No class Mon

Wed

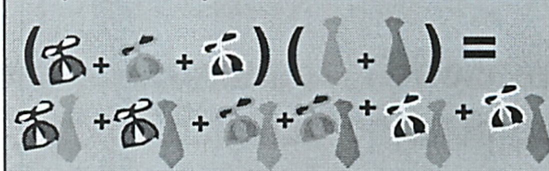
4/20

Mathematics for Computer Science

MIT 6.042J/18.062J

Binomial Theorem, Combinatorial Proof

Polynomials Express Choices & Outcomes



Products of Sums = Sums of Products

expression for c_k?

$$(1+X)^n =$$

$$c_0 + c_1X + c_2X^2 + \ldots + c_nX^n$$

expression for c_k?

$$(1+X)^n = \overbrace{(1+X)(1+X)(1+X)(1+X)\ldots(1+X)}^{n \text{ times}}$$

multiplying gives 2^n product terms:

$11\cdots1 + X11X\cdots X1 + 1XX\cdots1X1 + \cdots + XX\cdots X$

a term corresponds to selecting 1 or X from each of the n factors

expression for c_k?

$$(1+X)^n = \overbrace{(1+X)(1+X)(1+X)(1+X)\ldots(1+X)}^{n \text{ times}}$$

the X^k coeff, c_k, is # terms with exactly k X's selected

$$c_k = \binom{n}{k}$$

The Binomial Formula

binomial expression

$$(1+X)^n = \binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n$$

binomial coefficients

The Binomial Formula

$$(X+Y)^n =$$

$$\binom{n}{0}Y^n + \binom{n}{1}XY^{n-1} + \binom{n}{2}X^2Y^{n-2} + \dots + \binom{n}{k}X^kY^{n-k} + \dots + \binom{n}{n}X^n$$

The Binomial Formula

$$\boxed{(X+Y)^n = \sum_{k=0}^{n}\binom{n}{k}X^kY^{n-k}}$$

multinomial coefficients

What is the coefficient of

$$X_1^{r_1}X_2^{r_2}X_3^{r_3}\dots X_k^{r_k}$$

in the expansion of

$$(X_1+X_2+X_3+\dots+X_k)^n \; ?$$

$$\binom{n}{r_1, r_2, r_3, \dots, r_k}$$

The Multinomial Formula

$$(X_1+X_2+\dots+X_k)^n =$$

$$\sum_{r_1+\dots+r_k=n}\binom{n}{r_1, r_2, \dots, r_k}X_1^{r_1}X_2^{r_2}X_3^{r_3}\dots X_k^{r_k}$$

multinomial coefficients

binomial a special case:

$$\binom{n}{k} = \binom{n}{k, n-k}$$

Preceding slides adapted from:

- *Great Theoretical Ideas In Computer Science*
  Carnegie Mellon Univ., CS 15-251, Spring 2004
  Lecture 10 Feb 12, 2004 by Steven Rudich
- *Applied Combinatorics*, by Alan Tucker

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Algebraic Proof : routine, using

$$\binom{n}{k} ::= \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k(k-1)!(n-k)!} = \frac{n}{k}\binom{n-1}{k-1}$$

Combinatorial Proof

classify subsets of {1,...,n}

\# size k subsets =
\# size k subsets with 1
\+ # size k subsets without 1

Combinatorial Proof

classify subsets of {1,...,n}

$$\underbrace{\binom{n}{k}}_{\text{\# size k subsets}} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Combinatorial Proof

classify subsets of {1,...,n}

$$\underbrace{\binom{n}{k}}_{\text{\# size k subsets}} = \underbrace{\binom{n-1}{k}}_{\text{\# size k subsets without 1}} + \binom{n-1}{k-1}$$

Combinatorial Proof

classify subsets of {1,...,n}

QED

$$\underbrace{\binom{n}{k}}_{\text{\# size k subsets}} = \underbrace{\binom{n-1}{k}}_{\text{\# size k subsets without 1}} + \underbrace{\binom{n-1}{k-1}}_{\text{\# size k subsets with 1}}$$

Combinatorial Proof, II

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Combinatorial Proof, II

classify subsets of {1,…,n,1,….,n}

$$\text{RHS} = \underbrace{\binom{2n}{n}}_{\substack{\#\text{ size n}\\ \text{subsets}}}$$

Combinatorial Proof, II

$$\text{LHS} = \sum_{i=0}^{n} \binom{n}{i}^2$$

$$= \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$$

Combinatorial Proof, II

LHS =

$$\sum_{i=0}^{n} \underbrace{\binom{n}{i}} \binom{n}{n-i}$$

Combinatorial Proof, II

LHS =

$$\sum_{i=0}^{n} \underbrace{\binom{n}{i}}_{\substack{\#\text{ size i}\\ \text{red subsets}}} \binom{n}{n-i}$$

Combinatorial Proof, II

LHS =

$$\sum_{i=0}^{n} \underbrace{\binom{n}{i}}_{\substack{\#\text{ size i}\\ \text{red subsets}}} \underbrace{\binom{n}{n-i}}_{\substack{\#\text{ size n-i}\\ \text{black subsets}}}$$

Combinatorial Proof, II

$$\sum_{i=0}^{n} \underbrace{\binom{n}{i}}_{\substack{\#\text{ size i}\\ \text{red subsets}}} \underbrace{\binom{n}{n-i}}_{\substack{\#\text{ size n-i}\\ \text{black subsets}}}$$

So LHS = # size n subsets of {1,…,n,1,….,n} by Sum Rule

Combinatorial Proof, II

Therefore
LHS = # size n subsets = RHS

$$\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{2n}{n}$$

QED

Team Problems

Problems 1–3

Miniquiz 5

(Thought was total fail
Mostly studied counting
Not other stuff - asymptotics
Should have studied more + wrote down)

Binomial Theorem + Combinatorial proofs

Multiplying out 2 sums
- algebra + counting
- multiply each term on left w/ right

$(A + B + C)(1 + 2)$ is

$$A1 + A2 + B1 + B2 + C1 + C2$$

$(1+X)^n =$

$$c_0 + c_1 X + c_2 X^2 + \dots + c_n X^n$$

Take every possible $\underbrace{(1+X)\dots}_{n \text{ times}}$

Multiplying gives 2^n product terms

②

So when simplify - just product of 1 and x

$1\,1\,1\,1\,1\,1\,x + x\,1\,1\,x \ldots 1 + 1\,x\,x \ldots 1\,x\,1 \ldots + $

↑ $\cancel{0}$ xs ↑ 2 x ↑ 3 xs

So select x k times

1 $(n-k)$ times

So $C_k = \binom{n}{k}$

↑ Algebraic / logical proof

$(1+x)^n =$

$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{k}x^k + \ldots + \binom{n}{n}x^n$

↑ binomial coefficent

Chose 1 to ↓ confusion in formula

$(x+y)^n =$

$\binom{n}{0}y^n + \binom{n}{1}xy^{n-1} + \binom{n}{2}x^2 y^{n-2} + \ldots$ etc

$= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

③

What is coefficent of

$$X_1^{r_1} X_2^{r_2} X_3^{r_3} \dots X_k^{r_k}$$

in expansion of

$$(X_1 + X_2 + X_3 + \dots + X_k)^n \,?$$

$$\binom{n}{r_1, r_2, r_3 \dots r_k}$$

So multinomial coefficent

. . .

(I used this on MQ5 - thought we already studied this)

Binomial is special case

$$\binom{n}{k} = \binom{n}{k, n-k}$$

~~56~~ Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Combintorial Proof

Clasify subsets of $\{1, \dots, n\}$

\# size k subsets = # size k subsets w/ 1 + # size k subsets w/o 1

(4)

~~#6~~ No algebra - just story

(Is in book)

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Classify as 2 subsets < red $\{1, \dots, n\}$, black $\{1, \dots, n\}$

~~RHS~~ RHS is just this ↗

LHS is

$$\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$$

same

$\underbrace{\quad}_{\text{i red els}}$ $\underbrace{\quad}_{\text{n-i black els}}$

So have i total els

So that is story ⇒ QED

In-Class Problems Week 11, Wed.

