{X,X}$ of all *bijections* from set X to set X. Show a bijection from $B_{X,X}$ to the set of all permuations of X (as defined in the notes). Using that, count $B_{X,X}$.

Problem 2.
In this problem, all graphs will have vertices $[1, n] ::= \{1, 2, \ldots, n\}$; equivalently, all binary relations are on this set $[1, n]$.

(a) How many simple undirected graphs are there?

(b) How many digraphs are there?

(c) How many asymmetric binary relations are there?

(d) How many path-total strict partial orders are there?

Problem 3.
There is a robot that steps between integer positions in 3-dimensional space. Each step of the robot increments one coordinate and leaves the other two unchanged.

(a) How many paths can the robot follow going from the origin $(0, 0, 0)$ to $(3, 4, 5)$?

(b) How many paths can the robot follow going from the origin (i, j, k) to (m, n, p)?

Problem 4.
Suppose you have seven dice — each a different color of the rainbow; otherwise the dice are standard, with

faces numbered 1 to 6. A *roll* is a sequence specifying a value for each die in rainbow (ROYGBIV) order. For example, one roll is $(3, 1, 6, 1, 4, 5, 2)$ indicating that the red die showed a 3, the orange die showed 1, the yellow 6,....

For the problems below, describe a bijection between the specified set of rolls and another set that is easily counted using the Product, Generalized Product, and similar rules. Then write a simple numerical expression for the size of the set of rolls. You do not need to prove that the correspondence between sets you describe is a bijection, and you do not need to simplify the expression you come up with.

For example, let A be the set of rolls where 4 dice come up showing the same number, and the other 3 dice also come up the same, but with a different number. Let R be the set of seven rainbow colors and $S ::= [1, 6]$ be the set of dice values.

pick 2 #

Define $B ::= P_{S,2} \times R_3$, where $P_{S,2}$ is the set of 2-permutations of S and R_3 is the set of size-3 subsets of R. Then define a bijection from A to B by mapping a roll in A to the sequence in B whose first element is an ordered pair consisting of the number that came up three times followed by the number that came up four times, and whose second element is the set of colors of the three matching dice.

For example, the roll

$$(4, 4, 2, 2, 2, 4, 2, 4) \in A$$

Others fall into place

R= colors
S= numbers

maps to

$$((2, 4), \{\text{yellow,green,indigo}\}) \in B.$$

Now by the Bijection rule $|A| = |B|$, and by the Generalized Product and Subset rules,

$$|B| = 6 \cdot 5 \cdot \binom{7}{3}.$$

(a) For how many rolls do *exactly* two dice have the value 6 and the remaining five dice all have different values?

Example: $(6, 2, 6, 1, 3, 4, 5)$ is a roll of this type, but $(1, 1, 2, 6, 3, 4, 5)$ and $(6, 6, 1, 2, 4, 3, 4)$ are not.

(b) For how many rolls do two dice have the same value and the remaining five dice all have different values?

Example: $(4, 2, 4, 1, 3, 6, 5)$ is a roll of this type, but $(1, 1, 2, 6, 1, 4, 5)$ and $(6, 6, 1, 2, 4, 3, 4)$ are not.

(c) For how many rolls do two dice have one value, two different dice have a second value, and the remaining three dice a third value?

Example: $(6, 1, 2, 1, 2, 6, 6)$ is a roll of this type, but $(4, 4, 4, 4, 1, 3, 5)$ and $(5, 5, 5, 6, 6, 1, 2)$ are not.

Problem 5.

Answer the following questions with a number or a simple formula involving factorials and binomial coefficients. Briefly explain your answers.

(a) How many ways are there to order the 26 letters of the alphabet so that no two of the vowels `a`, `e`, `i`, `o`, `u` appear consecutively and the last letter in the ordering is not a vowel?

Hint: Every vowel appears to the left of a consonant.

(b) How many ways are there to order the 26 letters of the alphabet so that there are *at least two* consonants immediately following each vowel?

(c) In how many different ways can $2n$ students be paired up?

(d) Two n-digit sequences of digits 0,1,...,9 are said to be of the *same type* if the digits of one are a permutation of the digits of the other. For $n = 8$, for example, the sequences `03088929` and `00238899` are the same type. How many types of n-digit integers are there?

Doing P-set 8

4/13

Counting -
- should be middle details

1. ~~a~~ - abstract qu

binary relation
- oh just relation between two sets aRb

If $|X| = |Y|$ then 1

If $|X| = |Y| - 1$

$a \to a$
$b \to b$
c

Then could circle through - but how?

~~A~~ Permutations - but only first part

But then also the other side, so

$$\binom{3}{2} = \frac{3!}{2! \cdot 1!}$$

If $|X| < |Y|$

$a \to a$
$b \to b$
c

Same

But could also not have -
?? could you
ignore

2)

I might be all turned around - oh well

b) Bij $[X \to Y]$ of all total fns $X \to Y$

and set $Y^{|X|}$

↑ cartesian product

↑ product of set

What is $|[X \to Y]|$

like card x suit

$\{(A, \text{Spade}) (K, \text{Spade}) \ldots$ etc

total - every el domain is fn ≥ 1 arrow at

Why/ how Bij ~~~ from X to Y and $Y^{|X|}$

a a
b b
c c

aaa
aab
aac
aba
abb
abc
aca
~~~
etc

↑ 27 items

" Just 3 since # out
~~~

(3)

But what if y is just a

a a

b aaa

c

Nope - 2

$y = ab$

a a

b b

c ab

ab

ba

bb

$\cancel{\max}\ \min\left(|X|, |Y| + |Y|^{|X|}\right)$

? I don't really get this

c) How many functions are there from $X \to Y$,

How does this grow as X grows?

- I think I hve wrong assumptions here...

3 6

After OH

a) -completly mis intepeted

-never looked at #3 on wed

-Should have done.

b) Ok got hint on this

-Understand, but don't know what to do

So 4 total and 4

aa
ab
ba
bb

9

?, C same

? same as a $2^{(3 \cdot 3)} = 2^6 = 64$

-no

-but how visualize before ?

-to each one or same side ?? -no

It is $|X \cdot Y|$ - but why 2^

Oh possible relations $a \to a$, b b, c c is possible

(4)

So - minimum of X of them

$X + X^2$

but could be < 2 of them

But how to get to $Y^{|X|}$?

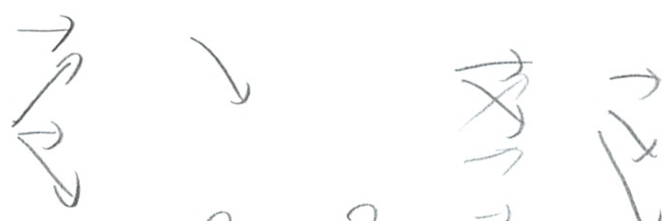
How about

should be $3^2 = 9$

6 arrows - but mixes of

6 arrows in d

remember must be total



3 3 3 3 3 = 16

but I got 16

Grrrr.... I have the $Y^{|X|}$ ans - but can't see why

(5)

c) Functions ≤ 1 arrow out

at most 1 arrow out

And they are not providing on ans

d) Power set

2^n sets

What are 0-1-valued total functions?

And need the total formula again

e) X to X

- how many permutations

- Straightforward - should get!

- So ≤ 1 arrow ≥ 1 arrow

- So permutations $X!$

What is the best way to write?

Think I got - seems too simple!

1a Well we want a bij from X to Y.
Sets - order does not matter.
So only depends on relative size.

If $|X| = |Y|$

then just 1 relationship between them
(assuming every one that can, will have a relation)

If $|X| > |Y|$
for example

$a \to a$
$b \to b$
c

Then we would randomly choose which elements could have a mapping. We would randomly choose $|Y|$ elements from X - leading to $\binom{n}{k}$ possible subsets via the subset rule

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

where $n = |X|$
$k = |Y|$

In this example $\binom{3}{2} = 3$ possible combos

②

<u>If $|X| < |Y|$</u>

example

a	a
b	b
	c

This is similar, except now we have to choose which of the subsets are included. We can also use the subset rule $\binom{n}{k}$ where $k = |X|$ and $n = |Y|$

3

b) So we can only define a bijection based on the minimum number of items on either side

On the left we have X and on the right we have Y and $Y^{|X|}$

$\cup$

So $|[X \to Y]| = \min(|X|, |Y| + |Y^{|X|}|)$

2. Have graph w/ verticles $[1, n] = \{1, 2, \ldots, n\}$
So on set $[1, n]$

a) How many simple undirected graphs are there?
Do we know an n?
Or are they asking generically?
based on n
Will be own graph
I think I am not getting this either!

b) How many diagraps are there
No edges - So no digraphs
That can't be right...

c) Skip rest for now

After OH

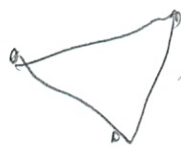
- totally misunderstand
- Possible ways to arrange edges so have that Condition w/ n-vertices

(2)

a) Why are there $\binom{n}{2}$ ways?

- First define: no loops
  no more than 1 edge b/w any 2 vertices

So $\binom{n}{2}$ for 3 is 3



but loop

$\binom{4}{2} = 6$

So just # of lines b/w each pt

Forget thing about no loops

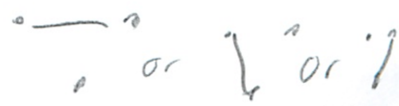
But where does the 2 come from?

Subset rule

- take first 2 elements
- why???????
- Since 2 edges b/w — no
- first 2 pts not special

Guess just pattern

Or wait it can also be individual pts



(3)

b) Dixgraphs
- directed

Actually that might be it
Then no loops
MCS says nothing about loops

? So here its same but dabled?
w/o fully understanding a - hard to get b

$\binom{n}{4}$?

But for 4 this = 1 - clearly wrong

$2\binom{n}{2}$

c) asym binary relations
$aRb \to$ NOT (bRa)
So only One way
Again can't build w/o understanding a

d) Path-total = total (email)
SPO = trasitive + asym
?
aRb and $bRc \to aRc$

I am so bad at these

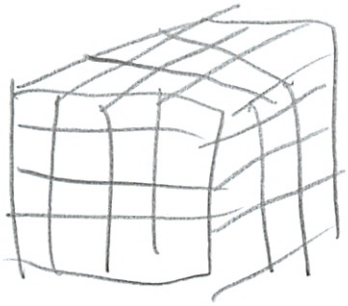
Michael Plasmeier
Oshani
Table 12
Pset #8

2.a. Each graph would have n vertexes. So it depends on what n is and how many "instances" of a graph you want - You will have 1 per instance.

2

b. No edges have been defined, so there are no digraphs.

#3 Robot in 3D space
- like 6.01!



a) How many paths from (0,0,0) to (3,4,5)

Can go ↑3 → 4 · 5

or any order

Or ↑1 → 1 ↑1 → 1 etc

So which rule is this

Like a chess game!

- Permutation

So treat each up down individually

- but over what?

Can also ∞ since can go out of the way

3 north 4 east 5 out in any order

$3 \cdot 4 \cdot 5$

(2)

b) easy to generalize

$$|m-i| \cdot |n-j| \cdot |p-k|$$

#4. 7 dice - each color of rainbow 1-6
roll is value of each die in ROYGBIV order

Define bij and write ~~exact~~ size
no proving ↑ no generalizing

Define $B = P_{5,2} \times R_3$ ← 3 subset of R (colors)
↑ 2 permutations of 5 (values)

Then $A \to B$ by mapping a roll in A to seq in B whose 1st el is an ordered pair consisting of # that come up 3 times, followed by # 4 times

So (4,4,2,2,4,2,4)

maps to $((2,4)\ \{\text{yellow, green, indigo}\}) \in B$
↑ 2 perm of 5 ↑ what are these??
colors of 3 matching die

- oh so don't need color of other 4 - since follows
- it should be locked in, =1
- otherwise made a mistake

(2) So lets get started

a) For how many rolls do 2 dice have 6 and others have diff values

So ; chance 2 are same · diff · diff ;;

(This is straightforward compared to others!)

$\frac{1}{6} \cdot \frac{1}{6}$

? no how many rolls

$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{5} \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1}$

So multinomial coefficent:

$\Big(\frac{7!}{6 \cdot 6}$

How many rolls of 2 die where both same

1st	2nd
can be anything	must be ~~same thi~~ same

$6 \cdot 1$

but then division

$\frac{6^2}{6} = 6$ rolls

Oh right - don't have to do division

(3)

So $\dfrac{6!}{6 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$ ← do need divison rule? Is it a k-to 1?

all k's (denom) are same - no

So no divison rule - I am over using
- should not be using

~~So~~

b) So 2 have same value

Opps I did b before
Fixed that fast

c) 2 dice have 1 value 2 have sencond 3rd
This is bookeeper since all defined

$\binom{7}{2,2,3}$

Once I got it, it went fast

#5 Last qv!

Ans w/ binomials + factorials

a) How many ways to order 26 letters so no 2 vowels appear consec, last letter not a vowel?
Hint every vowel appears to left of consonant

Vowl Const

So last condition auto satisfied
But does cond only hold in real words?
Do we only care about real words?
And how long can it be?

∞ many

Since no two vowels will appear consecutively if always to left of consonant

Or they want us to only use each letter once
So 26! - permutations
Minus condition with division rule
Can't have ae ea
ai ei
ao eo
au eu
? can repeat

②

but ae sea can appear anywhere

well 25 possibilities

b) How many ways so 2 consonts after every vowel

So now in addition to everything before

a _ _

do <u>not</u> again

So can't be a vowel here

So

a _ _

5 4 3 in any of the 24 positions

And also can't have vowel in 2nd to last (5)

This is that both vowels

Hope I got everything

c) This is completly different!

