The electromotive force is then

$$\mathcal{E} = -\frac{d}{dt}\Phi_{magnetic} = -B_x\frac{dz}{dt}w = -B_x v_z w > 0\,.$$

Note that the z-component of the velocity of the loop is negative, $v_z < 0$, so the electromotive force is positive.

The current that flows in the loop is therefore

$$I_{ind} = \frac{\mathcal{E}}{R} = -\frac{B_x v_z w}{R} > 0\,.$$

Note that a positive current corresponds to a counterclockwise flow of charge agreeing with our Lenz's Law analysis in part (a).

(c) Besides gravity, what other force acts on the loop in the $\pm\hat{\mathbf{k}}$ direction? Give its magnitude and direction in terms of the quantities given.

Solution: There is an induced magnetic force acting on the upper leg of the loop given by

$$\vec{\mathbf{F}}_{ind} = I\vec{\mathbf{w}} \times \vec{\mathbf{B}} = \frac{B_x v_z w}{R} w\,\hat{\mathbf{j}} \times B_x\,\hat{\mathbf{i}} = -\frac{B_x^{\,2} v_z w^2}{R}\hat{\mathbf{k}} > 0\,.$$

Note that this force is in the positive $\hat{\mathbf{k}}$-direction since $v_z < 0$.

(d) Assume that the loop has reached a "terminal velocity" and is no longer accelerating. What is the magnitude of that terminal velocity in terms of given quantities?

Solution: If terminal velocity (denote the z-component by $(v_z)_{term}$) is reached, some portion of the loop must still be in the magnetic field. Otherwise there will no longer be an induced magnetic force and the loop will accelerate uniformly downward due to the gravitational force. Terminal velocity is reached when the total force on the loop is zero, therefore

$$\vec{\mathbf{F}}_{ind,term} - mg\ \hat{\mathbf{k}} = \vec{\mathbf{0}}$$

The substitute our result for $(v_z)_{term}$ in the expression for the induced force to yield

$$-\frac{B_x^{\,2}(v_z)_{term}w^2}{R}\hat{\mathbf{k}} - mg\ \hat{\mathbf{k}} = 0\,.$$

We now solve this equation for the z-component of the terminal velocity:

$$(v_z)_{term} = -\frac{mgR}{B_x^2 w^2} < 0 .$$

Substitute the above results for $(v_z)_{term}$ into our expression for the induced current to find the induced current at terminal velocity,

$$I_{ind,term} = -\frac{B_x w}{R}(v_z)_{term} = \left(-\frac{B_x w}{R}\right)\left(-\frac{mgR}{B_x^2 w^2}\right) = \frac{mg}{B_x w}$$

(e) Show that at terminal velocity, the rate at which gravity is doing work on the loop is equal to the rate at which energy is being dissipated in the loop through Joule heating.

Solution: When the loop is moving at terminal velocity the power exerted by gravitational force is given by

$$P_{grav,term} = \vec{F}_{grav} \cdot \vec{v}_{term} = -mg\ \hat{\mathbf{k}} \cdot \left(-\frac{mgR}{B_x^2 w^2}\hat{\mathbf{k}}\right) = \frac{m^2 g^2 R}{B_x^2 w^2} .$$

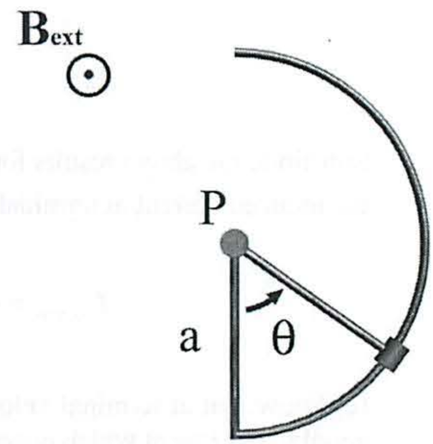
The power associated with the Joule heating at terminal velocity is given by

$$P_{joule,term} = (I^2)_{ind,term} R = \frac{m^2 g^2}{B_x^2 w^2} R .$$

Thus comparing these two last equations shows that

$$P_{grav,term} = P_{Joule,term} .$$

Problem 6: Generator



A "pie-shaped" circuit is made from a straight vertical conducting rod of length a welded to a conducting rod bent into the shape of a semi-circle with radius a (see sketch). The circuit is completed by a conducting rod of length a pivoted at the center of the semi-circle, *Point P*, and free to rotate about that point. This moving rod makes electrical contact with the vertical rod at one end and the semi-circular rod at the other end. The angle θ is the angle between the vertical rod and the moving rod, as shown. The circuit sits in a constant magnetic field $\mathbf{B}_{ext}$ pointing out of the page.

(a) If the angle θ is increasing with time, what is the direction of the resultant current flow around the "pie-shaped" circuit? What is the direction of the current flow at the instant shown on the above diagram? To get credit for the right answer, you must justify your answer.

The flux out of the page is increasing, so we want to generate a field into the page (Lenz' Law). This requires a clockwise current (see arrows beside pie shaped wedge).