Problem 1.
Find the coefficients of

(a) x^5 in $(1+x)^{11}$

(b) $x^8 y^9$ in $(3x+2y)^{17}$

(c) $a^6 b^6$ in $(a^2+b^3)^5$

Problem 2.
You want to choose a team of m people for your startup company from a pool of n applicants, and from these m people you want to choose k to be the team managers. You took a Math for Computer Science subject, so you know you can do this in

$$\binom{n}{m}\binom{m}{k}$$

ways. But your CFO, who went to Harvard Business School, comes up with the formula

$$\binom{n}{k}\binom{n-k}{m-k}.$$

Before doing the reasonable thing—dump on your CFO or Harvard Business School—you decide to check his answer against yours.

(a) Give a *combinatorial proof* that your CFO's formula agrees with yours.

(b) Verify this combinatorial proof by giving an *algebraic* proof of this same fact.

Problem 3. (a) Give a combinatorial proof of the following, more interesting theorem:

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} \tag{1}$$

Hint: Let S be the set of all length-n sequences of 0's, 1's and a single $*$.

(b) Now prove (1) algebraically by applying the Binomial Theorem to $(1+x)^n$ and taking derivatives.

In Class 11 Wed

4/20

1.a) Find coefficents of x^5 in $(1+x)^{11}$

- So this is straight forward problem?

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{k}x^k + \dots + \binom{n}{n}x^n$$

So coefficent is $\binom{11}{5}$ ✓

1.b.) $(3x+2y)^{17}$

So

$3x \cdot 2y$

Coefficents to 8

$\binom{17}{8}$ also $3^8 \cdot 2^9$ ✓

c) $a^6 b^6$ $(a^2+b^3)^5$

whole thing to power of 3

$\binom{5}{3} a^{2^3} \cdot b^{3^2}$

$\boxed{a^{2^3} = a^6}$

So coefficent is just $\binom{5}{3}$

②

2. Prove

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$$

; so how to get started?

a) First choose k managers from n applicants. Then chose n-k applicans from those that are left

$$\binom{n}{k}\binom{n-k}{m-k}$$

Oh its that simple - stop + think!

b) $\binom{n}{m}\binom{m}{k} = \frac{n!}{m!(n-m)!} \cdot \frac{m!}{k!(m-k)!}$

$$= \frac{n!}{k!(n-m)!(m-k)!}$$

$$\binom{n}{k}\binom{n-k}{m-k} = \frac{n!}{k!\cancel{(n-k)!}} \frac{\cancel{(n-k)!}}{(m-k)!(n-m)!} = \frac{n!}{k!(n-m)!(m-k)!}$$

$=$

Once you see it, its easy

But how to see it in 1st place?

Prof: It's ~~a~~ easy -obvious

Just expand def + try to move them towards each other

I guess I just need to try it

(3)

3. $n2^{n-1}$ is the # of length $n-1$ binary strings w/ a $*$ inserted somewhere

$\sum_{k=1}^{k=n} k\binom{n}{k}$ is the # of len n binary strings w/ k 1s ~~and~~ with 1s replaced w/ a ~~1~~

And they are the same

$$n2^{n-1} = \sum_{k=1}^{k=n} k\binom{n}{k}$$

$$\frac{d}{dx}(1+x)^n = n(1+x)^{n-1}(1)$$

$$= \frac{d}{dx}\sum_{k=0}^{k=n}\binom{n}{k}x^k \quad \text{w/ } x=1,\ n(2)^{n-1}$$

$$= \sum_{k=0}^{n}\binom{n}{k}k\cdot x^{k-1} \quad \text{So } \sum_{k=1}^{k=n}\binom{n}{k}k = n2^{n-1}$$

w/ $x=1$ $\sum_{k=0}^{n}\binom{n}{k}k$

Solutions to In-Class Problems Week 11, Wed.

Problem 1.
Find the coefficients of

(a) x^5 in $(1+x)^{11}$

Solution.

$$\binom{11}{5} = 462$$

■

(b) x^8y^9 in $(3x+2y)^{17}$

Solution.

$$\binom{17}{8}3^8 2^9.$$

When $(3x+2y)^{17}$ is expressed as a sum of powers of the summands $3x$ and $2y$, the coefficient of $(3x)^8(2y)^9$ is $\binom{17}{8}$, so the coefficient of x^8y^9 is this binomial coefficient times $3^8 \cdot 2^9$. ■

(c) a^6b^6 in $(a^2+b^3)^5$

Solution. $a^6b^6 = (a^2)^3(b^3)^2$, so the coefficient is

$$\binom{5}{3} = 10$$

■

Problem 2.
You want to choose a team of m people for your startup company from a pool of n applicants, and from these m people you want to choose k to be the team managers. You took a Math for Computer Science subject, so you know you can do this in

$$\binom{n}{m}\binom{m}{k}$$

ways. But your CFO, who went to Harvard Business School, comes up with the formula

$$\binom{n}{k}\binom{n-k}{m-k}.$$

Before doing the reasonable thing—dump on your CFO or Harvard Business School—you decide to check his answer against yours.

(a) Give a *combinatorial proof* that your CFO's formula agrees with yours.

Solution. Instead of choosing first m from n and then k from the chosen m, you could alternately choose the k managers from the n people and then choose $m-k$ people to fill out the team from the remaining $n-k$ people. This gives you $\binom{n}{k}\binom{n-k}{m-k}$ ways of picking your team. Since you must have the same number of options regardless of the order in which you choose to pick team members and managers,

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}.$$

Formally, in the first method we count the number of pairs (A, B), where A is a size m subset of the pool of n applicants, and B is a size k subset of A. By the Generalized Product Rule, there are

$$\binom{n}{m}\cdot\binom{m}{k}$$

such pairs.

In the second method, we count pairs (C, D), where C is a size k subset of the applicant pool, and D is a size $(m-k)$ subset of the pool that is disjoint from C. By the Generalized Product Rule, there are

$$\binom{n}{k}\cdot\binom{n-k}{m-k}$$

such pairs.

These two expressions are equal because there is an obvious bijection between the two kinds of pairs, namely map (A, B) to $(B, A-B)$. ■

(b) Verify this combinatorial proof by giving an *algebraic* proof of this same fact.

Solution.

$$\begin{aligned}
\binom{n}{m}\binom{m}{k} &= \frac{n!}{m!(n-m)!}\frac{m!}{k!(m-k)!} \\
&= \frac{n!}{(n-m)!k!(m-k)!} \\
&= \frac{n!(n-k)!}{(n-m)!k!(m-k)!(n-k)!} \\
&= \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(n-m)!(m-k)!} \\
&= \frac{n!}{k!(n-k)!}\frac{(n-k)!}{((n-k)-(m-k))!(m-k)!} \\
&= \binom{n}{k}\binom{n-k}{m-k}.
\end{aligned}$$

■

Problem 3. **(a)** Give a combinatorial proof of the following, more interesting theorem:

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} \tag{1}$$

Hint: Let S be the set of all length-n sequences of 0's, 1's and a single *.

Solution. Let $P ::= \{0, \ldots, n-1\} \times \{0, 1\}^{n-1}$. On the one hand, there is a bijection from P to S by mapping (k, x) to the word obtained by inserting a * just after the kth bit in the length-$n-1$ binary word, x. So

$$|S| = |P| = n2^{n-1} \tag{2}$$

by the Product Rule.