$2n \cdot (2n-1)$

Order does not matter

Or is it $\binom{2n}{2}$

(3)

But do you then have to say
# of possible combos of that

$\binom{2n}{2}!$??

b/c $\binom{2n}{2}$ is one partnership.

$\binom{2n}{2}\binom{2n-2}{2}$

How to expand to long form?
- Just leave
- prob better way

---

d) Totally diff again

2-n digit seq are "Same type" if
Permutation. How many "types" are there.
So the 2 digits must be together
(These qu are fun - but hard!)

I totally mis understand
n = # of digits = lenght
How many permutations with that lenght
So this is straightforward permutations

(4)

So n element = n!

But every digit not there once

So every possible arrangement of 9 digits

9^n

? 10 w/ 0

Oshani OH 1

4/14

1a. ~~[illegible]~~ $|X \times Y|$

- thats how many relations

b) I was right $Y^{|X|}$

Mapping X to Y

Y^X X

$X \xrightarrow{total} Y$ big Y^X

should be same size

total <1 arrows out

Find that $[X \to Y]$

- size
- define mapping so sizes are the same

c) function - check

5.2b.

2. Defined binary relation

# of relations

Wed's In-Class

Can define edge

- up to you

- how many connections possible

Vertex either has edge or not

So $2n^2$

Counting same edge twice?

$\binom{n}{2}$ undirected graphs

# of edges

There are $\binom{n}{2}$ ways to arrange edges so have un direct graphs.

Student's Solutions to Problem Set 8

Your name: Michael Plasmeier

Due date: April 15

Submission date: 4/15

Circle your TA/LA: Ali Nick Oscar Oshani **Table number** 12

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1] Oshani OH

   and referred to:[2] Wikipedia: Cartesian product

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	5
2	
3	
4	
5	
Total	

/10

[1]People other than course staff.

[2]Give citations to texts and material other than the Spring '11 course materials.

5

1a. From In Class Week 10 Wed 3d.

$2^{|X||Y|}$

Since $|X \times Y| = |X||Y|$

Since line from every point in one to every point in other (would be "complete" if a graph)

b) Got a hint on this from OH. This was written poorly. We want a bij between

left side: the total relations $X \to Y$

right side $Y^{|X|}$

Basically we are trying to show that there are $|Y|^{|X|}$ number of total relations from X to Y

total ≥ 1 arrow out.

This is a relation between every element in X to at least 1 element each in Y.

so show one..?

e) $Bij_{X,X}$

So this is simply what are all the possible permutations of X, since we must have 1 arrow out and 1 arrow in to every point. So this is the same as reordering the sequence in every permutation - Since every item in the set must be used once.

Permutations of a sequence are simply $|X|!$

This is the number of bij from X to X $= B_{X,X}$

ok

Michael Plasmeier

Oshani

Table 12

P-Set 8

1

#2 This question was also poorly written. It is asking how many possible ways are there to arange the edges between the n vertices so the condition is met.

a. Simple undirected graphs

$\binom{n}{2}$

see solutions

b) Now arrows can go both directions

$2\binom{n}{2}$

see solutions
self-loops?

~~c)~~ ?

~~d)~~ ?

Michael Plasmeier
Oshani
Table 12
P-Set 8

#3. Well the robot can really go ∞ possible paths Since the robot could go to (1000,0,0), etc before going back - nothing says the path must be efficent (ie no overshooting or heading in the wrong direction). There are ∞ly many possible "detours".

5

But if you assume it must go efficently:

It must go 3 north, 4 east, and 5 in, in any order. This is the product rule. We don't care about where north_1 is vs north_3 - so we don't care about permutations. But we do care

$\uparrow \rightarrow$ vs $\rightarrow \uparrow$

$\uparrow_1 \rightarrow_1$
vs
$\uparrow_3 \rightarrow_1$
means nothing

So $3 \cdot 4 \cdot 5 = 60$

X

(2)...

b) Generalize.

Again this is technically ∞, but if you care for an "efficent" answer (no overshoots and no moving in wrong direction) you get

$$|m-i| \cdot |n-j| \cdot |p-k|$$

abs value of difference

We can test a)

$= |3-0| \cdot |4-0| \cdot |5-0|$

$= 3 \cdot 4 \cdot 5$

$= 60$ as before ✓ ✗

Michael Plasmeier
Oshani
Table 12
P-Set 8

1

#4a. $1 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \binom{7}{2}$ −2

must be a 6
must be one of 1-5
must be something different

b) $6 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \binom{7}{2}$ −2

can be any of the digits
must be same
must be other from 1st one
must be other from 2nd

c) This is book keeper rule

$$\binom{7}{2,2,3} \cdot \binom{6}{2} 4$$ −2

bijections?? −3

Michaël Plasmeier
Oshani
Table 12
P-Set 8

5

#5 Since every vowel appears to the left of a consonant this satisfys both conditions; that the word not end in a vowel and that 2 vowels do not appear consecuitily. (well really we are intersted in other way around)

First there are 26! permutations of the seq of 26 letters (each letter only once)

But then we have to remove the following possibilities

two vowels in a row

ae ea ; o u
ai ei ... etc
ao eo
au eu

↑ can appear anywhere though
So each has 25 possible locations
(can start in pos 26)

last char only

a
e
i
o
u

↑ 5 permutations no longer possible

So $5 \cdot 4 \cdot 25$ + 5 must be removed
– via divison rule

(2)

So $\frac{26!}{5\cdot 4\cdot 25+5}$ X

b) So now in addition to everything that was not possible before, some additional things are not possible

if its a vowel
this can't be a vowel
___ ___ ___ or this

in any position (except last 2)

5 possible vowels in 1st position

So if 2nd and 3rd blanks were both vowels

$5\cdot 4\cdot 3$ actually we already have this covered from two vowels next to each other

If only 2nd was vowel
- we already covered

If only 3rd was vowel

$5\cdot \underline{21}\cdot 4$ 4 remaining vowels

any of the 21 consonants

So also need to worry about end

Vowel ___ |end| is legal

Consonant

but we only care about stuff that is illegal

So don't enumerate

(3) ...

So disallowed

$4 \cdot 5 \cdot 25 \quad + \quad 5 \quad + \quad \cancel{5 \cdot 4 \cdot 3 \cdot 24} \quad + \quad 5 \cdot 21 \cdot 4 \cdot 24$

can't be vowel after vowel

no vowels in last pos

~~no 2 vowels after vowel~~

Or wait do we already have that covered from 2 vowels next to each other

- So don't need

Vowel · consant vowel is illegal in 24 places

So

$$\frac{26!}{4 \cdot 5 \cdot 25 + 5 + 5 \cdot 21 \cdot 4 \cdot 24}$$

X

(4)

c) This asks how many ways can 2n students be paired up. This is the same as asking the number of 2-element subsets in an 2n-element set or $\binom{2n}{2}$.

But this is # of possible ways to do 1 partnership. We want to know the possibilities for multiple partnerships

So

$$\underbrace{\binom{2n}{2}}_{\text{1st partnership}} \cdot \underbrace{\binom{2n-2}{2}}_{\text{2nd partneship}} \cdot \binom{2n-4}{2} \overset{n \text{ times}}{\ldots} \text{ etc}$$

$$= \prod_{i=0}^{n} \binom{2n-i}{2}$$

That is $\displaystyle\prod_{i=0}^{n} \frac{(2n-i)!}{2!(2n-i-2)!}$

(5) ...

d) First - once you have an n-digit sequence you will have n! permutations of that n-lenght sequence by the permutation rule. However you can have many possible n-lenght sequences (assuming can have leading 0s - like the example) Since you can have digits more than once you have $10 \cdot 10 \cdot 10 \cdot \overset{n \text{ times}}{\ldots}$ or 10^n possible sequences, so

$$10^n \cdot n!$$ X

types of n-digit integers exist.

Solutions to Problem Set 8

Reading: Chapter ??–??, Counting Rules

Problem 1.
Let X and Y be finite sets.

(a) How many binary relations from X to Y are there?

Solution. The set of all pairs $X \times Y$ has $|X| \cdot |Y|$ elements. Any subset of $X \times Y$ can be the graph of a relation, hence there are $2^{|X| \cdot |Y|}$ relations. ■

(b) Define a bijection between the set $[X \to Y]$ of all total functions from X to Y and the set $Y^{|X|}$. (Recall Y^n is the cartesian product of Y with itself n times.) Based on that, what is $|\,[X \to Y]\,|$?

Solution. We can encode a given function from X to Y by first giving an ordering to elements in X, say, calling them $x_1, x_2, \cdots, x_{|X|}$.

Now given an element $f \in [X \to Y]$ we can associate it with and element $g \in Y^{|X|}$ by following the rule $g[i] = f(x_i)$, where $g[i]$ is the ith entry of the vector.

This is a total, bijective function, since it is defined for every $f \in [X \to Y]$. It is also surjective and injective, as we show next.

To prove it is surjective, suppose $(y_1, y_2, y_3, \ldots, y_{|X|}) \in Y^{|X|}$. Now, the function $h \in X$ with $h(x_i) ::= y_i$ will map to it under our definition. To prove it is injective, suppose $g, h \in X$ map to the same vector $(y_1, y_2, y_3, \ldots, y_{|X|}) \in Y^{|X|}$. Then based on our rule we know $g(x_i) = y_i = h(x_i)$ for all $x_i \in X$. Hence $g = h$.

Based on this bijection we can easily count the number of total functions $[X \to Y]$ by counting the elements of $Y^{|X|}$. Since we know how to count cartesian products, we know the answer is $|Y|^{|X|}$. In fact, in many book, the set of all total functions from a set X to a set Y is often denoted as Y^X. ■

(c) Using the previous part how many *functions*, not necessarily total, are there from X to Y? How does the fraction of functions vs. total functions grow as the size of X grows? Is it $O(1)$, $O(|X|)$, $O(2^{|X|})$,...?

Solution. We can model this by adding a dummy element to Y, which indicates whether a given $x \in X$ has an actual image or not. After using the previous part, we get there are $(|Y|+1)^{|X|}$ functions, not necessarily total. By taking the ratio of this answer and the previous questions, we see the ratio is $\left(\frac{|Y|+1}{|Y|}\right)^{|X|}$ so it is not $O(1)$ nor $O(|X|)$ but exponential in $|X|$. Also, since $|Y| + 1 \leq 2|Y|$, then the ratio above is indeed $O(2^{|X|})$ ■

(d) Show a bijection between the powerset, $\mathcal{P}(X)$, and the set $[X \to \{0, 1\}]$ of 0-1-valued total functions on X.

Solution. Consider bijection $b : \mathcal{P}(X) \to [X \to \{0,1\}]$ defined as follows. For $s \in \mathcal{P}(X)$, then let $b_s(x_i) ::= 1$ iff $x_i \in S$. We make $b_s(x_i) ::= 0$ otherwise. It can be shown this correspondence is a bijection. Firstly, to show it is injective, we can consider two different elements in $\mathcal{P}(X)$, call them s_1 and s_2. According to the definition, these two are distinct sets with all their elements in X. Therefore we can assume without losing generality there is an $x_0 \in s_1$ but $x_0 \notin s_2$. So according to our mapping $b_{s_1}(x_0) = 1$ but $b_{s_2}(x_0) = 0$, so the two functions are not equal. Now we need to show it is surjective, and we know this is the case because given any such binary function, we can construct a subset of X that maps to it. Namely, $\{x \in X \mid f(x) = 1\}$.

This and the previous part show why $\mathcal{P}(x)$ is sometimes denoted as 2^X. ■

(e) Let $X ::= \{1, 2, \ldots, n\}$. In this problem we count how many bijections there are from X to itself. Consider the set $B_{X,X}$ of all *bijections* from set X to set X. Show a bijection from $B_{X,X}$ to the set of all permuations of X (as defined in the notes). Using that, count $B_{X,X}$.

Solution. The main idea is we can encode a bijective function from X to X with an ordered list. For example, if we let $X = \{1, 2, 3\}$, the function $f : X \to X$ with $f(1) ::= 3$, $f(2) ::= 1$ and $f(3) ::= 2$ is bijective. We can encode it as (3, 1, 2). In this case $f(i) = v_i$.

This is a valid bijection, because if we have an arbitrary bijective function we can always write down its images in order, and two different bijections will have a different image. Also, given an ordered list, we can reconstruct a bijection which when encoded produces the list.

■

Problem 2.
In this problem, all graphs will have vertices $[1, n] ::= \{1, 2, \ldots, n\}$; equivalently, all binary relations are on this set $[1, n]$.

(a) How many simple undirected graphs are there?

Solution. There are $\binom{n}{2}$ potential edges, each of which may or may not appear in a given graph. Therefore, the number of graphs is:

$$2^{\binom{n}{2}}$$

■

(b) How many digraphs are there?

Solution. There are n^2 potential edges, each of which may or may not appear in a given graph. Therefore, the number of graphs is:

$$2^{n^2}$$

■

(c) How many asymmetric binary relations are there?

Solution. There are no self-loops in an asymmetric relation and for each of the $\binom{n}{2}$ pairs of distinct elements a and b, either

1. $a\ R\ b$, or
2. $b\ R\ a$, or

3. neither,

but not both. Therefore, the number of asymmetric binary relations is

$$3^{\binom{n}{2}}.$$

■

(d) How many path-total strict partial orders are there?

Solution. $n!$.