For the next two parts, assume that the angle θ is increasing at a constant rate, $d\theta(t)/dt = \omega$.

(b) What is the magnitude of the rate of change of the magnetic flux through the "pie-shaped" circuit due to $\mathbf{B}_{ext}$ only (do **not** include the magnetic field associated with any induced current in the circuit)?

$$\frac{d}{dt}\Phi_B = \frac{d}{dt}(B_{ext}A) = B_{ext}\frac{d}{dt}\left(\pi a^2 \cdot \frac{\theta}{2\pi}\right) = \frac{B_{ext}a^2}{2}\frac{d\theta}{dt} = \frac{B_{ext}a^2}{2}\omega$$

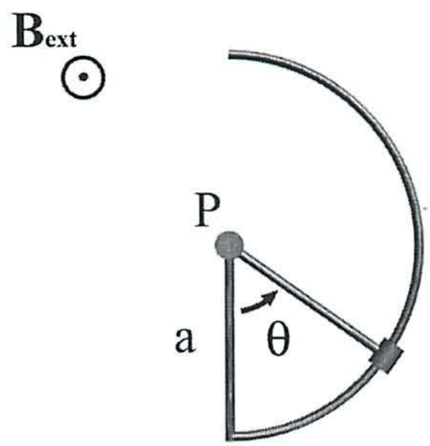
(c) If the "pie-shaped" circuit has a constant resistance R, what is the magnitude and direction of the magnetic force due to the external field on the moving rod in terms of the quantities given. What is the direction of the force at the instant shown on the above diagram?

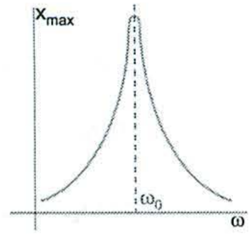
The magnetic force is determined by the current, which is determined by the EMF, which is determined by Faraday's Law:

$$\varepsilon = \frac{d}{dt}\Phi_B = \frac{B_{ext}a^2}{2}\omega \quad \Rightarrow \qquad I = \frac{\varepsilon}{R} = \frac{B_{ext}a^2\omega}{2R}$$

$$\Rightarrow F = IaB_{ext} = \frac{B_{ext}^2 a^3\omega}{2R}$$

The force opposes the motion, which means it is currently down and to the left (the cross product of a radially outward current with a B field out of the page).





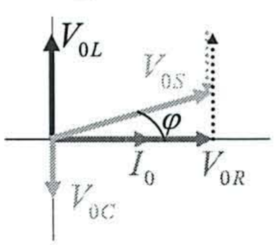
Driven LRC Circuits

We can also add a force to our circuits – the AC power supply. In this case the current responds at the drive frequency. However, depending on the frequency of the drive, the current may be out of phase (either *leading* or *lagging* the drive) and its amplitude will also vary. This is easily seen in mechanical systems. For a fantastic example, play with the pendula at the Kendall T station. Depending on how fast you drive them they will respond either in or out of phase with your drive, and they will either move a little or a lot. When you drive at the natural frequency, the amplitude increases greatly, and the system is "in resonance."

One Element at a Time

In order driven LRC circuits its easiest to start thinking about individual circuit elements. A resistor obeys Ohm's law: V=IR. Neither the amplitude nor phase of the voltage depends on the frequency (the response voltage is always in phase with the current).
A capacitor is different. At low frequencies the capacitor "fills up," but at high frequencies it begins discharging before "filling up." The voltage is frequency dependent and the current *leads* the voltage (current flows before the charge/potential on the capacitor appears).
An inductor is the opposite – it hates the change of high frequencies and thus has large voltages there – and the current *lags* – the inductor fights before current flows.
When put together in LRC circuits, the capacitor "dominates" at low frequencies, the inductor at high ones. At resonance ($\omega_0 = \sqrt{1/LC}$) the frequency is such that these two effects balance and the current will be largest in the circuit. Also at this frequency the current is in phase with the driving voltage (the AC power supply).

Seeing it Mathematically – Phasors



To do this mathematically we will use phasor diagrams. A phasor is a vector whose magnitude is the amplitude of either voltage or current and whose angle corresponds to its phase. Phasors rotate CCW about the origin with time as their phase evolves, and their current amplitude is their component along the y-axis, which oscillates as it should.
Phasors allow us to add voltages that are not in phase with each other. For example, the phasor diagram above illustrates the relationship of voltages in a series LRC circuit. The current I is assigned to be at "0 phase" (along the x-axis). The phase of the voltage across the resistor is the same. The voltage across the inductor L leads (is ahead of I) and the voltage across the capacitor C lags (is behind I). If you add up (using vector arithmetic) the voltages across R, L & C (the red and dashed blue & green lines respectively) you arrive at the voltage across the power supply. This then gives you a rapid way of understanding the phase between the drive (the power supply) and the response (the current) – here labeled ϕ.