On the other hand, every sequence in S contains between 1 and n nonzero entries since the $*$, at least, is nonzero. The mapping from a sequence in S with exactly k nonzero entries to a pair consisting of the set of positions of the nonzero entries and the position of the * among these entries is a bijection, and the number of such pairs is $\binom{n}{k}k$ by the Generalized Product Rule. Thus, by the Sum Rule:

$$|S| = \sum_{k=1}^{n} k\binom{n}{k}$$

Equating this expression and the expression (2) for $|S|$ proves the theorem. ■

(b) Now prove (1) algebraically by applying the Binomial Theorem to $(1+x)^n$ and taking derivatives.

Solution. By the Binomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Taking derivatives, we get

$$\begin{aligned} n(1+x)^{n-1} &= \sum_{k=0}^{n} k\binom{n}{k} x^{k-1} \\ &= \frac{1}{x}\sum_{k=0}^{n} k\binom{n}{k} x^k. \end{aligned} \tag{3}$$

Letting $x = 1$ in (3) yields (1).

■

Problem Set 9

Due: April 22

Reading: Chapter 15.10–15.13, Inclusion-exclusion, Pigeon Hole Principle, and Combinatorial Proof

Problem 1.
The Magician can determine the 5th card in a poker hand when his Assisant reveals the other 4 cards. Describe a similar method for determining 2 hidden cards in a hand of 9 cards when your Assisant reveals the other 7 cards.

Problem 2.
Let's develop a proof of the Inclusion-Exclusion formula using high school algebra.

(a) Most high school students will get freaked by the following formula, even though they actually know the rule it expresses. How would you explain it to them?

$$\prod_{i=1}^{n}(1-x_i) = \sum_{I \subseteq \{1,\dots,n\}} (-1)^{|I|} \prod_{j \in I} x_j. \tag{1}$$

Hint: Show them an example.

For any set, S, let M_S be the *membership* function of S:

$$M_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

Let $S_1, \dots, S_n$ be a sequence of finite sets, and abbreviate M_{S_i} as M_i. Let the domain of discourse, D, be the union of the S_i's. That is, we let

$$D ::= \bigcup_{i=1}^{n} S_i,$$

and take complements with respect to D, that is,

$$\overline{T} ::= D - T,$$

for $T \subseteq D$.

(b) Verify that for $T \subseteq D$ and $I \subseteq \{1, \dots n\}$,

$$M_{\overline{T}} = 1 - M_T, \tag{2}$$

$$M_{(\bigcap_{i \in I} S_i)} = \prod_{i \in I} M_{S_i}, \tag{3}$$

$$M_{(\bigcup_{i \in I} S_i)} = 1 - \prod_{i \in I}(1 - M_i). \tag{4}$$

(Note that (3) holds when I is empty because, by convention, an empty product equals 1, and an empty intersection equals the domain of discourse, D.)

(c) Use (1) and (4) to prove

$$M_D = \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \prod_{j \in I} M_j. \tag{5}$$

(d) Prove that

$$|T| = \sum_{u \in D} M_T(u). \tag{6}$$

(e) Now use the previous parts to prove

$$|D| = \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \left| \bigcap_{i \in I} S_i \right| \tag{7}$$

(f) Finally, explain why (7) immediately implies the usual form of the Inclusion-Exclusion Principle:

$$|D| = \sum_{i=1}^{n} (-1)^{i+1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = i}} \left| \bigcap_{j \in I} S_j \right|. \tag{8}$$

Problem 3.
Give a combinatorial proof for this identity:

$$\sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \binom{n}{i,\,j,\,k} = 3^n$$

Doing P-set 9

Magican trick but for 2 hidden cards in 9

- Seems like simple extension

But I have to think of it

Keys

which card is kept hidden ↙ well really

Sets 5 $\xrightarrow{\text{bij}}$ seq 4

set size prove

Set 9 | seq 7

$\cancel{\frac{7!}{?}}$ | 7! of those 9 cards

how many are the same - permutation | 5040

oh

$\binom{52}{9}$ = ~~52~~

which is

$\frac{52!}{9!\,43!}$

3 billion + ← must be wrong

(2)

So - wait is it more than 7!
for hidden card?

Think for 4! - no we no hidden card by default
No - can be ~~2 or 5~~ any of 5 cards
- thought was wrong!

So right is 9! - no since no order on 2

So its

Left $\frac{9 \cdot 8}{2} \cdot 7!$

since $12 = 21$

So $\frac{9!}{2}$

? this is what magican picks

Right is possibilities for hidden card
$52 - 7 = 45$

~~Ok~~ I got what is on right + left wrong
- But they said to do that
- Must have done something wrong ...

(3)

Sets of 5 cards

$5 \cdot 4!$

I don't see how this matches

- emailed in

(have not emailed in long time)

seq 4 cards

48 poss for 5th

So

~~$\cancel{\;}$~~

$\frac{9!}{2}$

45

181440

Check degree constrained (Def 11.5.5)

$\deg(l) \geq \deg(r)$

That means there is a matching (Theorem 11.5.6)

Do I have to do "real trick" too?

Have $\binom{52}{9} = 3679075400$ subsets

Too much to remember! - even for a PC!

9

~~share~~
So before 2 had to have suit
- So first was suit

Then the orders

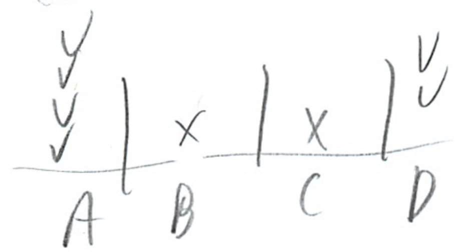
So 1st suit = one hidden
2nd suite = 2nd hidden

?but can't guarentee!?
~~need~~ 9 means ~~2 of each~~ No!
at least one has 3 cards same site
does not say everyone there > 2 x
~~ⓑ~~ but w/ limit of 4 - will be at least 3
cards = pigons
suites = holes
diff suits
?ah ha - so will alway share at least
2 of each suite
- still pigeonhole, but mod I belive
- well diff things holes?

Cards = pigons
Suites = holes
but how diff? max 4 in each hole
diff fill fn?

(5) but no - does not say if suite, will be 2

Can do



x = hidden card

- won't work!

We will be delt 3 suits - but may not have 2

How to Indicate?

Do we have to do?

Wait not 4~~8~~5 possibilities

45^2 since need 2

Emailed in asking do we need real trick

Can write up first part

Wait is it 45^2

First card is $52-7=45$ but 2nd must be diff

$52-8=44$ $= 45 \cdot 44$

Order does not matter

(6)

But did we say order matters

- permutations - so yes

So must divide by 2 ???

Yes - I think - magicion does not need order

Email back Julia

2 - 5 card tricks

3 cards w/ same suite - show 1 hide others 2

- oh

- but what about 4-4-1? ..

HLM

- ? so like each ~~code~~ half is seperate

Overlapping card

Jwang did something slightly different

Where sit of first card matches suit of hidden card

Ohhhh - you can choose which cards to hide.

~~I~~ Julia Right!

(7)

But no - will be ... Sometimes 3 or 4

Oh just need to match - can be 3 or 4 cards
1

#2 I freak out!
Seems to hand hold through

a) $\prod$ = product

$(1-x_1)(1-x_2) \dots (1-x_n)$

$$\sum_{I \subseteq \{1,\dots,n\}} (-1)^{|I|} \prod_{j \in I} x_j$$

size of set esentially n, right

Is a subset or =

Where are ( )?

$\left(\sum (-1)^n\right)\left(\prod x_j\right)$ ← guessing

Or

$\sum \left(-1 \prod x_j\right)$

Was explained in book
Or was it the lecture notes?

2

Binomial Theorm

- which I don't really get

Was in notes

2^n terms

- select either 1 or X from each of N factors

  - seems random??
  - how do you select?