Since the partial order is path-total, there is a unique listing of the elements in decreasing partial order. This listing defines a bijection between the path-total strict partial orders and the permutations of $[1, n]$.

■

Problem 3.
There is a robot that steps between integer positions in 3-dimensional space. Each step of the robot increments one coordinate and leaves the other two unchanged.

(a) How many paths can the robot follow going from the origin $(0, 0, 0)$ to $(3, 4, 5)$?

Solution.

$$\binom{12}{3, 4, 5}$$

■

(b) How many paths can the robot follow going from the origin (i, j, k) to (m, n, p)?

Solution.

$$\binom{m+n+p-(i+j+k)}{m-i, n-j, p-k}$$

■

Problem 4.
Suppose you have seven dice — each a different color of the rainbow; otherwise the dice are standard, with faces numbered 1 to 6. A *roll* is a sequence specifying a value for each die in rainbow (ROYGBIV) order. For example, one roll is $(3, 1, 6, 1, 4, 5, 2)$ indicating that the red die showed a 3, the orange die showed 1, the yellow 6,....

For the problems below, describe a bijection between the specified set of rolls and another set that is easily counted using the Product, Generalized Product, and similar rules. Then write a simple numerical expression for the size of the set of rolls. You do not need to prove that the correspondence between sets you describe is a bijection, and you do not need to simplify the expression you come up with.

For example, let A be the set of rolls where 4 dice come up showing the same number, and the other 3 dice also come up the same, but with a different number. Let R be the set of seven rainbow colors and $S ::= [1, 6]$ be the set of dice values.

Define $B ::= P_{S,2} \times R_3$, where $P_{S,2}$ is the set of 2-permutations of S and R_3 is the set of size-3 subsets of R. Then define a bijection from A to B by mapping a roll in A to the sequence in B whose first element is an ordered pair consisting of the number that came up three times followed by the number that came up four times, and whose second element is the set of colors of the three matching dice.

For example, the roll

$$(4, 4, 2, 2, 4, 2, 4) \in A$$

maps to

$$((2, 4), \{\text{yellow,green,indigo}\}) \in B.$$

Now by the Bijection rule $|A| = |B|$, and by the Generalized Product and Subset rules,

$$|B| = 6 \cdot 5 \cdot \binom{7}{3}.$$

(a) For how many rolls do *exactly* two dice have the value 6 and the remaining five dice all have different values?

Example: $(6, 2, 6, 1, 3, 4, 5)$ is a roll of this type, but $(1, 1, 2, 6, 3, 4, 5)$ and $(6, 6, 1, 2, 4, 3, 4)$ are not.

Solution. As in the example, map a roll into an element of $B ::= R_2 \times P_5$ where P_5 is the set of permutations of $\{1, \ldots, 5\}$. A roll maps to the pair whose first element is the set of colors of the two dice with value 6, and whose second element is the sequence of values of the remaining dice (in rainbow order). So $(6, 2, 6, 1, 3, 4, 5)$ above maps to $(\{\text{red,yellow}\}, (2, 1, 3, 4, 5))$. By the Product rule,

$$|B| = \binom{7}{2} \cdot 5!.$$

■

(b) For how many rolls do two dice have the same value and the remaining five dice all have different values?

Example: $(4, 2, 4, 1, 3, 6, 5)$ is a roll of this type, but $(1, 1, 2, 6, 1, 4, 5)$ and $(6, 6, 1, 2, 4, 3, 4)$ are not.

Solution. Map a roll into a triple whose first element is in S, indicating the value of the pair of matching dice, whose second element is the set of colors of the two matching dice, and whose third element is the sequence of the remaining five dice values (in rainbow order).

So $(4, 2, 4, 1, 3, 6, 5)$ above maps to $(4, \{\text{red,yellow}\}, (2, 1, 3, 6, 5))$. Notice that the number of choices for the third element of a triple is the number of permutations of the remaining five values, namely 5!. This mapping is a bijection, so the number of such rolls equals the number of such triples. By the Generalized Product rule, the number of such triples is

$$6 \cdot \binom{7}{2} \cdot 5!.$$

Alternatively, we can define a map from rolls in this part to the rolls in part (a), by replacing the value of the duplicated values with 6's and replacing any 6 in the remaining values by the value of the duplicated pair. So the roll $(4, 2, 4, 1, 3, 6, 5)$ would map to the roll $(6, 2, 6, 1, 3, 4, 5)$. Now a type a roll, r, is mapped to by exactly the rolls obtainable from r by exchanging occurrences of 6's and i's, for $i = 1, \ldots, 6$. So this map is 6-to-1, and by the Division rule, the number of rolls here is 6 times the number of rolls in part (a).

■

(c) For how many rolls do two dice have one value, two different dice have a second value, and the remaining three dice a third value?

Example: $(6, 1, 2, 1, 2, 6, 6)$ is a roll of this type, but $(4, 4, 4, 4, 1, 3, 5)$ and $(5, 5, 5, 6, 6, 1, 2)$ are not.

Solution. Map a roll of this kind into a 4-tuple whose first element is the set of two numbers of the two pairs of matching dice, whose second element is the set of two colors of the pair of matching dice with the smaller number, whose third element is the set of two colors of the larger of the matching pairs, and whose fourth element is the value of the remaining three dice. For example, the roll $(6, 1, 2, 1, 2, 6, 6)$ maps to the triple

$$(\{1, 2\}, \{\text{orange,green}\}, \{\text{yellow,blue}\}, 6).$$

There are $\binom{6}{2}$ possible first elements of a triple, $\binom{7}{2}$ second elements, $\binom{5}{2}$ third elements since the second set of two colors must be different from the first two, and 4 ways to choose the value of the three dice since their value must differ from the values of the two pairs. So by the Generalized Product rule, there are

$$\binom{6}{2} \cdot \binom{7}{2} \cdot \binom{5}{2} \cdot 4$$

possible rolls of this kind. ∎

Problem 5.
Answer the following questions with a number or a simple formula involving factorials and binomial coefficients. Briefly explain your answers.

(a) How many ways are there to order the 26 letters of the alphabet so that no two of the vowels `a`, `e`, `i`, `o`, `u` appear consecutively and the last letter in the ordering is not a vowel?

Hint: Every vowel appears to the left of a consonant.

Solution. The constraint on where vowels can appear is equivalent to the requirement that every vowel appears to the left of a consonant. So given a sequence of the 21 consonants, there are $\binom{21}{5}$ positions where the 5 vowels can be placed. After determining such a placement, we can reorder the consonants and vowels in any order. Thus, the number is:

$$\binom{21}{5} \cdot 21! \cdot 5!.$$

∎

(b) How many ways are there to order the 26 letters of the alphabet so that there are *at least two* consonants immediately following each vowel?

Solution. The pattern of consonants and vowels in any permutation of the 26 letters of the alphabet can be indicated by a binary string with 5 ones indicating where the vowels occur and 21 zeros where the consonants occur. Patterns where every vowel has at least two consonants to its right can be constructed by taking a sequence of 16 zeros and inserting "10" to the left of 5 of the 16 zeros. There are $\binom{16}{5}$ ways to do this. For any such pattern, there are 5! ways to place the vowels in the positions where ones occur and 21! ways to place the consonants where the ones occur. Thus, the final answer is:

$$\binom{16}{5} \cdot 5! \cdot 21!.$$

∎

(c) In how many different ways can $2n$ students be paired up?

Solution. Pair up students by the following procedure. Line up the students and pair the first and second, the third and fourth, the fifth and sixth, etc. The students can be lined up in $(2n)!$ ways. However, this overcounts by a factor of 2^n, because we would get the same pairing if the first and second students were swapped, the third and fourth were swapped, etc. Furthermore, we are still overcounting by a factor of $n!$, because we would get the same pairing even if pairs of students were permuted, e.g. the first and second were swapped with the ninth and tenth. Therefore, the number of pairings is:

$$\frac{(2n)!}{2^n \cdot n!}$$

■

(d) Two n-digit sequences of digits 0,1,...,9 are said to be of the *same type* if the digits of one are a permutation of the digits of the other. For $n = 8$, for example, the sequences `03088929` and `00238899` are the same type. How many types of n-digit integers are there?

Solution. The type of a string is determined by the numbers of occurrences of the 9 different digits in the string. So there is a bijection between types of strings and strings with n 0's and nine 1's: the length of the block of 0's before the ith 1 equals the number of occurrences of the digit i (and the length of the final block of 0's equals the number of occurrences of the digit 9). Therefore, the number of different types is $\binom{n+9}{9}$ ■

TP9 cont

TP 9.2

How many total fns $A \to B$ if $|A| = 3$
$|B| = 7$

This is something I did not really get on P-set

Did I figure this out on P-set?

total 1 arrow out

So min A, A to everything

$|A|^{|B|}$

or was it $|B|^{|A|}$?

7^3 ✓

~~9.4~~

Say $A = \{a_1, a_2, a_3\}$ Bij from total fn from A to B and lenght 3 vectors of els of B - namley a total fun f corresponds under the bij to vector $f(a_1), f(a_2), f(a_3)$ in B^3

If I would not have put tutor off - would hae seen,
But I don't get it at all - Why?

(2)

TP 9.4

Permutations of Missisippi ?

This is Bookkeeper rule $\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!}$

multinomial coefficent

TP 9.5 Canting Poker Hands

Indicate how many 5 card hands of following kinds

$C(n,k) = \binom{n}{k}$

$P(n,k) = \frac{n!}{(n-k)!}$ Notation

1. Sequence of 5 consec cards of any suit

5-6-7-8-9

Ace can be high or low

A-2-3-4-5 or 10-J-Q-K-A

but not around the coner Q K A 2 3

So $(\underset{\text{first place}}{10} \cdot 4)$ any suite $\cdot (\underset{\text{given \#}}{1} \cdot 4)$ any suite $\cdot (1 \cdot 4)(1 \cdot 4) \cdot (1 \cdot 4)$

A 2 3 ... 9 10 last possible

since ~~a~~ in any place

(3)

Only some possible ans

$6 \cdot 4^5$ ← what I had ✓

$\cancel{4!2}\ 4 \cdot 5 \cdot 6$

$4 \cdot 10^5$

$13 \cdot 4^5$

2. Matching suite - hands that are same suite in any order

$\cancel{4!}(4 \cdot 13) \cdot (1 \cdot 12) \cdot (1 \cdot 11) \cdot \cancel{444}(1 \cdot 10) \cdot (1 \cdot 9)$

any suite any #

1 suite

one less

(this method found on Pset 8 #3 works well

or $4 \cdot \frac{13!}{8!}$ ← since $\frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot \cancel{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}$

$4 \cdot P(13, 5)$ X

$4\binom{13}{5}$

Why is it this again? $\binom{13}{5} = \frac{13!}{5!\,(13-5)!}$ ← the 5 can also be reordered

~~8~~ 4

3. Straight Flush Seq and matching suit

5 cards in row

So must be 5 cards in row

$10 \cdot 1 \cdot 1 \cdot 1 \cdot 1$

Oh still 40 since can be any of the 4 suits

4. Straight Seq but not matching suite

$(4 \cdot 10) \cdot (3 \cdot 1) \cdot (4 \cdot 1) \cdot (4 \cdot 1) \cdot (4 \cdot 1)$

diff suit

now back to anything
- since now not all same

7680

That should be it!

The 3 can be anywhere - but that should be considered

5. Flush Matching suit but not seq

~~4240~~ $4 \cdot 13 \cdot 1 \cdot 11 \cdot 10 \cdot 9 \cdot 8$

anything

One less AND Not seq

$= 411840$

- but what about items that could never be seq!

(5)

Oh give up

4. 10 200

Oh ~~they do~~ via subtraction

- prob smarter

$10240 - 40 = 10200$

5. 5108

$5148 - 40 = 5108$

Should have just done the subtraction

TP 9.6 Magic Card Trick

Trying to communicate int b/w 1, n to partner by holding up 3 cards from 52 cards

Identify all expressions for largest n for which sure of this

- haven't read section
- skipped class

6

#2, 7

$\frac{52!}{49!}$ ~~[illegible]~~ $\binom{52}{3} \cdot 3!$

Are $52 \cdot 51 \cdot 50$ possible 3 card seq.

The n confused me - It is not asking how many possible seq from holding up 3 cards

- Not like in class magic tricks

4/16

TP. 9.7 Inclusion - Exclusion

$|A_1| = 100$

$|A_2| = 1000$

$|A_3| = 1000$

$|A_1 \cup A_2 \cup A_3|$

~~[illegible]~~ type in sizes |

a) What do they mean straight add?

oh ~~[illegible]~~ symbol missing

~~$A_1 \cap A_2 \cap A_3$~~

$A_1 \subseteq A_2 \subseteq A_3$

1

So $A_1 \subseteq A_2$

is a subset or equal

So $100 \subseteq 1000 \subseteq 10,000$

So is it 100?

or 100 + 1000 + 10000?

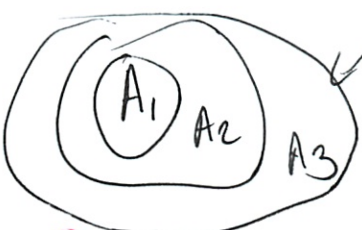
or 100 + 100 + 100

nd

What is this question asking

100 + 1000

Oh its saying



So size of largest circle

10000

(I was thinking what is this section about)

2. Pairwise disjoint

just add 11100

(8) 3. For any 2 sets, 1 el in both

- So 3 overlap

11100 - 3 = 11097

Or then is el in 3?
depends → variable
So nd (0)

(Think about all the possibilities!)