Important Equations

Natural Frequency of LC Circuit: $\omega_0 = \frac{1}{\sqrt{LC}}$

Current

Driving Voltage Source: $V(t) = V_0 \sin(\omega t)$

Current Response: $I(t) = I_0 \sin(\omega t - \phi)$

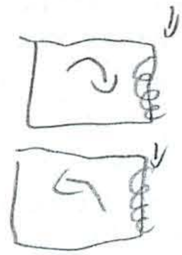
Amolitude: $I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}$; $X_L = \omega L$, $X_C = \frac{1}{\omega C}$

Phase: $\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$

With inductors

$-L\frac{dI}{dt}$ when in direction of current

$L\frac{dI}{dt}$ opposet dir of current



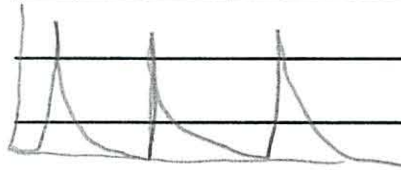
Don't say charging / discharging

4/14

Class 26: Outline

Hour 1 & 2:
Resonance & Driven LRC Circuits

C



inductor or capicator?
↑
Since current increases quickly
current not continuous

Driven Oscillations:
Resonance

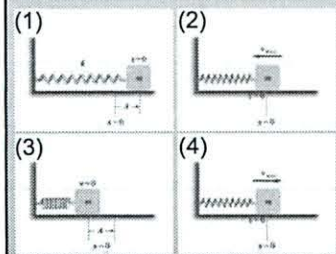
$\tau = \frac{1}{e} \approx 2.5$ fall in C
or about 5 mS in $= RC$
$R = \frac{V}{I} = \frac{2.5V}{5mA} = 5k\Omega$
$C = \frac{5ms}{5k\Omega} = 1\mu F$
one of 18% who got it right

Mass on a Spring:
Simple Harmonic Motion
A Second Look

← Last time did undriven

Undriven

Mass on a Spring



We solved this:

$$F = -kx = ma = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

Simple Harmonic Motion

$$x(t) = x_0 \cos(\omega_0 t + \phi)$$

Moves at natural frequency

What if we now move the wall?
Push on the mass?

↳ $\sqrt{\frac{k}{m}}$

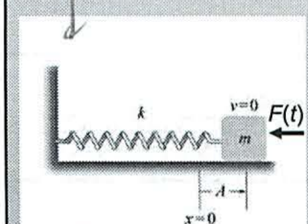
$\sqrt{\frac{1}{LC}}$

inertia in system

Demonstration: Driven Mass on a Spring Off Resonance

move wall sinusoidally

Driven Mass on a Spring



Now we get:

$$F = F(t) - kx = ma = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + kx = F(t)$$

Assume harmonic force:

$$F(t) = F_0 \cos(\omega t)$$

Simple Harmonic Motion

$$x(t) = x_{max} \cos(\omega t + \phi)$$

Moves at driven frequency

takes a while (30sec) to settle in

freq rod ↓ = ball ↑

turn off drive → freq goes ↓

- was moving at drive freq

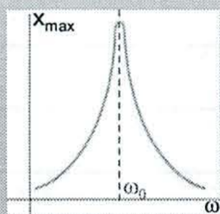
? If Force harmonic

- could solve w/ 18.03
- just know the solution

Resonance

$$x(t) = x_{max} \cos(\omega t + \phi)$$

Now the amplitude, x_{max}, depends on how close the drive frequency is to the natural frequency



Let's See...

x_{max} related to drive + natural freq

* will move a lot if moving it at its natural freq *

= resonance

has damper to prevent it from going too far

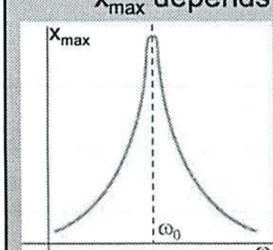
\# resistance sets as far as it can go in resonance

Demonstration: Driven Mass on a Spring

Resonance

$$x(t) = x_{max} \cos(\omega t + \phi)$$

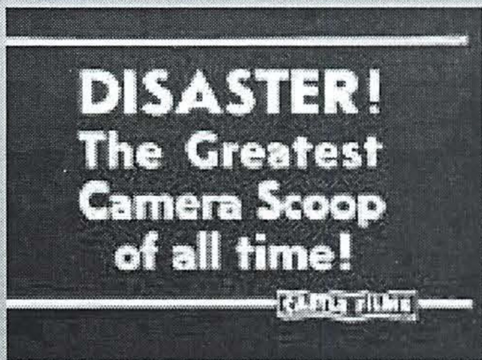
x_{max} depends on drive frequency



Many systems behave like this:

- Swings – move legs at top + bottom of motion
- Some cars – that are crappy
- Kendall T Station
- …