C_k = # terms w/ exactly k X's selected

$$C_k = \binom{n}{k}$$

emailed in

but think my self

$$(1+x)(1+x)$$ ← so $n=2$

$$1\cdot 1 + 1\cdot x + 1\cdot x + x\cdot x$$

$$1 + 2x + x^2$$

$$(1+x)(1+x)(1+x)$$ $n=3$

$$1\cdot 1\cdot 1 + 1\cdot 1\cdot x + 1\cdot x\cdot x + xxx$$

↑3 ↑3 more

1 x x
x 1 x
x x 1

$$x^3 + 3x^2 + 3x + 1$$ no ?

③

$(1+x)^4$

~~WA~~: $x^4+4x^3+6x^2+4x+1$

So how does this relate to our formula

$$\sum (-1)^4 \prod_{1,2,3,4} x_i$$

flips?

Oh no

- all subsets of - including ~~whole~~ thing
  whole

  - but $(-1)^{|I|}$ flips

So think it's

$$\sum \left((-1) \prod x\right)$$

So ~~set~~ $I = \{1,2,3,\cancel{4}\}$ do 3 - easier

So $\{1\}$

$$\sum (-1)^1 \prod x_1$$

$\{2\}$ always -1

-1 ~~xxx~~ x_2

$\{3\}$ $-1\ x_3$

(f)

~~$\{1\}$~~ always -1

~~$-1\ x$~~

$\{1, 2\}$

$1 \quad x_1 \cdot x_2$

$\{1, 3\}$

$1 \quad x^2$

$\{2, 3\}$

$1 \quad x^2$

$\{1, 2, 3\}$

$-1 \quad x^3$

So

$$-3x + 3x^2 - x^3$$

And $\{\ \}$ empty set

$$1$$

So $-x^3 + 3x^2 - 3x + 1$

So was wrong on $-$ signs

Slides say

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n$$

5

But where does Θ go in that formula?

Oh $(1-x)$ - opps!

? Θ

But how to describe - they just say example

It was $(1+x)$ in class

What do we do about the subscripts?

---

b) $M_S(x)$ is membership

- 1 if is
- 0 if not

$S_1, \ldots, S_n$ are seq of finite sets

$D = \bigcup_{i=1}^{n} S_i$

? so union

Then M_i is if in

$\overline{T} = D - T$

What is T? $T \subseteq D$

? every possible subset

Verify for $T \subseteq D$ and $I \subseteq \{1, \ldots, n\}$

does not mean anything

⑥

$M_{\bar{T}} = 1 - M_T$

(by definition?

(or is it)

Rearrange

$M_T + M_{\bar{T}} = 1$

(since must be in or out

But T is whole set

- does not matter?

T is a set

- some subset of D

$T + \bar{T} = D$

$(D = \bigcup_{i=1}^{n} S_i$

So basically split into 2 categories

But then $M_{S(x)}$ is true only if <u>everything</u> is in set?

---

(3) $M_{(\bigcap_{i \in I} S_i)} = \prod_{i \in I} M_{S_i}$

So what is this?

(7) ~~[illegible]~~

$\wedge$ = And

Yes because if any one is 0 - its false

~~(3a)~~ (4)

$$M\left(\bigvee_{i \in I} S_i\right) = 1 - \prod(1 - m_i)$$

¿ This is combo of above?

$M(\vee)$ true if any one is true

$1 - M_i$ means flip result

$1 \to 0$

$0 \to -1$ ← not quite

then multiply

if any true $= 0$

So

$1 - 0 = 1$

Complicated - who thought of this?

8

c) So now prove

(1) $\prod_{i=1}^{n}(1-x_i) = \sum_{I \subseteq \{1,\dots,n\}} (-1)^{|I|} \prod_{j \in I} x_j$

(4) $M_{(\bigcup_{i \in I} S_i)} = 1 - \prod_{i \in I}(1-M_i)$

$M_D = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \prod_{j \in I} M_j$

? what is this?

; all except $\emptyset$

? M_D is ~~one~~ is whole domain included?

$D = \bigcup S_i$

So just ~~a add~~ multiply M_i

but why need sets?

Oh $(1-M_i)$ is like $(1-x_i)$

$M_{\bigcup S_i} = 1 - (1)$ formula

? which is D

but add 1 so $+1$

nice

(I still don't see how it works)

(9)

Why exclude $\emptyset$?

?

---

d) Prove that

$$|T| = \sum_{v \in D} M_T(v)$$

what is fn for?

T is $\subseteq D$

random subset of d

1 if is in set

So add up?

Oh fn is defied

~~x is test S is wher~~

test if x is in S

" " v " " T there

Hopefully my proofs are good enough
- never know what to do
- Putting a lot of work/thought into this P-set

(10)

e) Prove

$$|D| = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \left|\bigcap_{i \in I} S_i\right|$$

So this is just pulling stuff togeter ;

$\bigcap S_i$ means all must be true.

Goes trough every subset and returns 1 if everything is true

?subject to even/odd from (1)

(I am starting to just write what I think verbally - which I think is what he wants)

No $\left|\bigcap_{i \in I} S_i\right|$ is just # of eles in I

Notice its $|D|$ - size of domain of discourse

oh inclusion exclusion

But that is # 8 - how go ~~to~~ to 7

M is replaced by S

~~D~~ M_d ~~Md~~ D

Hope good enough

11

f) Kinda already did

#3

Comnitorial proof ?

Well it should not be so bad - walk through it like other problems

Tell a story

Multinomial

$$\binom{n}{i,j,k} = \frac{n!}{i!\,j!\,k!}$$ ← book keeper rule

But $\sum_{\substack{i+j+k=n \\ i,j,k \geq 0}}$ ← So what are we summing over?

are just conditions

Make up a story

What is 3^n

$$\underbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdots 3}_{n \text{ times}}$$

$\sum_i$ mean the # of combos?

$\binom{1}{.25 \; .25 \; .5}$ is 1.37 not 3

- but sum - over what?

②

Matt's Help

i are same
j same
k are rest

3 diff els

So every combo of 3 elements
i, j, k

3^n - n boxes 3 diff values

Or n boxes, i have 1 value, j have another
k has third value

Can arrang in any diff ways $\Sigma()$

So =

(Matt saved me again!)

$3^n = n$ boxes 3 diff ~~ways~~

I wrote values

Say $n=2$

AB BA
AC BB
AA BC

③ What rule is this?

Multiplication rule

- I am figuring stuff out myself!

Why does it not work when I try

$$\frac{6!}{2!2!2!} = 90$$

but $3^6 = 729$

Re asked Matt

all possible combos!

I had 2, 2, 2

but 0, 0, 6

0, 6, 0

1, 1, 4

0, 0, 5 etc

Massachusetts Institute of Technology
6.042J/18.062J, Spring '11: Mathematics for Computer Science
Prof. Albert R Meyer

Solutions cover sheet
April 15

Student's Solutions to Problem Set 9

Your name: Michael Plasmeier

Due date: April 22

Submission date: 4/22

Circle your TA/LA: Ali Nick Oscar Oshani **Table number** 12

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

got help from:[1] Matt Fauk, Albert Meyer (email), Julia Zimmerman (via 6042-forum)

and referred to:[2]

jwang7

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	10
2	
3	
Total	

/10

[1] People other than course staff.

[2] Give citations to texts and material other than the Spring '11 course materials.

#1 We can check if it is possible.
On the left side, write all of the possible sequences - including which cards are missing.
This is simply $9!$, where we arbitrarily define the last 2 slots as hidden. However we do not care about the order of these last 2 cards. A hidden card is a hidden card. This means we have a 2-to-1 relationship because either order is the same thing. So via the divison rule, we have $\frac{9!}{2}$ on the left. This is 181440.