4. 2 els in common to each pair, one in all 3
? is it not det
- no here they clearly defined it I think
  - don't just G+V

11100 - 2 - 2 - 2 + 1

TP 9.8

6.042 table 8 students
- So same as knights were arrangement is order →
- NOT where each person sits

a) $(n-1)!$ (from book)

$7!$ ✓

9

b) How many arrangements w/ A next to B

~~Can be~~ $\div$ divison rule $\rightarrow$ furter restrictions

So

$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ is base

$\uparrow$ 1 less

No - don't care about this - divison rule

But A next to B means

$7 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

if A

$\frac{7!}{6}$ ⊗ Not on list

$= 840$

I think I am blending 2 of each method

$2 \cdot 6! = 1440$

$\uparrow$ why??

I figured out later AB is a unit or BA

So 6 remaining

(10)

Look at what is aignally written

$$\frac{|A|}{n} = \frac{n!}{n}$$

↑ thats why ~~ia~~ since 4 diff arrangements map to same thing "4 to 1"

So here - thats same
But extra restrictions!

c) How man if ~~B~~ next to A AND C

So now

A $\underline{B}$ C or C $\underline{B}$ A as a group so 3, 5 others

So $2 \cdot 5!$ ✓

← oh that must be how b was got

d) B next to ~~AB~~ next to A or C

either or
$2 \cdot 6! + 2 \cdot 6!$
$2 \cdot 2 \cdot 6!$
$4 \cdot 6! = 2880$
not a choice

$22 \cdot 5! = 2640$

~~Bo~~ So was BA or AB as unit ✓

11

inclusion-exclusion principle

$$2 \cdot 6! + 2 \cdot 6! \underbrace{- 2 \cdot 5!}_{\text{Oh forgot this!}}$$

(darn that was point of this section)

TP. 9.9 Pigeonhole Principle

Give min # of people that must be in a group so it holds the property

1. a. At least 2 people born same day of year

~~366~~ 365+1 366

b) at least 2 people Jan first

365·2

736

No could have 3 in April 14

So np

(Thought it sounded fishy — follow through on hunches!)

(12)

3.) At least 3 people born same day of week

nh ✗

^{r}Oh any day of week

15 ✓

4) At least 4 people born same month

$12+12+12+1=37$ ✓

5) At least 2 people born 1 week apart

nh

no guarentee on specific day ✓ woot!

MCS 15.10 - on

- can't write on pages, so notes here
- 15.10 Inclusion - Exclusion
- This is union of sets

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$$

get rid of 1 copy of overlap

More complex when 3

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 + S_3|$$

add back the triple overlap

So more complex get full inclusion - exclusion rule

(not writing)

Can use to calc Euler's function $\phi(n)$

rel prime

$$S = \bigcup_{i=1}^{m} C_{p_i}$$

Complex ...

4/15

Skipped Fri's class

go though afterwards

Mathematics for Computer Science
MIT 6.042J/18.062J

Magic Trick
~~Inclusion-exclusion~~

A Magic Trick

audience chooses 5 cards

Assistant reveals 4 of them

Magician announces 5^{th} card!

A Magic Trick

Let's do it!

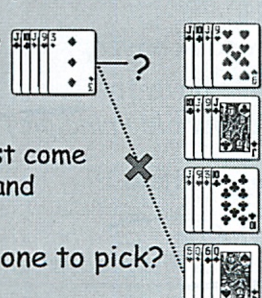
Assistant's Choices

Decide the order of the 4 cards:
 4! = 24 orderings
 -- but 48 cards remain
Decide which 4 cards to list

Map hands to 4-Card lists

5-card hands *(no order)*

4-card lists (ordered)



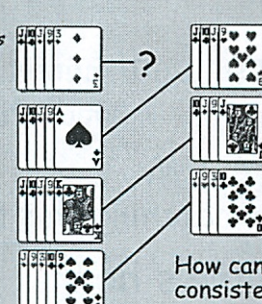
list must come from hand

Which one to pick?

Map hands to 4-Card lists

5-card hands *(no order)*

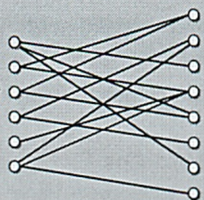
4-card lists (ordered)



How can we ensure consistency?

Map hands to 4-Card lists

5-card hands (no order)

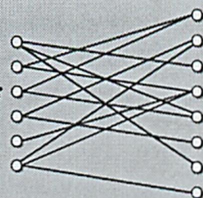


4-card lists (ordered)

Every hand must have an identifying list!

perfect matching of the hands ...is what we need

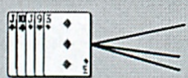
5-card hands (no order)



4-card lists (ordered)

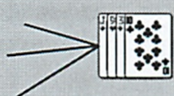
Every hand must have an identifying list!

Match hands with 4-Card lists



$\deg = 5 \cdot 4 \cdot 3 \cdot 2 = 120$

$\deg = 52 - 4 = 48$



Match hands with 4-Card lists

So graph is degree-constrained and hence has a matching that A & M can use

A Memorable Matching?

$\binom{52}{5} = 2{,}598{,}960$ hands to match

How will A & M learn any matching this big?

Here's how:

Not in book

Magic Trick Revealed (I)

Among 5 cards chosen:

at least 2 have the same suit (*Pigeonhole Principle*)

A lists one of them 1st

> *Aha! The first card has the same suit as the hidden card!*

Clever

Magic Trick Revealed (II)

How does M figure out the rank of the hidden card?

Aha! Look at the order of the other 3 cards!

Magic Trick Revealed (II)

Fix ordering of the deck

A♣ < A♦ < A♥ < A♠ <

2♣ < 2♦ < 2♥ < 2♠ <

⋮

K♣ < K♦ < K♥ < K♠

the whole deck?

Magic Trick Revealed (II)

Possible orders for the remaining 3 cards:

{ SML, SLM, MSL, MLS, LSM, LMS }

small medium large?

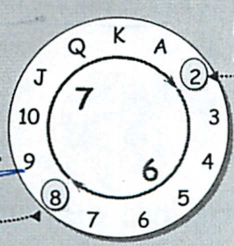
Magic Trick Revealed (II)

***Wait!* Only have 6 sequences of the remaining 3 cards, but 12 possible hidden cards of the known suit!**

Of two cards with the same suit, choosing which to reveal can give 1 more bit of information! Aha!

Clockwise Distance

The *smaller* clockwise distance between 2 card ranks is at most 6:



Reveal the other card

Hide card with smaller offset.

Magic Trick Revealed (Finally)

- The first card determines the hidden suit (♠ ♥ ♦ ♣).
- Hidden rank (A … K) = first-card rank + offset ($\leq$ 6).
- Offset given by order of remaining 3 cards:

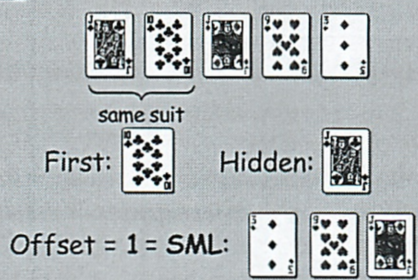
  SML = 1, SLM = 2, MSL = 3, MLS = 4, LSM = 5, LMS = 6.

the order of the 3

clever - who figured this all out?

how to make sure have all "bits" of info?

Example



won't work with 4-card hands

audience can pick	A can reveal
$\binom{52}{4} = 270{,}725$	$\frac{52!}{49!} = 132{,}600$
possible 4-card hands	possible 3-card lists

won't work with 4 card hands

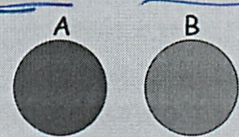
so at least

$$\left\lceil \frac{270{,}725}{132{,}600} \right\rceil = 3$$

hands map to the same list
– M can't tell which!

Sum Rule

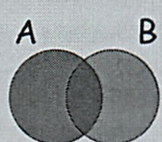
$$|A \cup B| = |A| + |B|$$



for disjoint sets A, B

Sum Rule

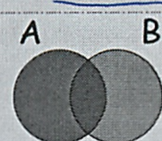
$$|A \cup B| = ?$$



What if not disjoint?

Inclusion-Exclusion

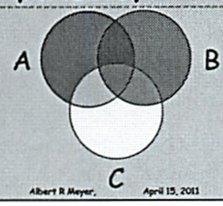
$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$$



What if not disjoint?

Inclusion-Exclusion (3 Sets)

$|A \cup B \cup C| =$
$|A| + |B| + |C|$
$- |A \cap B| - |A \cap C| - |B \cap C|$
$+ |A \cap B \cap C|$



Team Problems

Problems
1—3

gets complicated ? he does not show generalize

Massachusetts Institute of Technology
6.042J/18.062J, Spring '11: Mathematics for Computer Science — April 15
Prof. Albert R Meyer — revised Tuesday 12th April, 2011, 20:45

In-Class Problems Week 10, Fri.

Problem 1.
Section 15.13.3 explained why it is not possible to perform a four-card variant of the hidden-card magic trick with one card hidden. But the Magician and her Assistant are determined to find a way to make a trick like this work. They decide to change the rules slightly: instead of the Assistant lining up the three unhidden cards for the Magician to see, he will line up all four cards with one card face down and the other three visible. We'll call this the *face-down four-card trick.*

For example, suppose the audience members had selected the cards $9\heartsuit$, $10\diamondsuit$, $A\clubsuit$, $5\clubsuit$. Then the Assistant could choose to arrange the 4 cards in any order so long as one is face down and the others are visible. Two possibilities are:

$A\clubsuit$	?	$10\diamondsuit$	$5\clubsuit$
?	$5\clubsuit$	$9\heartsuit$	$10\diamondsuit$

(a) Explain why there must be a bipartite matching which will in theory allow the Magician and Assistant to perform the face-down four-card trick.

(b) There is actually a simple way to perform the face-down four-card trick.[1]

Case 1. *there are two cards with the same suit*: Say there are two $\spadesuit$ cards. The Assistant proceeds as in the original card trick: he puts one of the $\spadesuit$ cards *face up as the first card.* He will place the second $\spadesuit$ card *face down.* He then uses a permutation of the face down card and the remaining two face up cards to code the offset of the face down card from the first card.

Case 2. *all four cards have different suits*: Assign numbers $0, 1, 2, 3$ to the four suits in some agreed upon way. The Assistant computes, s, the sum modulo 4 of the ranks of the four cards, and chooses the card with suit s to be placed *face down as the first card.* He then uses a permutation of the remaining three face-up cards to code the rank of the face down card.

Explain how in Case 2. the Magician can determine the face down card from the cards the Assistant shows her.

(c) Explain how any method for performing the face-down four-card trick can be adapted to perform the regular (5-card hand, show 4 cards) with a 52-card deck consisting of the usual 52 cards along with a 53rd card call the *joker.*

Problem 2.
A certain company wants to have security for their computer systems. So they have given everyone a name and password. A length 10 word containing each of the characters:

[1]This elegant method was devised in Fall '09 by student Katie E Everett.

a, d, e, f, i, l, o, p, r, s,

is called a *cword.* A password will be a cword which does not contain any of the subwords "fails", "failed", or "drop".

For example, the following two words are passwords:

adefiloprs, srpolifeda,

but the following three cwords are not:

adropeflis, failedrops, dropefails.

(a) How many cwords contain the subword "drop"?

(b) How many cwords contain both "drop" and "fails"?

(c) Use the Inclusion-Exclusion Principle to find a simple formula for the number of passwords.

tricky, but normal

Problem 3.

We want to count step-by-step paths between points in the plane with integer coordinates. Ony two kinds of step are allowed: a right-step which increments the x coordinate, and an up-step which increments the y coordinate.

(a) How many paths are there from $(0,0)$ to $(20,30)$?

(b) How many paths are there from $(0,0)$ to $(20,30)$ that go through the point $(10,10)$?

(c) How many paths are there from $(0,0)$ to $(20,30)$ that do *not* go through either of the points $(10,10)$ and $(15,20)$?

Hint: Let P be the set of paths from $(0,0)$ to $(20,30)$, N_1 be the paths in P that go through $(10,10)$ and N_2 be the paths in P that go through $(15,20)$.

Solutions to In-Class Problems Week 10, Fri.

Problem 1.
Solve the following problems using the pigeonhole principle. For each problem, try to identify the *pigeons*, the *pigeonholes*, and a *rule* assigning each pigeon to a pigeonhole.

(a) In a certain Institute of Technolofy, Every ID number starts with a 9. Suppose that each of the 75 students in a class sums the nine digits of their ID number. Explain why two people must arrive at the same sum.

Solution. The students are the pigeons, the possible sums are the pigeonholes, and we map each student to the sum of the digits in his or her MIT ID number. Every sum is in the range from $9 + 8 \cdot 0 = 9$ to $9 + 8 \cdot 9 = 81$, which means that there are 73 pigeonholes. Since there are more pigeons than pigeonholes, there must be two pigeons in the same pigeonhole; in other words, there must be two students with the same sum. ■

(b) In every set of 100 integers, there exist two whose difference is a multiple of 37.

Solution. The pigeons are the 100 integers. The pigeonholes are the numbers 0 to 36. Map integer k to $\text{rem}(k, 37)$. Since there are 100 pigeons and only 37 pigeonholes, two pigeons must go in the same pigeonhole. This means $\text{rem}(k_1, 37) = \text{rem}(k_2, 37,)$, which implies that $k_1 - k_2$ is a multiple of 37. ■

(c) For any five points inside a unit square (not on the boundary), there are two points at distance *less than* $1/\sqrt{2}$.