Famous Resonance Examples



galliping girdy

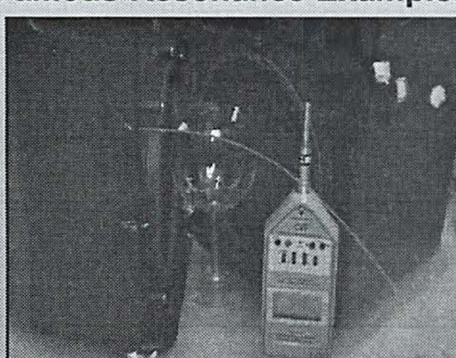
That wasn't really Resonance but Aeroelastic Flutter (It's impressive anyway)

This next is really resonance

driving at freq - same as freq as motion

blowing over playing card
wind - instramints

Famous Resonance Examples



glass breaking

Electronic Analog: RLC Circuits

Analog: RLC Circuit

Recall:

- Inductors are like masses (have inertia)
- Capacitors are like springs (store/release energy)
- Batteries supply external force (EMF)

Charge on capacitor is like position,
Current is like velocity – watch them resonate

Now we move to "frequency dependent batteries:"
AC Power Supplies/AC Function Generators

What is eq of moving wall
- function generator/power supply AC
- supplies time dependant voltage

Demonstration: RLC with Light Bulb

PRS: RLC Circuit w/ Light bulb

As I slide the core into the inductor the light bulb changes brightness. Why?

I am driving the circuit through resonance by…

0%	1. continuously increasing the frequency of current oscillations in the circuit
0%	2. continuously decreasing the frequency of current oscillations in the circuit
0%	3. continuously increasing the natural frequency of oscillations in the circuit
0%	4. continuously decreasing the natural frequency of oscillations in the circuit
0%	5. I don't know

:20

as out
- increasing osilation freq
↳ driving freq 60Hz
* changing inductor so changing L → so changing natural freq could also C
- decreasing inductance
- pulling core out of inductor - makes it harder to store charge (decreasing inductance)

$\omega_0 = \frac{1}{\sqrt{LC}}$

Start at Beginning: AC Circuits

P26-17

Group Problem: Discovery Mathlet Driven RLC Circuits

P26-18

Alternating-Current Circuit

- direct current (dc) – current flows one way (battery)
- alternating current (ac) – current oscillates — wall power

- sinusoidal voltage source

$$V(t) = V_0 \sin \omega t$$

$\omega = 2\pi f$: angular frequency

V_0 : voltage amplitude

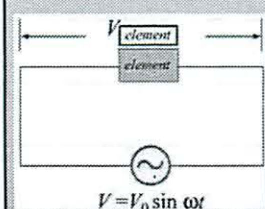
P26-19

ossilates

Edison vs Tesla

AC
- can use transformers
- generators make AC

AC Circuit: Single Element



$$V_{element} = V$$
$$= V_0 \sin \omega t$$

$$I(t) = I_0 \sin(\omega t - \phi)$$

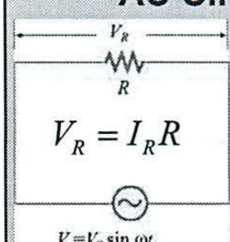
Questions:

1. What is I_0?
2. What is ϕ?

↳ for each element

P26-20

AC Circuit: Resistors

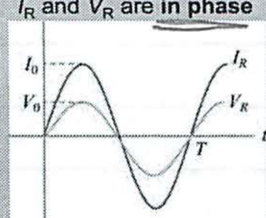


$$V_R = I_R R$$

$$I_R = \frac{V_R}{R} = \frac{V_0}{R} \sin \omega t$$
$$= I_0 \sin(\omega t - 0)$$

$$I_0 = \frac{V_0}{R}$$
$$\varphi = 0$$

I_R and V_R are **in phase**



P26-21

oh from book I spent a lot of time w/

AC Circuit: Capacitors

$V_C = \frac{Q}{C}$

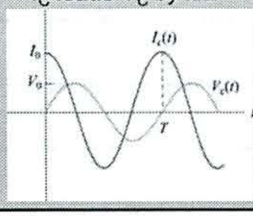
$V = V_0 \sin \omega t$

$Q(t) = CV_C = CV_0 \sin \omega t$

$$I_C(t) = \frac{dQ}{dt} = \omega C V_0 \cos \omega t = I_0 \sin(\omega t - \pi/2)$$

$$I_0 = \omega C V_0 \qquad \varphi = -\pi/2$$

I_C leads V_C by $\pi/2$



c b/w charge + voltage

must take derivitive $\frac{dQ}{dt}$

what is amp current as function of charge?