On the right we write all of the possible sets of what the 2 missing cards could be. If it was 1 card missing (from 8 inital chosen) - we would have 45 possible cards (since we know 7 cards) $(52-7=45)$. However we also have another card missing. This must be one of $52-8=44$ cards since we now know 8 cards. We also need to divide by 2 since the magican does not need to get the "order" of the two hidden cards - since there is no order. So we have

②

$$\frac{45 \cdot 44}{2} = 990$$

So on the left 181440, right 990

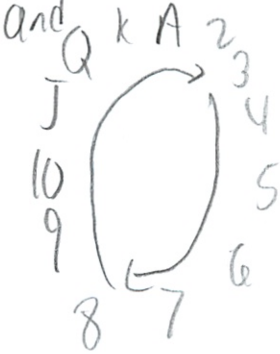
This meets our requirements, since
Def 11.5.5 says a graph is degree constrained
When $\deg(l) \geq \deg(r)$
This def. is met.

Theorm 11.5.6 says that if G is a degree constrained bipartite graph, then there is a matching that covers L(G)
(proof in book)

This theorm is satisfied, so there is a matching
Because there is a matching, a particular seq which the assistant arranges can map to 1 specific set of missing cards

(3) To actually do the mapping, so that the magican can do the trick w/o requiring the memory of a computer - picture this as 2 5-card tricks.

However, by the Pigon Hole principle we know that there must be a suit with at least 3 cards. (It could have 4 cards) Choose the 2 cards to hide such that they are the same suit. Choose a third card that is also the same suit (The suit must have ≥ 3 cards). put this card as the first visable card. Now 6 cards remain. Organize the next 3 to give the rank of the first hidden card (well order does not matter) using the same SML and



scheme as described in class.

Do the same for the 2nd hidden card.

ok

Michael Plasmeier
Oshani
Table 12

14.5

P-Set 9

#2a. Start by breaking it down and defining the pieces.

$\prod$ means multiplication, kinda like Σ, except $\otimes$ not $\oplus$

2.5

$\prod_{i=1}^{n}(1-x_i)$ also means $(1-x)^n$ since we are simply multiplying this formula. The formula also tries to "identify" xs with subscripts. For now lets assume that all x_is are the same and later we can break it out.

The summation has a subset symbol $\subseteq$. This means add over all possible subsets. For the set $\{1,2,3\}$ all subsets include

$\{1,2,3\}$,
$\{1,2\}, \{2,3\}, \{1,3\}$,
$\{1\}, \{2\}, \{3\}$,
$\emptyset$ empty set

②

We will try each possible subset.

$(-1)^{|I|}$ is next. $|I|$ is size of set I. - This is how many items are currently in the set we are dealing with. $(-1)^{\text{this}}$ is basically an even/odd $\oplus/\ominus$ alternator. When $|I|$ is odd then we are negitive, If $|I|$ is even, then we are positive. This is because 2 negitive signs make a positive as we will show in the example.

$\prod x_i$ means multiply the xs together that are in our current set. When all xs are the same this is basically $x^{|I|}$ - but when xs are different, each one is multiplied together sepertly

So now an example where $n=3$ and x_is are the same

$$\prod_{i=1}^{n}(1-x) = (1-x)^n = (1-x)(1-x)(1-x)$$

Normally we would go through this equation and make sure we get every possible combo. For 2, we used the expression "FOIL" for first, outside, inside, last.

(3) Things are a bit more complicated — we have a lot more terms

But lets try to list them out

$1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot -x + 1 \cdot -x \cdot 1 + -x \cdot 1 \cdot 1 + 1 \cdot -x \cdot -x + -x \cdot 1 \cdot -x + -x \cdot -x \cdot 1$
$+ -x \cdot -x \cdot -x$

We can simplify this

$1 - 3x + 3x^2 - x^3$

Remember 2 (or even) # of ⊖ means ⊕

The formula gives us the same thing. Lets go through every possible subset - and add them together

$\{1,2,3\}$

$(-1)^3 x^3$

$= -x^3$

$\{1,2\}$ — and same for 2 other lenght 2 subsets $\{1,3\}$ $\{2,3\}$

$(-1)^2 x^2$

$= x^2$

$\{1\}$

$(-1) x$ — and same for $\{2\} \{3\}$

$= -x$

$\emptyset$ Don't forget

$(-1)^0 = 1$

(9) ...

Added together we have

$$-x^3 + 3x^2 - 3x + 1$$

Same as formula!

(5)

b) (2) $M_{\bar{T}} = 1 - M_T$

Rearrange

$M_T + M_{\bar{T}} = 1$

Since M_T is either 1 or 0, the complement $M_{\bar{T}}$ must also be 1 or 0

M_T		$M_{\bar{T}}$	$M_T + M_{\bar{T}}$
1	$\rightarrow$	0	$1+0=1$
0	$\rightarrow$	1	$0+1=1$

So sum always $= 1$

T is a subset of D. $\bar{T}$ is the complement, So basically

$$T + \bar{T} = D$$

So items from D either go in T or $\bar{T}$

M_T is only true (1) if <u>all</u> items are in S?

This is either 1 or 0, see above

b) (3) $M_{(\bigcap_{i \in I} S_i)} = \prod_{i \in I} M_{S_i}$

With both - all of the S_i (for $i \in I$) must be true. If any one of them is false - then the whole thing is false. The LHS uses intersection also called AND - it combines all the S_is together. If one is false than the test is false. The RHS tests each S_i individually. However it then multiplies the tests together. If any test is false, then
$\uparrow = 0$

*by test I mean membership function

the string of multiplication will be false
$\uparrow = 0$.

This is because anything $\cdot 0 = 0$

Also note that this holds when I is empty because by convention an empty product = 1 and an empty intersection equals the entire domain of discourse D.

8 . .

b) (4) $M_{(\bigcup_{i \in I} S_i)} = 1 - \prod_{i \in I} (1 - M_i)$

This is basically saying that if any 1 of them is true, then the statement is true,

$\boxed{\begin{array}{l}\text{true means } = 1 \\ \text{false means } = 0\end{array}}$

$1 - M_i$ will be

$1 - 1 = 0$ if true
$0 - 1 = -1$ if false.

All of these will be multiplied together. If any one of the M_{is} is true, then the result will be 0 for reasons described above. If all are false then $\prod 1 - M_i$ will be -1 since $(-1)^n$ will be either 1 or -1

Then this result is subtracted from 1. If at least one M_i is true $1 - 0 = 1$ and the RHS = LHS. If all M_is are false then either $1 - 1 = 0$, $1 - -1 = 2$ both indicate that the statement is false.

8.

Although strictly speaking $M_{S(x)}$ should be redefined as

$$M_{S(x)} = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

8 ..

c) Prove $M_D = \sum_{\emptyset \neq I \{1,\dots,n\}} (-1)^{|I|+1} \prod_{j \in I} M_j$

Using

(1) $\prod_{i=1}^{n} (1-x_i) = \sum_{I \subseteq \{1,\dots,n\}} (-1)^{|I|} \prod_{j \in I} x_j$

(4) $M_{(\bigcup_{i \in I} S_i)} = 1 - \prod_{j \in I} (1-M_i)$

First M_D is M_i of whole domain D

$D = \bigcup_{i=1}^{n} S_i$ which is $\bigcup_{i \in I} S_i$

$\prod_{j \in I} (1-M_i)$ is $\prod_{i=1}^{n} (1-x_i)$ which is (1)

So we have

$$M_{(\bigcup_{i=1}^{n} S_i)} = 1 - \sum_{I \subseteq \{1,\dots,n\}} (-1)^{|I|} \prod_{j \in I} M_j$$

Now put the 1 inside the summation symbol

And exclude the $\emptyset$

why??

(9) . .

Now ya have it

$$M_D = M\left(\bigcup_{i=1} S_i\right) = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \prod_{j\in I} M_j$$

(10) ..

d) Prove $|T| = \sum_{v \in D} M_T(v)$

T is $\subseteq D$
some random subset of D

$M_T(v)$ means test if v is in T

So this just goes through (iterates) every item in D (calling it v) and seeing if it is in T. If v is in T $M_T(v)$ returns 1, which is added to the sum (accumulator). Because we try every element in D and $T \subseteq D$, the sum will be = to the # of elements in T or $|T|$. $\square$

(11)

e) Prove $|D| = \sum_{\emptyset \neq I \subseteq \{1, ..., n\}} (-1)^{|I|+1} \left| \bigcap_{i \in I} S_i \right|$

This builds on previous formulas, namly (5) and (6).