Solution. The pigeons are the points. The pigeonholes are the four subsquares of the unit square, each of side length 1/2.

Pigeons are assigned to the subsquare that contains them, except that if the pigeon is on a boundary, it gets assigned to the leftmost and then lowest possible subsquare that includes it (so the point at $(1/2, 1/2)$ is assigned to the lower left subsquare).

There are five pigeons and four pigeonholes, so more than one point must be in the same subsquare. The diagonal of a subsquare is $1/\sqrt{2}$, so two pigeons in the same hole are at most this distance. But pigeons must be inside the unit square, so two pigeons cannot be at the opposite ends of the same subsquare diagonal. So at least one of them must be inside the subsquare, so their distance is less than the length of the diagonal. ■

(d) Show that if $n + 1$ numbers are selected from $\{1, 2, 3, \dots, 2n\}$, two must be consecutive, that is, equal to k and $k + 1$ for some k.

Solution. The pigeonholes will be the n sets $\{1, 2\}, \{3, 4\}, \{5, 6\}, \dots, \{2n - 1, 2n\}$. The pigeons will be the $n + 1$ selected numbers. A pigeon is assigned to the unique pigeon hole of which it is a member. By the Pigeonhole Principle, two pigeons must assigned to some hole, and these are the two consecutive numbers required. Notice that we've actually shown a bit more: there will be two consecutive numbers with the smaller being odd. ■

Problem 2.
A certain company wants to have security for their computer systems. So they have given everyone a name and password. A length 10 word containing each of the characters:

a, d, e, f, i, l, o, p, r, s,

is called a *cword*. A password will be a cword which does not contain any of the subwords "fails", "failed", or "drop".

For example, the following two words are passwords:

adefiloprs, srpolifeda,

but the following three cwords are not:

a**drop**eflis, **failedrop**s, **drop**e**fails**.

(a) How many cwords contain the subword "drop"?

Solution. Such cwords are obtainable by taking the word "drop" and the remaining 6 letters in any order. There are 7! permutations of these 7 items. ■

(b) How many cwords contain both "drop" and "fails"?

Solution. Take the words "drop" and "fails" and the remaining letter "e" in any order. So there are 3! such cwords.

■

(c) Use the Inclusion-Exclusion Principle to find a simple formula for the number of passwords.

Solution. There are 7! cwords that contain "drop", 6! that contain "fails", and 5! that contain "failed". There are 3! cwords containing both "drop" and "fails". No cword can contain both "fails" and "failed". The cwords containing both "drop" and "failed" come from taking the subword "failedrop" and the remaining letter "s" in any order, so there are 2! of them. So by Inclusion-exclusion, we have the number of cwords containing at least one of the three forbidden subwords is

$$(7! + 6! + 5!) - (3! + 0 + 2!) + 0 = 5!(49) - 8.$$

Among the 10! cwords, the remaining ones are passwords, so the number of passwords is

$$10! - 7! - 6! - 5! + 3! + 2! = 3,622,928.$$

■

Problem 3.
Let's develop a proof of the Inclusion-Exclusion formula using high school algebra.

(a) Most high school students will get freaked by the following formula, even though they actually know the rule it expresses. How would you explain it to them?

$$\prod_{i=1}^{n}(1 - x_i) = \sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|} \prod_{j\in I} x_j. \tag{1}$$

Hint: Show them an example.

Solution. Let's do an example. To "multiply out"

$$(1 - x_1)(1 - x_2)(1 - x_3), \tag{2}$$

you would form *monomial* products by selecting some of the $(-x_i)$'s to multiply together. For example, selecting $(-x_i)$'s with

- $i \in \{1, 3\}$ leads to the monomial $(-x_1)(-x_3) = (-1)^2 x_1 x_3 = x_1 x_3$,
- $i \in \{1, 2, 3\}$ leads to the monomial $(-x_1)(-x_2)(-x_3) = (-1)^3 x_1 x_2 x_3 = -x_1 x_2 x_3$, and
- $i \in \emptyset$ leads (by convention) to the monomial 1.

Then you sum up the monomials from *all possible* selections to get

$$(1 - x_1)(1 - x_2)(1 - x_3) = 1 - x_1 - x_2 - x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3 - x_1 x_2 x_3.$$

Now we can decipher (1) as saying to do the same thing for the product of n different $(1 - x_i)$'s: for any selection of $(-x_i)$'s with i in some subset, $I \subseteq \{1, \dots, n\}$, multiply the $(-x_i)$'s to get the monomial

$$\prod_{i \in I} (-x_i) = \prod_{i \in I} (-1)^{|I|} x_i,$$

and sum up all such monomials obtained by every possible selection, I, to get the right hand side of equation (1). ■

For any set, S, let M_S be the *membership* function of S:

$$M_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

Let $S_1, \dots, S_n$ be a sequence of finite sets, and abbreviate M_{S_i} as M_i. Let the domain of discourse, D, be the union of the S_i's. That is, we let

$$D ::= \bigcup_{i=1}^{n} S_i,$$

and take complements with respect to D, that is,

$$\overline{T} ::= D - T,$$

for $T \subseteq D$.

(b) Verify that for $T \subseteq D$ and $I \subseteq \{1, \dots n\}$,

$$M_{\overline{T}} = 1 - M_T, \tag{3}$$

$$M_{(\bigcap_{i \in I} S_i)} = \prod_{i \in I} M_{S_i}, \tag{4}$$

$$M_{(\bigcup_{i \in I} S_i)} = 1 - \prod_{i \in I} (1 - M_i). \tag{5}$$

(Note that (4) holds when I is empty because, by convention, an empty product equals 1, and an empty intersection equals the domain of discourse, D.)

Solution. To prove (3), we have for all $u \in D$,

$$\begin{aligned}
M_{\overline{T}}(u) = 1 \quad &\text{iff} \quad u \in \overline{T} \quad \text{iff} \quad M_T(u) = 0 \quad \text{iff} \quad 1 - M_T(u) = 1, \\
M_{\overline{T}}(u) = 0 \quad &\text{iff} \quad u \notin \overline{T} \quad \text{iff} \quad u \in T \quad \text{iff} \quad M_T(u) = 1 \quad \text{iff} \quad 1 - M_T(u) = 0,
\end{aligned}$$

so $M_{\overline{T}}(u) = 1 - M_T(u)$.

Similarly, to prove (4),

$$M_{(\bigcap_{i \in I} S_i)}(u) = 1 \quad \text{iff} \quad u \in \bigcap_{i \in I} S_i \quad \text{iff} \quad \bigwedge_{i \in I} u \in S_i \quad \text{iff} \quad \bigwedge_{i \in I} [M_i(u) = 1] \quad \text{iff} \quad \left(\prod_{i \in I} M_i(u) \right) = 1.$$

Finally, (5) follows from (3) and (4) by DeMorgan's Law. ■

(c) Use (1) and (5) to prove

$$M_D = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \prod_{j \in I} M_j. \tag{6}$$

Solution.

$$\begin{aligned}
M_D &= M_{(\bigcup_{i=1}^n S_i)} \\
&= 1 - \prod_{i=1}^{n} (1 - M_i) && \text{by (5)} \\
&= 1 - \sum_{I \subseteq \{1,\dots,n\}} (-1)^{|I|} \prod_{j \in I} M_j && \text{by (1)} \\
&= 1 - \left(1 + \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|} \prod_{j \in I} M_j \right) && \Big(\prod_{j \in \emptyset} M_j ::= 1\Big) \\
&= \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \prod_{j \in I} M_j.
\end{aligned}$$

■

(d) Prove that

$$|T| = \sum_{u \in D} M_T(u). \tag{7}$$

Solution.

$$\sum_{u \in D} M_T(u) = \sum_{u \in T} M_T(u) + \sum_{u \in \overline{T}} M_T(u) = \left(\sum_{u \in T} 1 \right) + \left(\sum_{u \in \overline{T}} 0 \right) = |T| + 0 = |T|,$$

■

(e) Now use the previous parts to prove

$$|D| = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \Big| \bigcap_{i \in I} S_i \Big| \tag{8}$$

Solution. Summing both sides of (6) over $u \in D$, we have

$$\begin{aligned}
|D| &= \sum_{u \in D} M_D(u) && \text{(by (7))} \\
&= \sum_{u \in D} \left(\sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \prod_{j \in I} M_j(u) \right) && \text{(by (6))} \\
&= \sum_{u \in D} \left(\sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} M_{\bigcap_{i \in I} S_i}(u) \right) && \text{(by (4))} \\
&= \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \left(\sum_{u \in D} M_{\bigcap_{i \in I} S_i}(u) \right) && \text{(reversing the order of sums)} \\
&= \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} |\bigcap_{i \in I} S_i| && \text{(by (7)).}
\end{aligned}$$

■

(f) Finally, explain why (8) immediately implies the usual form of the Inclusion-Exclusion Principle:

$$|D| = \sum_{i=1}^{n} (-1)^{i+1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=i}} |\bigcap_{j \in I} S_j|. \tag{9}$$

Solution. We obtain (9) from (8) by breaking up the sum over nonempty subsets, $I \subseteq \{1, \ldots, n\}$, into separate sums over all the subsets of size i, for $1 \leq i \leq n$. ■

Problem 4. (a) How many solutions over the *positive* integers are there to the inequality:

$$x_1 + x_2 + \ldots + x_{10} \leq 100$$

Solution.

$$\binom{90+10}{10}.$$

There is a bijection between solutions and bit-strings $0^{x_1-1}10^{x_2-1}1 \ldots 0^{x_9-1}10^{x_{10}-1}10^k$ with $x_i > 0$ and $k + \sum_1^{10} x_i = 100$. So the number of solutions is the same as the number of bit-strings with ten 1's and number of 0's equal to

$$k + \sum_1^{10}(x_i - 1) = \left(k + \sum_1^{10} x_i\right) - 10 = 100 - 10 = 90.$$

■

(b) We want to count step-by-step paths between points in the plane with integer coordinates. Ony two kinds of step are allowed: a right-step which increments the x coordinate, and an up-step which increments the y coordinate.

(i) How many paths are there from $(0, 0)$ to $(20, 30)$?

Solution. $\binom{50}{20}$.
There is a bijection from 50-bit sequences with 20 zeros and 30 ones. The sequence $(b_1, \ldots, b_{30})$ maps to a path where the i-th step is right if $b_i = 0$ and up if $b_i = 1$. Therefore, the number of paths is equal to $\binom{50}{20}$. ■

(ii) How many paths are there from $(0,0)$ to $(20,30)$ that go through the point $(10,10)$?

Solution. $\binom{20}{10} \cdot \binom{30}{10}$.
There is a bijection between the paths from $(20,30)$ that go through $(10,10)$ and set of pairs of paths consisting of path from $(0,0)$ to $(10,10)$ and a path from $(10,10)$ to $(20,30)$. So the number of paths through $(10,10)$ is the product of the sizes of these two sets of paths. ■

(iii) How many paths are there from $(0,0)$ to $(20,30)$ that do *not* go through either of the points $(10,10)$ and $(15,20)$?
Hint: Let P be the set of paths from $(0,0)$ to $(20,30)$, N_1 be the paths in P that go through $(10,10)$ and N_2 be the paths in P that go through $(15,20)$.

Solution.

$$\binom{50}{20} - \binom{20}{10} \cdot \binom{30}{10} - \binom{30}{15} \cdot \binom{15}{5} + \binom{20}{10} \cdot \binom{15}{5} \cdot \binom{15}{5}.$$

$N_1 \cap N_2$ is the set of paths from $(0,0)$ to $(20,30)$ that go through both $(10,10)$ and $(15,20)$. So $P - (N_1 \cup N_2)$ is the set of paths to be counted. Now we have

$$\begin{aligned} |P - (N_1 \cup N_2)| &= |P| - |N_1 \cup N_2| \\ &= |P| - |N_1| - |N_2| + |N_1 \cap N_2| \qquad \text{by Inclusion-Exclusion.} \end{aligned}$$

Part (ii) shows how to calculate $|N_i|$. Also, there is a bijection between $N_1 \cap N_2$ and the set of triples consisting of a path $(0,0)$ to $(10,10)$, a path from $(10,10)$ to $(15,20)$, and a path from $(15,20)$ to $(20,30)$. So the size of $N_1 \cap N_2$ is the product of the sizes of these three sets of paths.

■

(c) In how many ways can Mr. and Mrs. Grumperson distribute 13 identical pieces of coal to their three children for Christmas so that each child gets at least one piece??