Steady-state - no current flowing in circuit - also in DC circuit?
Once assilator settles into pattern (after 30 sec) * keep flipping what is +/-

↑ freq = ↑ current flowing
current peaks 90° before voltage

AC Circuit: Inductors

$V_L = L \frac{dI_L}{dt}$

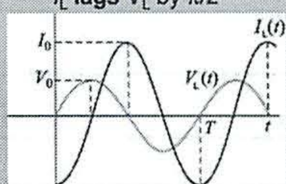
$V = V_0 \sin \omega t$

$\frac{dI_L}{dt} = \frac{V_L}{L} = \frac{V_0}{L} \sin \omega t$

$$I_L(t) = \frac{V_0}{L} \int \sin \omega t \, dt = -\frac{V_0}{\omega L} \cos \omega t = I_0 \sin(\omega t - \pi/2)$$

$$I_0 = \frac{V_0}{\omega L} \qquad \varphi = \frac{\pi}{2}$$

I_L lags V_L by $\pi/2$



AC Circuits: Summary

Element	I_0	Current vs. Voltage	Resistance Reactance
Resistor	$\frac{V_{0R}}{R}$	In Phase	$R = R$
Capacitor	$\omega C V_{0C}$	Leads	$X_C = \frac{1}{\omega C}$
Inductor	$\frac{V_{0L}}{\omega L}$	Lags	$X_L = \omega L$

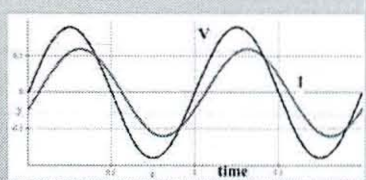
Although derived from single element circuits, these relationships hold generally!

in phase
90°
+ Impedance } 3 terms for same thing
↳ any phase relationship

← amp voltage + current

PRS: Leading or Lagging?

The plot shows the driving voltage V (black curve) and the current I (red curve) in a driven RLC circuit. In this circuit,



0% 1. The current leads the voltage ✓

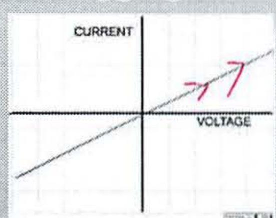
0% 2. The current lags the voltage

0% 3. Don't have a clue.

:20

PRS: Leading or Lagging?

The graph shows current versus voltage in a driven RLC circuit at a given driving frequency. In this plot



← inphase

very baddly plotted

0% 1. The current leads the voltage by about 45°

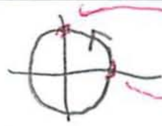
0% 2. The current lags the voltage by about 45°

0% 3. The current and the voltage are in phase

0% 4. Don't have a clue

20

similar q✓



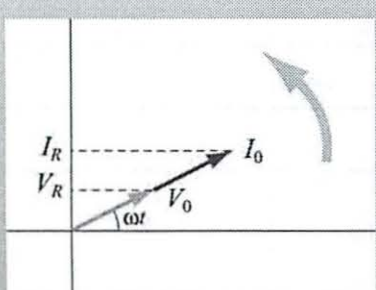
not enough info - but not in phase

voltage max → current 0

ramp down voltage → current peaks 90°

voltage → leads current 90°

Phasor Diagram: Resistor



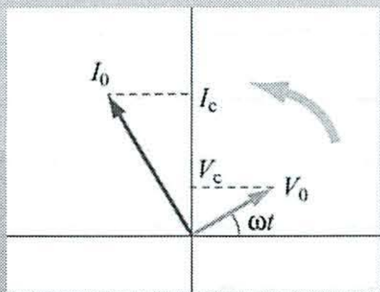
$V_0 = I_0 R$

$\varphi = 0$

I_R and V_R are **in phase**

relative amplitudes worthless

Phasor Diagram: Capacitor

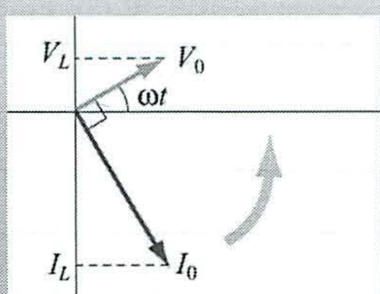


$$V_0 = I_0 X_C = I_0 \frac{1}{\omega C}$$
$$\varphi = -\pi/2$$

I_C **leads** V_C by $\pi/2$

P26-28

Phasor Diagram: Inductor



$$V_0 = I_0 X_L = I_0 \omega L$$
$$\varphi = \frac{\pi}{2}$$

I_L **lags** V_L by $\pi/2$

P26-29

This makes it easy to put them all together

Put it all together:
Driven RLC Circuits

P26-30

Question of Phase

We had fixed phase of voltage:

$$V_{\boxed{element}} = V_0 \sin \omega t \qquad I(t) = I_0 \sin(\omega t - \phi)$$

It's the same to write:

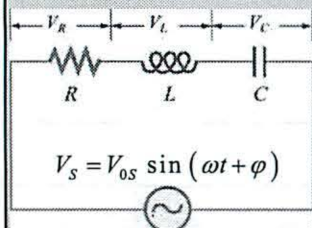
$$V_{\boxed{element}} = V_0 \sin(\omega t + \phi) \qquad I(t) = I_0 \sin \omega t$$

(Just shifting zero of time)

P26-31

current all the same
- all have same phase
- so put the phase into V.
- redefined time)

Driven RLC Series Circuit



$I(t) = I_0 \sin(\omega t)$

$V_R = V_{R0} \sin(\omega t)$

$V_L = V_{L0} \sin\left(\omega t + \frac{\pi}{2}\right)$

$V_C = V_{C0} \sin\left(\omega t + \frac{-\pi}{2}\right)$

What is I_0 (and $V_{R0} = I_0 R$, $V_{L0} = I_0 X_L$, $V_{C0} = I_0 X_C$)?
What is φ? Does the current lead or lag V_s?