The $\left| \bigcap_{i \in I} S_i \right|$ part is the size of the intersection (overlapping) parts of the set that is being tested at the particular moment. A particular item must be in every set.

Notice also that the LHS is $|D|$ - we seem to be building up the inclusion-exclusion principle

As we include more items in our particular subset for the moment - we alternate adding and subtracting - just like to inclusion-exclusion principle.

But first we want to know how to get here. This is like (5) but we replace M_i by S_i
M_D by D

We no longer check if something is a member - we just include it. We saw this in (6) - that adding up all of the members gives ~~you~~ the size.

2. see solution

(12) ..

f) Explain how this is Inclusion - Exclusion

$$|D| = \sum_{i=1}^{n} (-1)^{i+1} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I| = i}} \left|\bigcap_{j \in I} S_j\right|$$

I kinda already started in the last problem - we are adding and removing alternating "layers" - subsets of increasing size.

We get (8) from (7) by rewriting some of the set notation into simple summation.

✓

Michael Plasmeier
Oshani
Table 12
P-Set 9

4

#3 For 3^n LHS we can arrange n boxes having 3 different values. This is the multiplication rule. The first box can be an A, B, or C. The second box can be an A, B, or C. Repeats are allowed. This repeats n times, hence 3^n

For RHS imagine we have n boxes. i of them have 1 value (say A), j of them will have another value (say B) and k of them will have a third value (say C). You can arrange these in different ways, equal to $\binom{n}{i,j,k}$ via the "Bookkeeper" rule. But there are different ways of designating the boxes. The Bookkeeper rule just considers the way to arrange the boxes once they have been labled A,B,C. We must also consider other lablings

This is confusing

2 For example if $n=3$ could have i,j,k

$$\begin{array}{lll} 0,0,3 & 1,0,2 & 1,1,1 \\ 0,3,0 & 0,1,2 & \\ 3,0,0 & 2,0,1 & \end{array}$$

So we must add all of these combos as well so that we have n boxes, each randomly labled A, B, C. Then this would $= 3^n$, so both ways of adding the boxes are the same! $\square$

Solutions to Problem Set 9

Reading: Chapter ??–??, Inclusion-exclusion, Pigeon Hole Principle, and Combinatorial Proof

Problem 1.
The Magician can determine the 5th card in a poker hand when his Assisant reveals the other 4 cards. Describe a similar method for determining 2 hidden cards in a hand of 9 cards when your Assisant reveals the other 7 cards.

Solution. Since there must be $\lceil 9/4 \rceil = 3$ cards with the same suit, our collaborator chooses to hide two of them and then use the third one as the first card to be revealed. So this first revealed card fixes the suit of the two hidden cards; it will also be used as the origin for the offset position of the first hidden card. This first hidden card will in turn be used as the origin for the offset of the other hidden card. There are six cards to code the two offset positions. These suffice to code two offsets of size from one to six. That is, our collaborator can choose one of the $3! = 6$ orders in which to reveal the first three cards and thereby tell us the offset position of the first hidden card. Our collaborator can then choose the order of the final three cards to describe the offset position of the second hidden card from the first. Note that the first revealed card must be chosen so that both offsets are ≤ 6; since the sum of the offsets between successive cards ordered in a cycle from Ace to King is 13, it is not possible for more than one offset between successive cards to exceed seven, so this is always possible.

■

Problem 2.
Let's develop a proof of the Inclusion-Exclusion formula using high school algebra.

(a) Most high school students will get freaked by the following formula, even though they actually know the rule it expresses. How would you explain it to them?

$$\prod_{i=1}^{n}(1 - x_i) = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \prod_{j \in I} x_j. \tag{1}$$

Hint: Show them an example.

Solution. Let's do an example. To "multiply out"

$$(1 - x_1)(1 - x_2)(1 - x_3), \tag{2}$$

you would form *monomial* products by selecting some of the $(-x_i)$'s to multiply together. For example, selecting $(-x_i)$'s with

- $i \in \{1, 3\}$ leads to the monomial $(-x_1)(-x_3) = (-1)^2 x_1 x_3 = x_1 x_3$,
- $i \in \{1, 2, 3\}$ leads to the monomial $(-x_1)(-x_2)(-x_3) = (-1)^3 x_1 x_2 x_3 = -x_1 x_2 x_3$, and
- $i \in \emptyset$ leads (by convention) to the monomial 1.

Then you sum up the monomials from *all possible* selections to get

$$(1-x_1)(1-x_2)(1-x_3) = 1 - x_1 - x_2 - x_3 + x_1x_2 + x_1x_3 + x_2x_3 - x_1x_2x_3.$$

Now we can decipher (1) as saying to do the same thing for the product of n different $(1-x_i)$'s: for any selection of $(-x_i)$'s with i in some subset, $I \subseteq \{1,\ldots,n\}$, multiply the $(-x_i)$'s to get the monomial

$$\prod_{i\in I}(-x_i) = \prod_{i\in I}(-1)^{|I|}x_i,$$

and sum up all such monomials obtained by every possible selection, I, to get the right hand side of equation (1). ■

For any set, S, let M_S be the *membership* function of S:

$$M_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

Let $S_1, \ldots, S_n$ be a sequence of finite sets, and abbreviate M_{S_i} as M_i. Let the domain of discourse, D, be the union of the S_i's. That is, we let

$$D ::= \bigcup_{i=1}^{n} S_i,$$

and take complements with respect to D, that is,

$$\overline{T} ::= D - T,$$

for $T \subseteq D$.

(b) Verify that for $T \subseteq D$ and $I \subseteq \{1, \ldots n\}$,

$$M_{\overline{T}} = 1 - M_T, \tag{3}$$

$$M_{(\bigcap_{i\in I} S_i)} = \prod_{i\in I} M_{S_i}, \tag{4}$$

$$M_{(\bigcup_{i\in I} S_i)} = 1 - \prod_{i\in I}(1 - M_i). \tag{5}$$

(Note that (4) holds when I is empty because, by convention, an empty product equals 1, and an empty intersection equals the domain of discourse, D.)

Solution. To prove (3), we have for all $u \in D$,

$$\begin{aligned} M_{\overline{T}}(u) = 1 \quad &\text{iff} \quad u \in \overline{T} \quad \text{iff} \quad M_T(u) = 0 \quad \text{iff} \quad 1 - M_T(u) = 1, \\ M_{\overline{T}}(u) = 0 \quad &\text{iff} \quad u \notin \overline{T} \quad \text{iff} \quad u \in T \quad \text{iff} \quad M_T(u) = 1 \quad \text{iff} \quad 1 - M_T(u) = 0, \end{aligned}$$

so $M_{\overline{T}}(u) = 1 - M_T(u)$.