Solution.

$$\binom{12}{2}.$$

There is an obvious bijection between distributions of coal to children and bit strings $0^{a+1}10^{b+1}10^{c+1}$ where $(a+1)+(b+1)+(c+1) = 13$, namely such a string corresponds to distributing $a+1$ coals to the first child, $b+1$ coals to the second, and $c+1$ coals to the third. There is also an obvious bijection between such bit strings and bitstrings of the form $0^a10^b10^c$ where $a+b+c = 10$, that is, bit-strings with ten 0's and two 1's. ■

Sums

closed form ↓

$1+2+3+\dots+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$1+x+x^2+x^3+\dots+x^n = \frac{1-x^{n+1}}{1-x}$

for product → take log to convert to sum

to find → Pertubation Method

$V = \sum_{i=1}^{n} \frac{m}{(1+p)^{i-1}} = m\sum_{j=0}^{n-1}\left(\frac{1}{1+p}\right)^j$ sub $j=i-1$

$= m\sum_{i=0}^{n-1} x^i$ sub $x = \frac{1}{1+p}$

$S = 1+x+x^2+\dots+x^n$

$xS = x+x^2+\dots+x^{n+1}$

Subtract $S - xS \to = 1-x^n$

Solve for $S \to S = \frac{1-x^{n+1}}{1-x}$

$V = m\left(\frac{1-x^n}{1-x}\right)$

$= m\left(\frac{1+p-(1/(1+p))^{n-1}}{p}\right)$

If $|x|<1$: $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$

find by taking lim $n\to\infty$

$V = m\sum_{i=0}^{\infty} x_i$

$= m\cdot\frac{1}{1-x}$

$= m\cdot\frac{1+p}{p}$

$1+\frac12+\frac14+\dots = \sum_{i=0}^{\infty}\left(\frac12\right)^i = \frac{1}{1-\frac12} = 2$

$.999\dots = .9\sum_{i=0}^{\infty}\left(\frac{1}{10}\right)^i = .9\left(\frac{1}{1-1/10}\right) = .9\,\frac{10}{9} = 1$

$1-\frac12+\frac14-\dots = \sum_{i=0}^{\infty}\left(-\frac12\right)^i = \frac{1}{1-(-1/2)} = \frac23$

$1+2+4+\dots+2^{n-1} = \sum_{i=0}^{n-1}2^i = \frac{1-2^n}{1-2} = 2^n-1$

$1+3+9+\dots+3^{n-1} = \sum_{i=0}^{n-1}3^i = \frac{1-3^n}{1-3} = \frac{3^n-1}{2}$

6.042 Cheat Sheat 5

Can also differentiate/integrate

$\sum_{i=1}^{n-1} ix^i = \frac{x-nx^n+(n-1)x^{n+1}}{(1-x)^2}$

$\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$

Sum of Powers $\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6}$

Approximating - find closed-form upper/lower bounds

Weakly ↑ $S = \sum_{i=1}^{n} f(i) \to I = \int_1^n f(x)\,dx$

$I + f(1) \le S \le I + f(n)$

Weakly ↓ $I + f(n) \le S \le I + f(1)$

nth Harmonic # $H_n = \sum_{i=1}^{n}\frac{1}{i}$

So $S_n = \frac{H_n}{2}$ - no closed form - so can get first few terms

- or upper/lower bounds

$\int_1^n \frac{1}{x}\,dx = \ln(x)\big|_1^n = \ln(n)$

$\ln(n)+\frac{1}{n} \le H_n \le \ln(n)+1$

So $\gamma = .577215664$

Asymptotic Inequality

~ leading term = iff $\lim \frac{f(x)}{g(x)} = 1$

Products

$P = \prod_{i=1}^{n} f(i)$ take log $\ln(P) = \sum_{i=1}^{n}\ln(f(i))$

to approximate $= \sum_{i=1}^{n}\ln(i)$

$n\ln(n)-n+1 \le \cancel{?} \le n\ln(n)-n+1+\ln(n)$

exponentiate

$n^n/e^{n-1} \le n! \le n^{n+1}/e^{n-1}$

Stirling's Formula for $n \ge 1$

$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\epsilon(n)}$

where

$\frac{1}{12n+1} \le \epsilon(n) \le \frac{1}{12n}$ but $\epsilon \to 0$ so $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Little Oh asy. smaller 4/19

$f = o(g(x))$ iff $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 0$

Big Oh upper bound on growth

$f = O(g(x))$ iff $\limsup_{x\to\infty}\frac{f(x)}{g(x)} < \infty$ finite

There exists constant c, such that for $\forall x \ge x_0$

$|f(x)| \le c\,g(x)$

- ignoring some leading term

Theta - precise up to constant terms

- upper + lower bounds

$f = \Theta(g)$ iff $f = O(g)$ and $g = O(f)$

every constant is 1

base of exponent matters

big O always upper bound

never = to, bad notation

Omega lower bound of running time

$f = \Omega(g)$ is $g = O(f)$

Little Omega - one grows strictly faster than other

$f = \omega(g)$ is $g = o(f)$

Cardinality/Counting Rules

- Count one thing by counting another
- that is related as a bij
- like encode w/ 1s and 0s

Product Rule size of product of sets

if finite just multiply sizes

2^n = # bit string subsets

Sum Rule - if disjoint - just add

# of possible arrangements of 3 prizes = n^3

but if prizes must go unique people = $n(n-1)(n-2)$

or $\frac{n!}{(n-3)!}$

Permutations (order matters) each item once = $n!$

$\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Division Rule k-to-1 function

- like fingers to person = 10 to 1 relation

$|A| = k\cdot|B|$

So $|B| = \frac{|A|}{k}$

Knights of Round Table - only who next to who matters

$|B| = \frac{|A|}{n} = \frac{n!}{n} = (n-1)!$, n cyclic shifts ok

Counting Subsets
How many k-el subsets from n-el set.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Also by division rule $n! = k!(n-k)!\binom{n}{k}$

example # n bit seq w/ k 1s $= \binom{n}{k}$

its like putting k in 1 subset and n-k in another

n from permutations - then $k!(n-k)!$ to 1 fn

Can have m subsets — multinomial coefficient

$$\binom{n}{k_1, k_2, \ldots k_m} = \frac{n!}{k_1! \cdot k_2! \cdots k_m!}$$

So # of splits of n el subset

Subset split rule

example Bookkeeper rule

$$\frac{10!}{1!2!2!3!1!1!1!} \rightarrow \binom{k_1+k_2+\ldots+k_m}{k_1, k_2, \ldots k_m}$$

Binomial Theorem = sum of 2 terms

$(a+b)^4 = 2^4$ terms

terms w/ k copies of b is

$$\frac{n!}{k!(n-k)!} = \binom{n}{k}$$

So $a^{n-k}b^k = \binom{n}{k}$

$(a+b)^4 = \binom{4}{0}a^4b^0 + \ldots \binom{4}{4}a^0b^4$

So

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

Multinomial
- Extension above
- want coefficients for $bo^2k^2e^3pr$

$$(z_1+z_2+\ldots+z_m)^n = \sum_{\substack{k_1,\ldots k_m \in N \\ k_1+\ldots+k_m=n}}\binom{n}{k_1, k_2, \ldots k_m}$$

$$z_1^{k_1}z_2^{k_2}\ldots z_m^{k_m}$$

Solver
think of clever rules to represent what is specified or not

Inclusion - Exclusion
adding non-disjoint sets

$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$

$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|$

- must remove duplicates

Proving
1. Define S
2. Show $|S| = n$ by counting 1 way
3. Show $|S| = m$ " " other "
4. Conclude $n = m$

Pascal's Identity
Boxer story

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pigeonhole Principle
If more pigeons than holes, then at least 2 pigeons in some hole

Identify A = Pigeons $\rightarrow n$

B = Pigeonholes $\rightarrow h$

$f \; A \rightarrow B$

$\lceil \frac{n}{h} \rceil$ pigeons in some hole

Prove size of both

Magic Trick
- info on which card kept hidden
- what is degree constrained?

log	$O(\log n)$	
poly	n^x	n is what ?
exp	x^n	

non neg integer solutions

$x_1 + x_2 + \ldots + x_m = k \rightarrow \binom{m+k-1}{k}$

$x_1 + x_2 + \ldots + x_m \leq k \rightarrow \binom{m+k}{k}$

length $k+m-1$ bit strings w/ k 0s

add m+1

length-m weakly ↑ seq non neg int ≤ k

$\binom{m+k}{k}$

L bij to seq $x_1, x_1+x_2, x_1+x_2+x_3, \ldots$

So ans follows from

Miniquiz 5

4/18

- This is mostly a Counting quiz
  - with a bit asy. notation
- know all the Counting rules
  - and when to use!
- Is it all of 14 - sums + asymptotes?
- Never was very good at sums
- Another cheat sheet!
  - 3rd in a week almost

Hanging over edge - Think about slowly!

$$\frac{(n-1)\cdot 1 + 1\cdot \frac{1}{2}}{n}$$

$$\frac{n-1+\frac{1}{2}}{n}$$

$$\frac{n-\frac{1}{2}}{n} \quad \frac{n}{n} - \frac{\frac{1}{2}}{n} = 1 - \frac{1}{2n}$$ ✓ got it

- See not so hard / opaque

Think through diff cases

Best solution - just **think** through the problem

②

Harmonics - just think about it
generallize stuff to variables

Other chaps more memorizing
- this is more thinking

I think I convert bij to word problem to solve!
- Not other way around

Need to find the trick
- like 01 bit streams

Divison Rule if 10 fingers for every person
10-to-1
divide by 10

If so many combos same, divide out
or 3! of the items are the same
So must eliminate
- like if order does not matter

like awards n·n·n w/ repeats
? that's repeats

n·n·n ← is $\binom{n}{3}$ any order of the first 3
- those chosen can rearrange } inside themselves only!
- " not " " "

(3)

No! - remember division is sameness equivilants

- remove because these are equiv
- since order of what we pick does not matter

(everything makes a lot more sense now)

So for pizza divide by 3! - because these outcomes are all the same!

diff ways of doing the same thing

Oh that was mapping rule used (inj)

$$\frac{\geq 1 \text{ out}}{\leq 1 \text{ in}} \rightarrow |A| \leq |B|$$

I never really understood these

If $|A| > |B|$ then no $\frac{\geq \text{out}}{\leq \text{in}}$

Bookkeeper is 2 to 1 since

$B O_1 O_2 k$
$B O_2 O_1 k$ $\rightarrow$ BOOk

$k O_1 B O_2$
$k O_2 B O_1$ $\rightarrow$ kOBO

$O_1 B O_2 k$
$O_2 B O_1 k$ $\rightarrow$ OBOk

etc

ah-ha! :)

(4)

So # arrangements $\binom{4}{1,2,1} = \frac{4!}{1!\,2!\,1!}$ ← multinomial

Which is just $\frac{4!}{2!}$ which is "normal" divison rule

Note carefully # pigon holes + pigons + mapping!

Think about!

Draw! ↑ Test usually goes by fast

---

lenght m words from n alpha - no more than 1

$m = 1$
$n = 5$

$\frac{5!}{(5-4)!} = 5!$ ok

↑ this was $5-4$

I was picturing just 4

↑ see, good to write down

But how to find fresh

$m = 2$
$n = 5$

↑ join as unit ↑

$\frac{5!}{5-2}$

5

No did wrong

$$\frac{5!}{(5-1)!} = \frac{5!}{4!} = 5$$

oh a, b, c, d, e

Does make sense

L Then why did I write other makes sense

Think!

$n=5$
$m=2$

$$\frac{5!}{(5-2)!} = \frac{5!}{3!} = 5 \cdot 4$$

ab ba ca
ac bc cb
ad bd etc $5 \cdot 4$
ae be

– I actually found that before!

Write examples + build up!

6

m balls
n holes

Say $m=4$
$n=3$

$$\begin{array}{ccc} & & 4 \\ 1 & 2 & 3 \end{array}$$

$$\begin{array}{ccc} & 4 & \\ 1 & 2 & 3 \end{array}$$

$$\begin{array}{ccc} 4 & & \\ 1 & 2 & 3 \end{array}$$

but then also $\cdot 4$

so $3 \cdot 4 = 12$

<u>Ans</u> $3^4 =$ but that is more

How did they get that ?

Oh so # seq

1 2 3 1

can repeat

1 1 1 1 works

So $n \cdot n \cdot n \cdot n$ m times

Also think like this

— not an example
— example bad?

Mini-Quiz Apr. 20

Your name: Michael Plasmeier

Circle the name of your TA and write your table number:

Ali   Nick   Oscar   Oshani   **Table number** 12

- This quiz is **closed book**. Total time is 30 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	5	2	ORM
2	7	4	OS
3	3	1	NJ
4	5	∅	AK
Total	20	7	

Every Card is different except 2 pair

Problem 1 (5 points). (a) Suppose two identical 52-card decks[1] are mixed together. Write a simple expression for the number of different arrangements of the 104 cards that could possibly result from such a mixing.

$104!$ Permutations

$104 \cdot 103 \cdot 102 \cdots$

— Oh but 2 cards will be same 2-to-1

$\frac{104!}{2}$ −1

(b) Using only integers from the interval $[1, n]$, how many different **strictly increasing** length-m sequences can be formed?

Correction $m \le n$ on cheat sheet

$\binom{m+n}{n}$ −1

bij to seq $x_1, x_1+x_2, x_1+x_2+x_3, \ldots,$
ans follows from $x_1 + x_2 + \cdots + x_m \le n \rightarrow \binom{m+n}{n}$

[1] Standard decks of playing cards, without jokers.

Problem 2 (7 points).
For each pair of functions, $f : \mathbb{N}^+ \to \mathbb{N}$ and $g : \mathbb{N}^+ \to \mathbb{N}$, in the table below, indicate which of the listed asymptotic relations hold **and** which do not.

Fill **every** cell in the table. You may use checkmarks and crosses, “T” and “F”, “TRUE” and “FALSE”, “Y” and “N”, or “YES” and “NO”. Look at largest term

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$\log_4 n$	$\sqrt[3]{n}$	✗ ✗	✓	✓ ✗	✗
$n^2 + 3^n$	$n^3 + 2^n$	✓	✗	✓	✓
$n \ln n!$	$n^2 \log_{10} n^2$	✗ ✗	✓ ✗	✓	✗
$n^{2\cos(\pi n/2)+3}$	$5n^5 + 3n^3 + n$	✓	✓ ✗	✓ ✗	✗

10/16

didn't write which smaller
exp > log “
poly > log “
on cheat sheet

$f = O(g)$ iff $\lim \frac{f(x)}{g(x)} < \infty$ finite

$n^2 + 3^n = f$
$n^3 + 2^n = g$

f bigger

g bigger means $f = o(g)$
f " " $g = o(f)$

¿how tell if finite?

guessing

Problem 3 (3 points).
Give an example of a pair of strictly increasing total functions, $f : \mathbb{N}^+ \to \mathbb{N}^+$ and $g : \mathbb{N}^+ \to \mathbb{N}^+$, that satisfy $f \sim g$ but **not** $3^f = O(3^g)$.

kinda remember

asy = to

upper bound on growth

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$$

if leading terms =

correct.

how to do?