Must Solve: $V_S = V_R + V_L + V_C$

P26-32

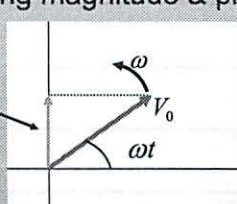
↳ power supply voltage

paper + pencil = a pain

Recall: Phasor Diagram

Nice way of tracking magnitude & phase:

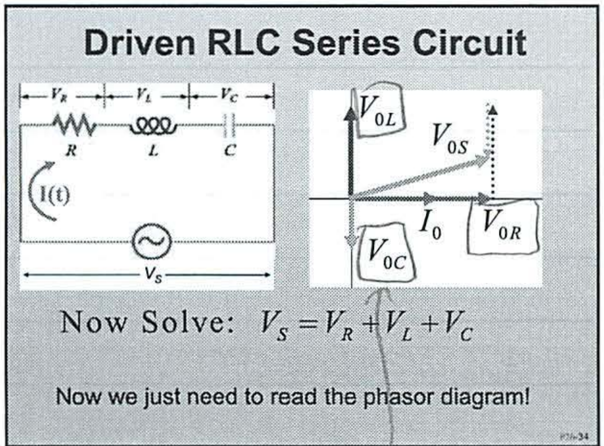
$V(t) = V_0 \sin(\omega t)$



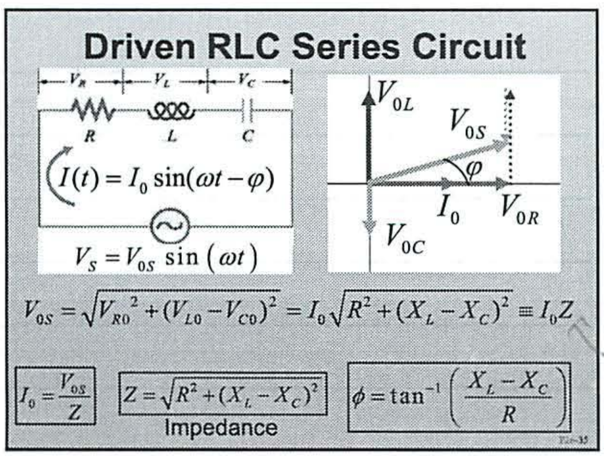
Notes: (1) As the phasor (red vector) rotates, the projection (pink vector) oscillates
(2) Do both for the current and the voltage

P26-33

← easier - since can add vectors visually



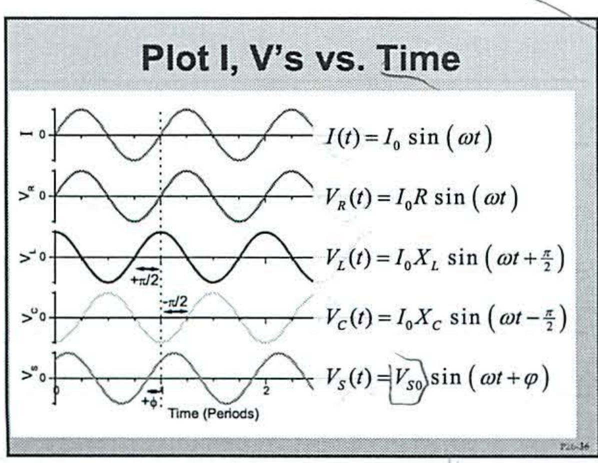
back to back
inductor + capacitor



Z is the impedance

Resistances + Reactances add in quadrature
This is the only one we are doing

phase

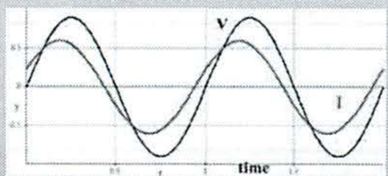


plot as a function
of time

how to look at plots and identify
which is which

- Inductors lead 90°
- Conductors lag 90°
- Power supply anything

PRS: Who Dominates?



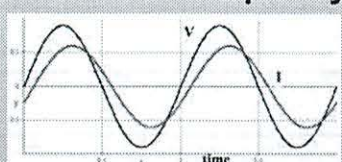
The graph shows current & voltage vs. time in a driven RLC circuit at a particular driving frequency. At this frequency, the circuit is dominated by its

0% 1. Inductance
0% 2. Capacitance
0% 3. I don't know

20

I leading V → Conductor

PRS: What Frequency?



The graph shows current & voltage vs. time in a driven RLC circuit at a particular driving frequency. Is this frequency above or below the resonance frequency of the circuit?

0% 1. Above the resonance frequency
0% 2. Below the resonance frequency
0% 3. I don't know

:20

Did we discuss yet?