Similarly, to prove (4),

$$M_{(\bigcap_{i\in I} S_i)}(u) = 1 \quad \text{iff} \quad u \in \bigcap_{i\in I} S_i \quad \text{iff} \quad \bigwedge_{i\in I} u \in S_i \quad \text{iff} \quad \bigwedge_{i\in I}[M_i(u) = 1] \quad \text{iff} \quad \left(\prod_{i\in I} M_i(u)\right) = 1.$$

Finally, (5) follows from (3) and (4) by DeMorgan's Law. ■

(c) Use (1) and (5) to prove

$$M_D = \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \prod_{j \in I} M_j. \tag{6}$$

Solution.

$$\begin{aligned}
M_D &= M_{(\bigcup_{i=1}^n S_i)} \\
&= 1 - \prod_{i=1}^{n} (1 - M_i) && \text{by (5)} \\
&= 1 - \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \prod_{j \in I} M_j && \text{by (1)} \\
&= 1 - \left(1 + \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \prod_{j \in I} M_j \right) && (\prod_{j \in \emptyset} M_j ::= 1) \\
&= \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \prod_{j \in I} M_j.
\end{aligned}$$

■

(d) Prove that

$$|T| = \sum_{u \in D} M_T(u). \tag{7}$$

Solution.

$$\sum_{u \in D} M_T(u) = \sum_{u \in T} M_T(u) + \sum_{u \in \overline{T}} M_T(u) = \left(\sum_{u \in T} 1\right) + \left(\sum_{u \in \overline{T}} 0\right) = |T| + 0 = |T|,$$

■

(e) Now use the previous parts to prove

$$|D| = \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \left| \bigcap_{i \in I} S_i \right| \tag{8}$$

Solution. Summing both sides of (6) over $u \in D$, we have

$$\begin{aligned}
|D| &= \sum_{u \in D} M_D(u) && \text{(by (7))} \\
&= \sum_{u \in D} \left(\sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \prod_{j \in I} M_j(u) \right) && \text{(by (6))} \\
&= \sum_{u \in D} \left(\sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} M_{\bigcap_{i \in I} S_i}(u) \right) && \text{(by (4))} \\
&= \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \left(\sum_{u \in D} M_{\bigcap_{i \in I} S_i}(u) \right) && \text{(reversing the order of sums)} \\
&= \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \left| \bigcap_{i \in I} S_i \right| && \text{(by (7))}.
\end{aligned}$$

■

(f) Finally, explain why (8) immediately implies the usual form of the Inclusion-Exclusion Principle:

$$|D| = \sum_{i=1}^{n} (-1)^{i+1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=i}} \left| \bigcap_{j \in I} S_j \right|. \tag{9}$$

Solution. We obtain (9) from (8) by breaking up the sum over nonempty subsets, $I \subseteq \{1, \ldots, n\}$, into separate sums over all the subsets of size i, for $1 \le i \le n$. ■

Problem 3.
Give a combinatorial proof for this identity:

$$\sum_{\substack{i+j+k=n \\ i,j,k \ge 0}} \binom{n}{i, j, k} = 3^n$$

Solution. Number of sequences of length n composed of digits 0,1, and 2. ■

TP 10

4/21

TP 10.1 Binomial Coefficents

What is coefficent of x^4?
\# in front of

a) $(x+1)^9$

? how to expand easily the ~~(lm~~ $\binom{n}{k}$ thing

$\binom{9}{4}$

126

b) $(3x+2)^6$

$\binom{6}{4}(3x)^4(2)^2$

$15 \cdot 81 \cdot 4 =$

4860

②

TP 10.2 Combinatorial Identity

n balls

L pick k and r −k

Could pick r then partition $\langle\begin{smallmatrix} k \\ r-k \end{smallmatrix}$

Could pick k

and pick r−k from n−k

(? Pascals Idenity ?) No

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

TP 10.3 Pascal's Theorm on Crack

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

1. $\binom{n}{n}$ = what

always 1

2 same $\binom{n+1}{n+1}$

3

2. Now which = $\binom{n}{n} + \binom{n+1}{n}$

$\uparrow 1$ $\uparrow n+1$ ← from WA - but how?

Say $\binom{3}{2}$ ABC

AB BC
AC = 3

So $\binom{n+2}{n+1}$ = to ?

3. Now = $\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n}$

$\uparrow 1$ $n+1$ $n+2$

So $\binom{n+3}{n}$

first tutor problem wrong
Jinxed it

$\binom{n+3}{n+1}$

↑ Oh right 1+2=3

4. No $\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \dots + \binom{n+k}{n}$

$\binom{n+k}{n}$ or $\binom{n+k+1}{n+1}$

↑ is this the same?

WA can't say

(4)

TP 10.4 Addition Rule

- going simple! - ~~0~~

P(roll 7 or 11) = P(roll 7) + P(roll 11)

2 die

16 65
25 56
34
43
54 ~~6~~
61

- oh is prob chap

$\frac{6}{36} + \frac{2}{36}$

$\frac{8}{36}$ = $\frac{2}{9}$

TP 10.5 Fun w/ coins

- fair coin

1. P(throw tails forever) = 0
   to the limit
2. P(at least 1 heads when flip forever) = 1
   will happen at least once

(5)

TP 10.6 Fun w/ inclusion-exclusion

If int [1,100] selected at random P(15 or 17)
divisable by

= P(15) + P(17)

20 14

just $\lfloor \frac{100}{5} \rfloor$

= ~~24~~ 34
duh

<u>Minus</u> overlap P(15 ∩ 17)

what are these

7 14 21 28 35 42 ~~4~~49 56

63 70 77 84 91 98

24 - 2 = 22

~~Oh Opps $\lceil \frac{100}{3} \rceil$ counted 15~~

~~So 23~~

34 - 2 = 32

Now P $\frac{32}{100}$

That was very easy tutorl

6

Look at this stale thing list

LSA looks all messed up

Oh I reuse LSA variable - bad idea!

- no

totally screwed up - how?

Most seem right

A's LSA

Or am I saving stuff wrong?

Its not appending #

Should it do that in make_ls_ad

sections onClick

Is pop copy by ~~[illegible]~~ ref?

- Ohhh 45 min on that

Link not really there

- still not clear

How do you get that?

Says LSA should include Links

Oh routes should include link - so self routes [✓]

7

That worked - I don't know why

Let me look through recitation

Recitation we used links B->A

Do we just track initial / next step link

Task complete!

Goes 100x faster when does not print

4/22

Mathematics for Computer Science
MIT 6.042J/18.062J

Introduction to Probability Theory

Counting in Probability

What is the *probability* of getting exactly two jacks in a poker hand?

Counting in Probability

Outcomes: $\binom{52}{5}$ 5-card hands

Event: $\binom{4}{2}\cdot\binom{52-4}{3}$ hands w/2Jacks

$$\Pr\{2\text{ Jacks}\} ::= \frac{\binom{4}{2}\cdot\binom{48}{3}}{\binom{52}{5}} \approx 0.04$$

Probability: 1st Idea

- A set of basic experimental outcomes
- A subset of outcomes is an event
- The probability of an event:

$$\Pr\{\text{event}\} ::= \frac{\#\text{outcomes in event}}{\text{total }\#\text{outcomes}}$$

The Monty Hall Game

Applied Probability:

Let's Make A Deal

(1970's TV Game Show)

Monty Hall Webpages

`http://www.letsmakeadeal.com`

The Monty Hall Game

- goats behind two doors
- prize behind third door
- contestant picks a door
- Monty reveals a goat behind an unpicked door
- Contest sticks, or switches to the other unopened door

Analyzing Monty Hall

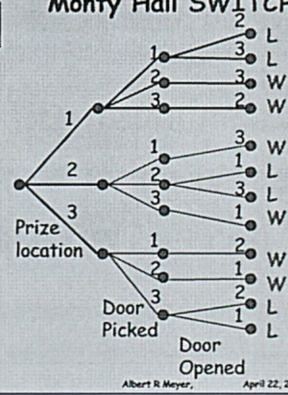
Marilyn Vos Savant explained Game in magazine -- bombarded by letters (even from PhD's) debating:

1) sticking & switching equally good
2) switching better

Analyzing Monty Hall

Determine the outcomes.
-- using a tree of possible steps can help

Monty Hall SWITCH strategy



SWITCH
Wins: 6

Lose: 6

Analyzing Monty Hall

A false conclusion:
sticking and switching have same # winning outcomes, so probability of winning is the same for both: 1/2.

Analyzing Monty Hall

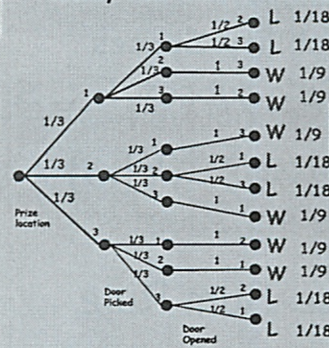
Another false argument:
after door opening, 1 goat and 1 prize are left. Each door is equally likely to have the prize (by symmetry), so

both strategies win with probability: 1/2.