$f = 2^x$

$g = 3^x$

$2^\infty = \infty$

$3^\infty = \infty$

$\frac{\infty}{\infty} = 1$

you can't take limits like that: $\lim_{x \to \infty} \frac{f}{g} = \lim_{x \to \infty} \left(\frac{2}{3}\right)^x = 0.$

$$\frac{3^{2^x}}{3^{3^x}}$$ Not $< \infty$

So not finite

actually $\lim_{x \to \infty} \frac{3^{2^x}}{3^{3^x}} < \infty.$

"common base rule."

$\frac{1}{3}$

guessing

not as much counting as thought

Problem 4 (5 points).
A spacecraft is traveling through otherwise-empty three-dimensional space. It can move along only one dimension at a time, stepping precisely one unit in the positive direction along that dimension with each movement. For any two points, P and Q, in space, let $p_{P,Q}$ denote the number of distinct paths the spacecraft can follow to go from P to Q.

(a) Let P and Q have coordinates (x_P, y_P, z_P) and (x_Q, y_Q, z_Q), respectively. Assuming that $p_{P,Q}$ is positive, express $p_{P,Q}$ as a **single multinomial coefficient**.

- did not write - was on pset; was not multinomial before?

Can be any order

$x_q - x_p = k_x \quad y_q - y_p = k_y \quad z_q - z_p = k_z$

$\emptyset$

$$P_{p,q} = \sum_{k_x, k_y, k_z \in N} \binom{k_x \cdot k_y \cdot k_z}{k_x, k_y, k_z} x_p^{k_x} \cdot y_p^{k_y} \cdot z_p^{k_z}$$

example

(b) Suppose there exist five points in space, A, B, C, D, and E, such that it is possible for the spacecraft to travel from A to B, from B to C, from C to D, and from D to E. Write an expression for the number of distinct paths the spacecraft can follow to go from A to E while **avoiding** B, C, and D. Your expression **must** be written entirely in terms of symbols of the form $p_{P,Q}$, where $P, Q \in \{A, B, C, D, E\}$.

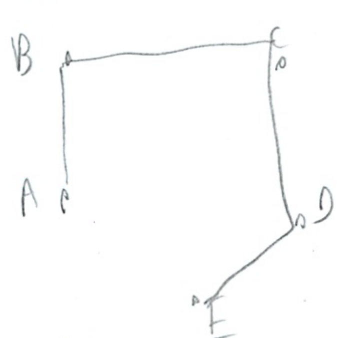
Hint: Inclusion-Exclusion. how is it a non disjoint set; might be able to go $A \to E$

TA: don't go there at all

Can go $A \to E$
$A \to D \to E$
$A \to C \to D \to E$
? but want to avoid

$\emptyset$

Can go



But how do we know if it can go $A \to E$ directly?

$|S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5| = |S_1| + |S_2| + |S_3| + |S_4| + |S_5| - |\text{the sets of 2} \; S_1 \cap S_2| \text{ etc}$
$+ |\text{sets of 3} \; S_1 \cap S_2 \cap S_3| - |\text{sets of 4} \; S_1 \cap S_2 \cap S_3 \cap S_4| + |S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5|$

Don't get what qu asking!

b) Oh related to previous problem

So can go direct

$P_{A,E}$ or through stops + 2 stop + 3 stop + 4 stop + 5 stop
but not allowed

So just direct $P_{A,E}$ which is multinomial from part a

Solutions to Mini-Quiz Apr. 20

Problem 1 (5 points). (a) Suppose two identical 52-card decks[1] are mixed together. Write a simple expression for the number of different arrangements of the 104 cards that could possibly result from such a mixing.

Solution. In the mixed deck, there are precisely two copies of each of 52 distinct cards. By the Bookkeeper Rule and the definition of multinomial coefficients, the number of possible arrangements of cards in the mixed deck is therefore just

$$\frac{104!}{(2!)^{52}}.$$

■

(b) Using only integers from the interval $[1, n]$, how many different **strictly increasing** length-m sequences can be formed?

Solution.

$$\binom{n}{m}$$

Justification: Given any m-element subset of $\{1, 2, ..., n\}$, listing its elements in increasing order yields a sequence that is strictly increasing and has length m. By collecting in a set the terms of any strictly increasing length-m sequence whose terms have been drawn from $\{1, 2, ..., n\}$, an m-element subset of $\{1, 2, ..., n\}$ is formed. Thus there is a bijection between the set of all strictly increasing length-m sequences with terms drawn from $\{1, 2, ..., n\}$ and the set of all size-m subsets of $\{1, 2, ..., n\}$.

■

Problem 2 (7 points).
For each pair of functions, $f : \mathbb{N}^+ \to \mathbb{N}$ and $g : \mathbb{N}^+ \to \mathbb{N}$, in the table below, indicate which of the listed asymptotic relations hold **and** which do not.

Fill **every** cell in the table. You may use checkmarks and crosses, "T" and "F", "TRUE" and "FALSE", "Y" and "N", or "YES" and "NO".

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$\log_4 n$	$\sqrt[3]{n}$				
$n^2 + 3^n$	$n^3 + 2^n$				
$n \ln n!$	$n^2 \log_{10} n^2$				
$n^{2\cos(\pi n/2)+3}$	$5n^5 + 3n^3 + n$				

[1]Standard decks of playing cards, without jokers.

Solution.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$\log_4 n$	$\sqrt[3]{n}$	*YES*	*YES*	*NO*	*NO*
$n^2 + 3^n$	$n^3 + 2^n$	*NO*	*NO*	*YES*	*YES*
$n \ln n!$	$n^2 \log_{10} n^2$	*YES*	*NO*	*YES*	*NO*
$n^{2\cos(\pi n/2)+3}$	$5n^5 + 3n^3 + n$	*YES*	*NO*	*NO*	*NO*

Justification:

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$\log_4 n$	$\sqrt[3]{n}$	*YES*	*YES*	*NO*	*NO*

Using either (1) l'Hôpital's Rule or (2) the fact that $\log n = o(n^\epsilon)$ for all $\epsilon > 0$ (see the Notes), conclude that $f = o(g)$. This implies that $f = O(g)$, $g \neq o(f)$, and $g \neq O(f)$.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$n^2 + 3^n$	$n^3 + 2^n$	*NO*	*NO*	*YES*	*YES*

Intuitively, 3^n grows far faster than n^2 and 2^n grows far faster than n^3, as n grows large. (Any power of n is asymptotically smaller than any increasing exponential in n.) Also, 3^n grows far faster than 2^n. (Given two increasing exponentials, the one with the smaller base will be asymptotically smaller.) A bit more rigorously,

$$\begin{aligned}
\lim_{n\to\infty} \frac{g(n)}{f(n)} &= \lim_{n\to\infty} \frac{n^3 + 2^n}{n^2 + 3^n} \\
&= \lim_{n\to\infty} \frac{\frac{n^3}{3^n} + \left(\frac{2}{3}\right)^n}{\frac{n^2}{3^n} + 1} \\
&= \frac{\lim_{n\to\infty} \frac{n^3}{3^n} + \lim_{n\to\infty} \left(\frac{2}{3}\right)^n}{\lim_{n\to\infty} \frac{n^2}{3^n} + \lim_{n\to\infty} 1} \\
&= \frac{0+0}{0+1} \\
&= 0
\end{aligned}$$

Where $\lim_{n\to\infty} \frac{n^3}{3^n}$ and $\lim_{n\to\infty} \frac{n^2}{3^n}$ can be found to be zero by l'Hôpital's Rule, and $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n$ is zero because $\left|\frac{2}{3}\right| < 1$. Thus $g = o(f)$, which implies $g = O(f)$, $f \neq o(g)$, and $f \neq O(g)$.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$n \ln n!$	$n^2 \log_{10} n^2$	*YES*	*NO*	*YES*	*NO*

Using Stirling's formula, $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, it is easy to show that $\ln n! \sim n \ln n$ and hence that $f(n) \sim n^2 \ln n$. Now,

$$\begin{aligned}
n^2 \log_{10} n^2 &= 2n^2 \log_{10} n \\
&= 2n^2 \frac{\ln n}{\ln 10}
\end{aligned}$$

It should be evident now that $g(n) \sim \frac{2}{\ln 10} f(n)$. Hence $f \neq o(g)$ and $g \neq o(f)$, but $f = O(g)$ and $g = O(f)$.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$n^{2\cos(\pi n/2)+3}$	$5n^5 + 3n^3 + n$	YES	NO	NO	NO

Notice that

$$f(n) = \begin{cases} n^5 & \text{if } n \equiv 0 \bmod 4 \\ n^3 & \text{if } n \equiv 1 \bmod 4 \text{ or } n \equiv 3 \bmod 4 \\ n & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

Because $f(n)$ is thus clearly bounded above by n^5 and $g(n)$ is a polynomial of degree 5, have $f = O(g)$. The behavior of $f(n)$ when n is not a multiple of 4 leads to $g \neq O(f)$. It is obvious that $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ and $\lim_{n\to\infty} \frac{g(n)}{f(n)}$ are both nonzero, so $f \neq o(g)$ and $g \neq o(f)$. ■

Problem 3 (3 points).
Give an example of a pair of strictly increasing total functions, $f : \mathbb{N}^+ \to \mathbb{N}^+$ and $g : \mathbb{N}^+ \to \mathbb{N}^+$, that satisfy $f \sim g$ but **not** $3^f = O(3^g)$.

Solution. The pair

$$f(n) = n^2 + n$$
$$g(n) = n^2$$

satisfies these criteria. Since n^2 is the term that dominates the behavior of $n^2 + n$ as n grows large, it is obvious that $n^2 + n \sim n^2$. (Applying the limit definition of asymptotic equality readily establishes this result.) Clearly, $3^{f(n)} = 3^{n^2+n} = 3^n 3^{n^2}$, while $3^{g(n)} = 3^{n^2}$. Thus $3^{f(n)} = 3^n 3^{g(n)}$. From this, it is obvious that $3^f \neq O(3^g)$. (It is very easy to check that, in fact, $3^g = o(3^f)$.) ■

Problem 4 (5 points).
A spacecraft is traveling through otherwise-empty three-dimensional space. It can move along only one dimension at a time, stepping precisely one unit in the positive direction along that dimension with each movement. For any two points, P and Q, in space, let $p_{P,Q}$ denote the number of distinct paths the spacecraft can follow to go from P to Q.

(a) Let P and Q have coordinates (x_P, y_P, z_P) and (x_Q, y_Q, z_Q), respectively. Assuming that $p_{P,Q}$ is positive, express $p_{P,Q}$ **as a single multinomial coefficient.**

Solution. Because each of the spacecraft's permissible atomic movements involves incrementing precisely one of its three position coordinates, $p_{P,Q} > 0$ implies that $x_Q - x_P$, $y_Q - y_P$, and $z_Q - z_P$ are all nonnegative integers. (The converse is also true.) To go from P to Q, the spacecraft must increment its first position coordinate $x_Q - x_P$ times, its second $y_Q - y_P$ times, and its third $z_Q - z_P$ times. So it must undergo precisely $(x_Q - x_P) + (y_Q - y_P) + (z_Q - z_P)$ atomic movements, $x_Q - x_P$ of them along the first dimension, $y_Q - y_P$ of them along the second, and $z_Q - z_P$ of them along the third.

So, number the spacecraft's atomic movements: $1, 2, \ldots, (x_Q - x_P) + (y_Q - y_P) + (z_Q - z_P)$. Partition the set $T = \{1, 2, \ldots, (x_Q - x_P) + (y_Q - y_P) + (z_Q - z_P)\}$ into three sets, T_x, T_y, and T_z, such that $|T_x| = x_Q - x_P$, $|T_y| = y_Q - y_P$, and $|T_z| = z_Q - z_P$. T_x then specifies which atomic movements are along the first dimension, T_y does the same for the second dimension, and T_z for the third. Each distinct partition corresponds to a single permissible path from P to Q, and each permissible path from P to Q corresponds to a single partition. So the number of permissible paths from P to Q is just the number of distinct partitions – that is, the number of $(x_Q - x_P, y_Q - y_P, z_Q - z_P)$-splits of the $((x_Q - x_P) + (y_Q - y_P) + (z_Q - z_P))$-element set T. And of course this number is just:

$$p_{P,Q} = \binom{(x_Q - x_P) + (y_Q - y_P) + (z_Q - z_P)}{x_Q - x_P,\, y_Q - y_P,\, z_Q - z_P}$$

Alternatively, consider a bijection between the set of possible paths from P to Q and the set of sequences of length $(x_Q - x_P) + (y_Q - y_P) + (z_Q - z_P)$ that contain $(x_Q - x_P)$ 1s, $(y_Q - y_P)$ 2s, and $(z_Q - z_P)$ 3s. The kth term of each sequence specifies the dimension associated with the kth atomic movement in the corresponding path. The Bookkeeper Rule then leads directly to the expression for $p_{P,Q}$. ■

(b) Suppose there exist five points in space, A, B, C, D, and E, such that it is possible for the spacecraft to travel from A to B, from B to C, from C to D, and from D to E. Write an expression for the number of distinct paths the spacecraft can follow to go from A to E while **avoiding** B, C, and D. Your expression **must** be written entirely in terms of symbols of the form $p_{P,Q}$, where $P, Q \in \{A, B, C, D, E\}$.