V leading I
I laging V → Inductor
When do inductors work hard
lots of change → high freq
so above resonance freq
* if inductor like *

RLC Circuits: Resonances

P26-39

If very slow, it just moves w/
0° out of phase → conductor like
ball is like charge in capacitor
At resonant freq - it is
opposet
rod ↑ ball ↓ so
it is like Inductor - 180°
out of phase

in middle Resonance
$X_L = X_R$

Resonance

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}; \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C}$$

At very low frequencies, C dominates (X_C>>X_L): it fills up and keeps the current low

At very high frequencies, L dominates (X_L>>X_C): the current tries to change but it won't let it

At intermediate frequencies we have **resonance**

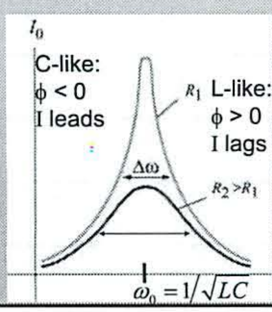
I_0 reaches maximum when $X_L = X_C$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

P26-40

Resonance

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}; \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C}$$



$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

P26-41

↑ in middle renosance

When $X_L = X_C$ They will cancle
solve for $w = \frac{1}{\sqrt{LC}}$ — natural freq
get resonance

Demonstration: RLC with Light Bulb

P26-42

Play w/ applet

PRS: RLC Circuit w/ Light bulb

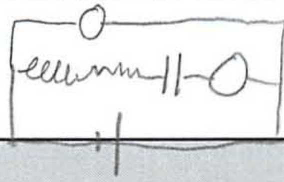
Imagine another light bulb connected in parallel to this LRC circuit. With the core pulled out that light bulb would be flashing:

0%	1. before the LRC light bulb (leading)
0%	2. after the LRC light bulb (lagging)
0%	3. in time with the LRC light bulb
0%	4. not at all
0%	5. I don't know

:20

P26-43

does not appear flashing - 60Hz
imagine this was 1/10 Hz



? no clue
? would it depend on relative strenghts?

Core pulled out, capicator dominating
↑ inductance. ↓
natural freq ↑
now below resonance
current leads voltage

RLC Circuits: leading/lagging

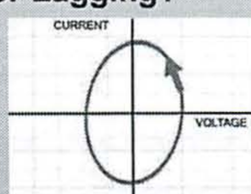
P26-44

↑ capicators
↑ lots of voltages but on power-supply

top light flashes w/ power supply
bottom flasks w/ I
#2

PRS: Leading or Lagging?

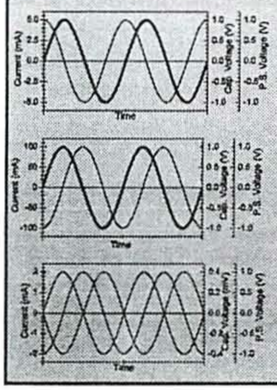
The graph shows current versus voltage in a driven RLC circuit at a given driving frequency. In this plot



0%	1. Current lags voltage by ~90°
0%	2. Current leads voltage by ~90°
0%	3. Current and voltage are almost in phase
0%	4. Not enough info (but they aren't in phase!)
0%	5. I don't know.

P26-45

In Class Problem: RLC Circuit



- Consider plots of V_C, V_S and I made at 3 frequencies:
- a very low angular frequency (100 s^{-1}), a very high one (10^5 s^{-1}) and the resonance frequency, which is somewhere in between
- Each plot allows you to find one of R,L,C. In that order, which plot do you use, which frequency is it and what are the values of R, L & C?

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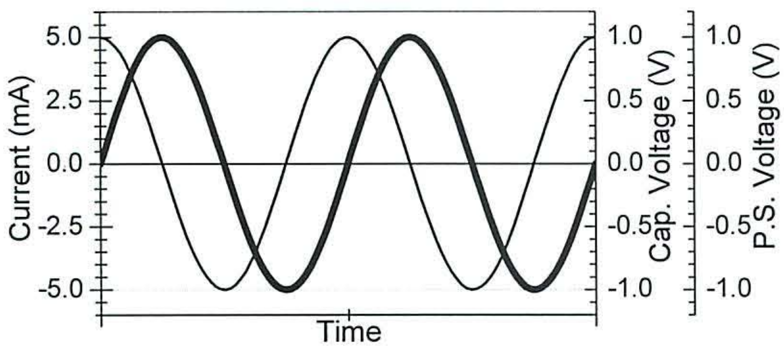
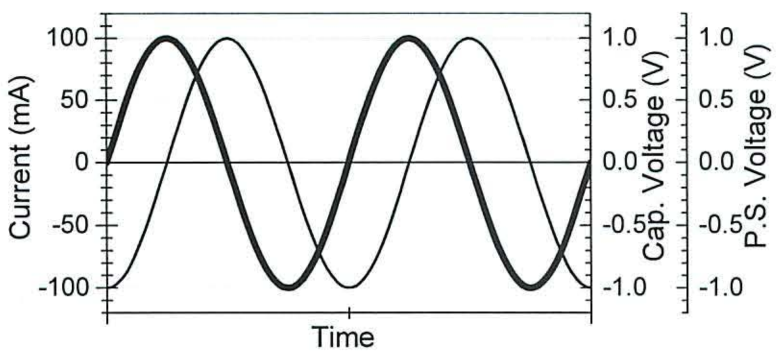
Department of Physics: 8.02