Analyzing Monty Hall

What's wrong?
Let's look at the outcome tree more carefully.

Monty Hall SWITCH strategy



W: 6/9 = 2/3

L: 6/18 = 1/3

Probability: 2nd Idea

Outcomes may have differing probabilities!
Not always uniform.

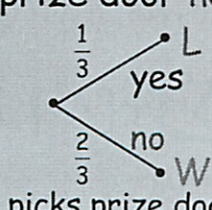
Finding Probability

Intuition is important but dangerous.
Stick with 4-part method:

1. Identify outcomes (*tree helps*)
2. Identify event (*winning*)
3. Assign outcome probabilities
4. Compute event probabilities

really simple analysis

SWITCH strategy wins <u>iff</u> prize door not picked:



Pr{switch wins} $= \frac{2}{3}$

Probability Spaces

1) Sample space: a countable set, $\mathcal{S}$, whose elements are called outcomes.
2) Probability function,

$$\Pr: \mathcal{S} \rightarrow [0,1], \text{ such that } \sum_{\omega \in \mathcal{S}} \Pr\{\omega\} = 1.$$

Probability Spaces

An event is a subset, $E \subseteq S$.

$$\Pr\left\{E\right\} ::= \sum_{\omega \in E} \Pr\{\omega\}$$

Cor: The Sum Rule

Sum Rule

For *pairwise disjoint* $A_0, A_1, \ldots$

$$\boxed{\begin{aligned}&\Pr\{A_0 \cup A_1 \cup A_2 \cup \cdots\} = \\ &\Pr\{A_0\} + \Pr\{A_1\} + \Pr\{A_2\} + \cdots\end{aligned}}$$

Difference Rule

$$\boxed{\begin{aligned}&\Pr\{A-B\} = \\ &\quad \Pr\{A\} - \Pr\{A \cap B\}\end{aligned}}$$

because by Sum Rule:

$$\Pr\{A\} = \Pr\{A \cap B\} + \Pr\{A-B\}$$

Inclusion-Exclusion

$$\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\}$$

The Union Bound

$$\boxed{\Pr\{A \cup B\} \leq \Pr\{A\} + \Pr\{B\}}$$

Boole's Inequality

for sets $A_0, A_1, \ldots$

$$\Pr\left\{\bigcup_{i \geq 0} A_i\right\} \leq \sum_{i \geq 0} \Pr\{A_i\}$$

Team Problems

Problems

1—4

Albert R Meyer, April 22, 2011 lec 11F.32

6.042

4/22

Take is different than 6.041

Prob starts in gambling - 17th century

What frac of hands have this property?

- Comes from combintorics

$\binom{4}{2}\binom{52-4}{3}$ exactly 2 Jacks

$$P(2 \text{ jacks}) = \frac{\binom{4}{2}\binom{52-4}{3}}{\binom{52}{5}} \approx .04$$

← total # of hands

Whats the fraction? addition to combinatorics

---

have outcomes

Subset of outcomes we want = event

So $P(\text{event}) = \ldots$

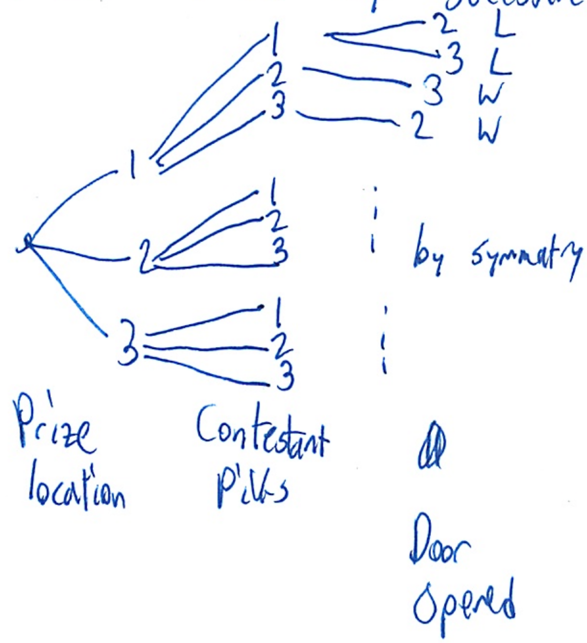
2

Monty Hall Game

(seen before ...)

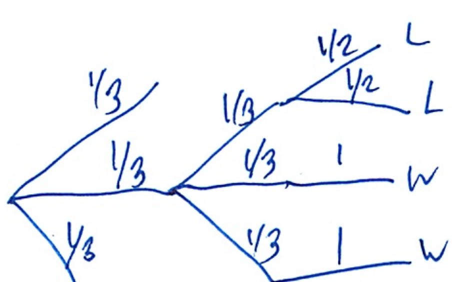
must open unpicked goat door

Build a tree w/ sucessive steps



So 2 L and 2 W - is that ~~[illegible]~~ mean it doesn't matter to switch?

- No! non-equal prob to get to this place
- Goes behind simple combintorics



$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{18}$

$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{18}$

$\frac{1}{3} \cdot \frac{1}{3} \cdot 1 = \frac{1}{9}$

$\frac{1}{3} \cdot \frac{1}{3} \cdot 1 = \frac{1}{9}$

assuming staff picks randomly

he picks randomly too
- best possible thing to do
- staff can't psyc him out

(3)

Bunch of assumptions made about modling + contestent's behavior

Now add all the W $3\left(\frac{1}{9}+\frac{1}{9}\right) = \frac{6}{9} = \frac{2}{3}$

L $3\left(\frac{1}{18}+\frac{1}{18}\right) = \frac{6}{18} = \frac{1}{3}$

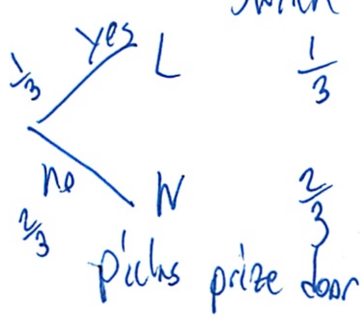
So <u>do</u> want to switch

<u>4-Step process</u>

1. Identify outcome
   - tree helpers
2. ~~Ass~~ Identify events
   - winning
3. Assign outcome prob
4. Compute event prob

Could also do simpler version

Switch strategy wins <u>iff</u> prize door <u>not</u> picked



(4)

Could understand by generalizing to extreem

100 doors

They open 98 doors except picked door and other

goat door or winning door
↳ if picked winner ↳ if picked lose

So likely you picked winner first is $\frac{1}{100}$ - not likely so $\frac{99}{100}$ times the other door left closed is winner

Always switch

---

1. Always a <u>Countable</u> $S =$ set # of outcomes in 6.042
   - Can have ∞ possible outcomes
   - like tutor problem
   - So summation, not integration
2. <u>Prob function</u> - assigns prob to outcomes
   - always $[0,1]$ b/w 0 and 1
   - " $\sum_{w \in S} P(w) = 1$ ← probs add to 1

⑤

3. Event is subset $E \subseteq S$

$$P(E) = \sum_{\omega \in E} P(\omega)$$

A lot of counting theory carrys over.

<u>Sum Rule</u> $P(A_0 \cup A_1 \cup A_2 \ldots \} =$

$P(A_0) + P(A_1) + P(A_2) + \ldots$

for <u>pairwise disjoint</u>

<u>Difference Rule</u>

$$P(A - B) = P(A) - P(A \cap B)$$

<u>Inclusion - Exclusion</u>

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

? if not disjoint

<u>Union Bound</u>

$$P(A \cup B) \leq P(A) + P(B)$$

<u>Boole's Inequality</u> for sets $A_0, A_1 \ldots$

... missed