Hint: Inclusion-Exclusion.

Solution. First, note that since it is possible for the spacecraft to travel from A to B, from B to C, from C to D, and from D to E, therefore paths exist from A to each of A, B, C, D, and E, from B to each of C, D, and E, ..., and from E to E. Thus, because of the way in which the spacecraft must move, positive-length paths cannot exist from E to A, B, C, D, or E, from D to A, B, C, or D, ..., or from A to A. (This is why, in what follows, terms like $p_{B,D}$ appear, but terms like $p_{D,B}$ do not. If B and D are distinct, $p_{B,D}$ is positive and $p_{D,B}$ is zero, so including $p_{D,B}$ would affect nothing. If B and D are coincident, both $p_{B,D}$ and $p_{D,B}$ are equal to one, but considering both would amount to counting every path through B twice.) In a very loose sense, and if cases involving coincident points are ignored, this essentially means that the spacecraft only moves "forward" and that B is "ahead" of A, C is "ahead" of B, D is "ahead" of C, and E is "ahead" of D.

Let S denote the set of all paths from A to E. Clearly, $|S| = p_{A,E}$.

Let S_X denote the set of all paths that go from A to E, through X, where $X \in \{B, C, D\}$. Evidently, $|S_X| = p_{A,X}\, p_{X,E}$.

Now, $S_X \cap S_Y$ is the set of paths that go from A to E, through both X and Y, where $X, Y \in \{B, C, D\}$. Obviously, $|S_B \cap S_C| = p_{A,B}\, p_{B,C}\, p_{C,E}$, $|S_B \cap S_D| = p_{A,B}\, p_{B,D}\, p_{D,E}$, and $|S_C \cap S_D| = p_{A,C}\, p_{C,D}\, p_{D,E}$. Also, $S_B \cap S_C \cap S_D$ is the set of all paths that go from A to E, through all three of B, C, and D. Obviously, $|S_B \cap S_C \cap S_D| = p_{A,B}\, p_{B,C}\, p_{C,D}\, p_{D,E}$.

Now, the set of paths that go from A to E and pass through at least one of B, C, and D, is just $S_B \cup S_C \cup S_D$. By inclusion-exclusion,

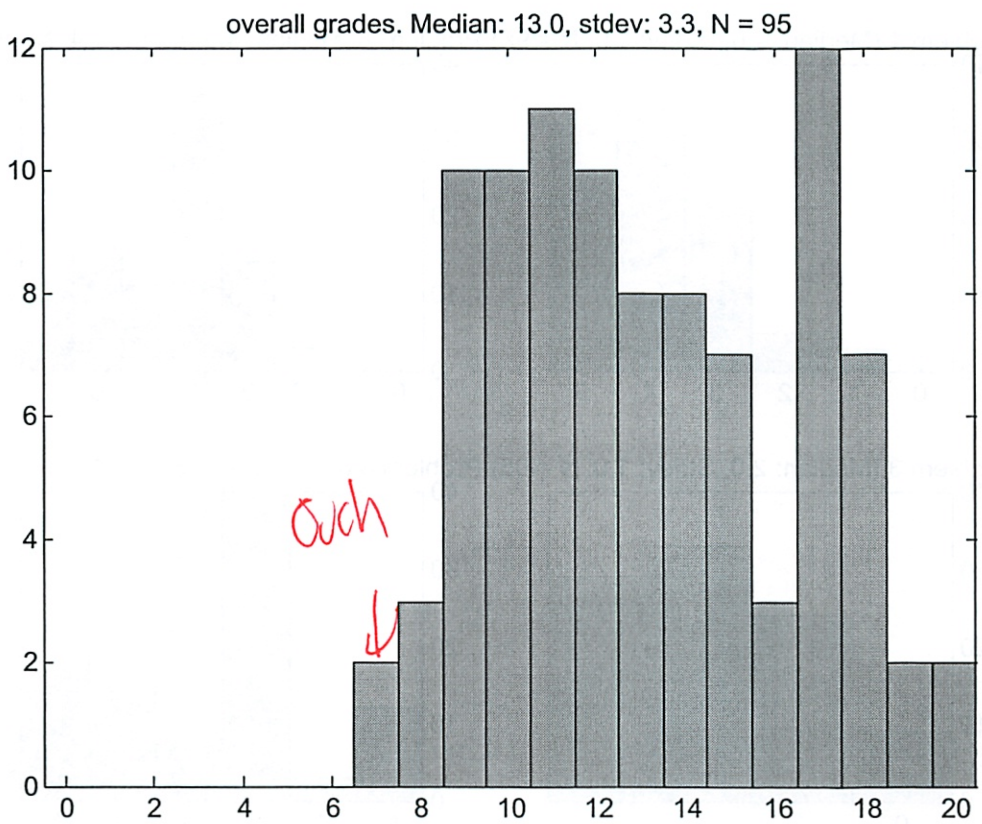
$$\begin{aligned}
|S_B \cup S_C \cup S_D| =& |S_B| + |S_C| + |S_D| - |S_B \cap S_C| - |S_B \cap S_D| - |S_C \cap S_D| + |S_B \cap S_C \cap S_D| \\
=& p_{A,B}\, p_{B,E} + p_{A,C}\, p_{C,E} + p_{A,D}\, p_{D,E} \\
& - p_{A,B}\, p_{B,C}\, p_{C,E} - p_{A,B}\, p_{B,D}\, p_{D,E} - p_{A,C}\, p_{C,D}\, p_{D,E} + p_{A,B}\, p_{B,C}\, p_{C,D}\, p_{D,E}
\end{aligned}$$

Let R denote the set of all paths from A to E that go through neither B, nor C, nor D. Evidently, $S = R \cup (S_B \cup S_C \cup S_D)$ and $R \cap (S_B \cup S_C \cup S_D) = \emptyset$. Therefore $|S| = |R| + |S_B \cup S_C \cup S_D|$, so the number of distinct paths the spacecraft can follow to go from A to E while avoiding B, C, and D is

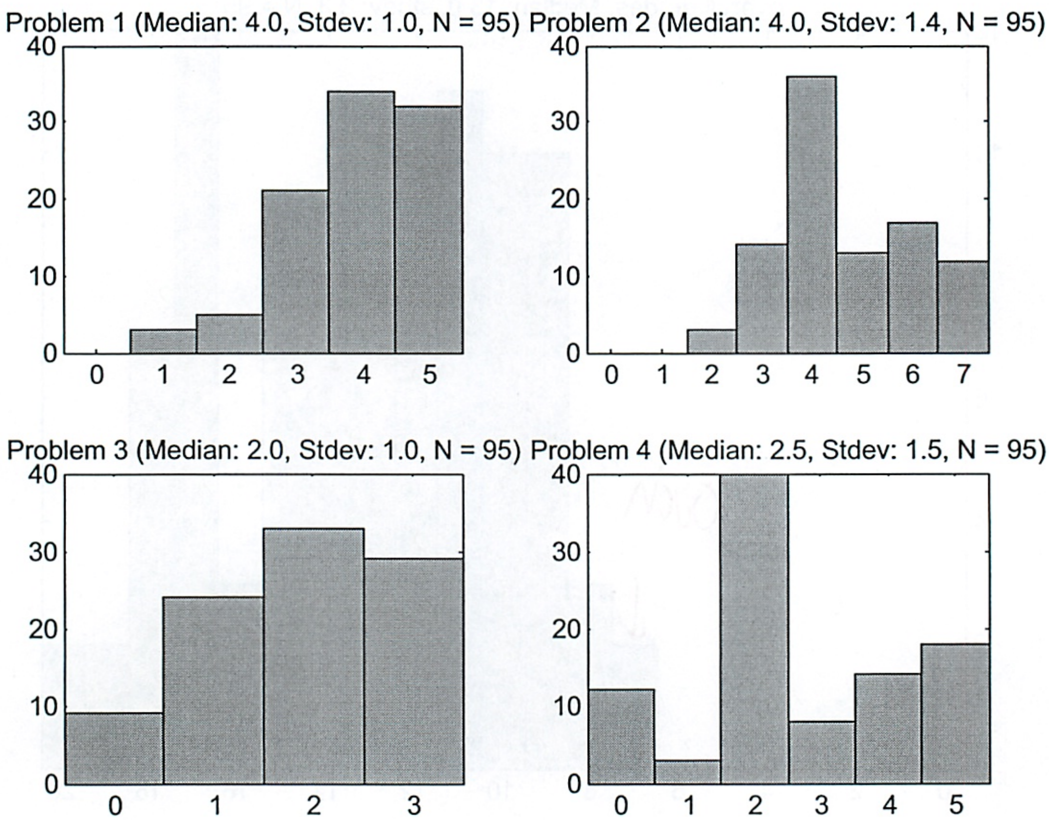
$$\begin{aligned}
|R| =& |S| - |S_B \cup S_C \cup S_D| \\
=& p_{A,E} - p_{A,B}\, p_{B,E} - p_{A,C}\, p_{C,E} - p_{A,D}\, p_{D,E} \\
& + p_{A,B}\, p_{B,C}\, p_{C,E} + p_{A,B}\, p_{B,D}\, p_{D,E} + p_{A,C}\, p_{C,D}\, p_{D,E} - p_{A,B}\, p_{B,C}\, p_{C,D}\, p_{D,E}
\end{aligned}$$

■

MQ 5


overall grades. Median: 13.0, stdev: 3.3, N = 95
Ouch
0
2
4
6
8
10
12
14
16
18
20
0
2
4
6
8
10
12

MQ5



Problem 1 (Median: 4.0, Stdev: 1.0, N = 95)
Problem 2 (Median: 4.0, Stdev: 1.4, N = 95)
Problem 3 (Median: 2.0, Stdev: 1.0, N = 95)
Problem 4 (Median: 2.5, Stdev: 1.5, N = 95)

6.042 Grade Report for *Plasmeier, Michael*

Problem Sets

id ▲	adjusted score	raw score	max	statistics
PS.01	**35.15**	28.00	50.00	link
PS.02	**35.98**	33.00	50.00	link
PS.03	**22.00**	18.50	40.00	link
PS.04	**26.02**	24.00	30.00	link
PS.05	**34.83**	32.20	40.00	link
PS.06	**36.82**	33.00	50.00	link
PS.07	**33.72**	29.00	50.00	link

Note: The psets' adjusted scores reflect the psets scores after being adjusted by its corresponding MQ's score. The adjusted scores will be further increased according to final exam's performance.

Mini Quizzes

id ▲	pts	max	statistics
MQ.01	**13.00**	20.00	link
MQ.02	**7.00**	20.00	link
MQ.03	**13.50**	20.00	link
MQ.04	**9.00**	20.00	link
MQ.05	**7.00**	20.00	link

Reading Assignments

No grades available yet.

Tutor Problems

id ▲	pts	max
T.01	**1.00**	1.00
T.02	**1.00**	1.00
T.03	**1.00**	1.00
T.04	**1.00**	1.00
T.05	**1.00**	1.00
T.06	**1.00**	1.00
T.07	**1.00**	1.00
T.08	**1.00**	1.00
T.09	**1.00**	1.00

Class Participation

id ▲	pts	max	pending makeup
CP.01	**2.00**	2.00	
CP.02	**2.00**	2.00	
CP.03	**2.00**	2.00	
CP.04	**2.00**	2.00	
CP.05	**2.00**	2.00	
CP.06	**2.00**	2.00	
CP.07	**2.00**	2.00	
CP.08	**2.00**	2.00	
CP.09	**2.00**	2.00	
CP.10	**1.00**	2.00	
CP.11	**2.00**	2.00	
CP.12	**2.00**	2.00	
CP.13	**1.00**	2.00	
CP.14	**1.00**	2.00	
CP.15	**2.00**	2.00	
CP.16	**1.00**	2.00	
CP.17	**2.00**	2.00	
CP.18	**2.00**	2.00	
CP.19	**2.00**	2.00	
CP.20	**2.00**	2.00	
CP.21	**2.00**	2.00	
CP.22	**1.00**	2.00	
CP.23	**1.00**	2.00	
CP.24	**2.00**	2.00	
CP.25	**2.00**	2.00	
CP.26	**2.00**	2.00	
CP.27	**0.00**	2.00	
CP.28	**1.00**	2.00	

4/21

Final Exam

No grades available yet.

Totals

id ▲	pts	max	weight	mean	median	stddev
Problem Set	228.59	300.00	0.25	255.65	269.58	41.00
Final Exam	0.00	0.00	0.30	0.00	0.00	0.00
Class participation	36.00	38.00	0.20	36.82	38.00	3.47
Miniquiz	42.50	80.00	0.17	57.65	57.50	12.37
Reading Comments	0.00	0.00	0.03	0.00	0.00	0.00
Tutorial	9.00	9.00	0.05	8.23	9.00	1.55
Grand Total	**52.03**	**67.00**	**1.00**	**57.51**	**59.04**	**6.94**

Note: The totals only reflect grades that have been completely entered for the class. A grade with gray background signifies that the grade has not been completely entered yet.

Note: A grade with red font signifies that the grade has been dropped.

Grade Quartile

Your current rank is: ***4th quartile (79th - 101th)*** out of 101 students.

Grades compiled at: 4/21/11 8:28 AM

Please contact your TA if there is any problem with the grade report.