In Class W03D2_1 Solutions: RLC Circuit



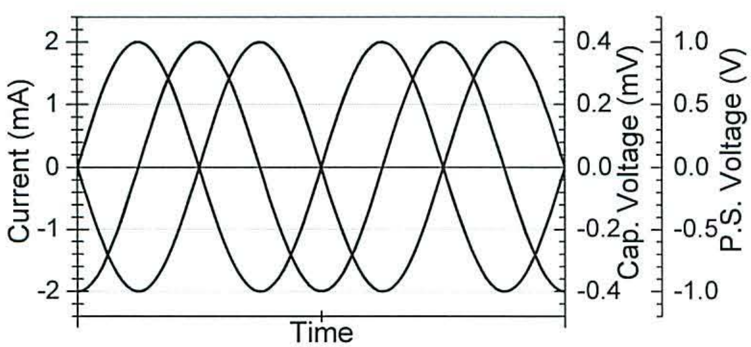
Problem: Consider plots of V_C, V_S and I made at 3 frequencies: a very low angular frequency (100 s^{-1}), a very high one (10^5 s^{-1}) and the resonance frequency, which is somewhere in between

Each plot allows you to find one of R,L,C. In that order, which plot do you use, which frequency is it and what are the values of R, L & C?



Solution:
The resistance is most easily determined from the curve on resonance (the middle one, where V_S and I are in phase and V_C lags). At this frequency $Z = R$, so

$$R = \frac{V_{PS}}{I} = \frac{1\text{ V}}{100\text{ mA}} = 10\ \Omega$$



For the inductor to dominate we need to be at high frequency, where the power supply will lead the current which leads the capacitor (bottom plot). Here:

$$V_0 = I_0 X_L = I_0 \omega L \Rightarrow$$

$$L = \frac{V_0}{\omega I_0} = \frac{1\text{V}}{10^5\text{s}^{-1} \cdot 2\text{ mA}} = 5\text{ mH}$$

At very low frequencies the capacitor will dominate. Now the power supply is the same voltage as the capacitor, and the plot shows one curve (current) leading the other two (top curve). Here:

$$V_0 = I_0 X_C = \frac{I_0}{\omega C} \Rightarrow C = \frac{I_0}{\omega V_0} = \frac{5\text{ mA}}{100\text{s}^{-1} \cdot 1\text{V}} = 50\ \mu\text{F}$$

Physics Math Review

4/15

Differential Equations

- using greek letters to be generic

$$\alpha \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + \gamma x = F(t)$$

α, β, γ constants — could be L, R, C
given by Physics

$F(t)$ given

$F(t) = F_0 \cos(\omega t)$ periodic source

Let's take

$$\beta \frac{dx}{dt} + \gamma x = 0$$

↑ exponential - LR / RC circuit
- just decays
- chemistry mixing
- β = rate constant

$$\frac{dx}{dt} = -\frac{\gamma}{\beta} x = -\frac{1}{\tau_1} x$$

define $\tau_1 = \frac{\beta}{\gamma}$

$x(t) = e^{-t/\tau_1}$ here $\tau_1 = RC$

(2) Now take

$$\alpha \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} = 0$$

$$Y(t) = \frac{dx}{dt}$$

$$\alpha \frac{dY}{dt} + \beta Y = 0$$

$$\frac{dy}{dt} = -\frac{\beta}{\alpha} Y$$

define $T_2 = \frac{\alpha}{\beta}$ — also a decay constant

$$Y(t) = \frac{dx}{dt} = Y_0 e^{-t/T_2}$$ here $T_2 = LR$

So $\alpha \frac{d^2x}{dt} \to L$

$\beta \frac{dx}{dt} \to R$

~~α~~ $\gamma x \to C$

Now take $\alpha \frac{d^2x}{dt^2} + \gamma x = 0$ — Harmonic oscilator 8.01, LC circuit 8.02

$$\frac{d^2x}{dt^2} = -\frac{\gamma}{\alpha} x$$

define $\omega_0 = \sqrt{\frac{\gamma}{\alpha}}$ — "natural freq"

- no steady value
- shakes back + forth

3) $x(t) = X_0 \cos(\omega_0 t + \phi)$

~~Sho~~ So show that equation <u>does</u> satisfy differential

$$x(t) = x_0 \cos(\omega_0 t + \phi)$$

$$\frac{dx}{dt} = -\omega_0 X_0 \sin(\omega_0 t + \phi)$$

$$\frac{d^2x}{dt^2} = -\omega_0^2 x_0 \cos(\omega_0 t + \phi)$$

↳ $\omega_0^2 = \frac{\gamma}{\alpha}$ ← remember

and $x = x_0 \cos(\omega_0 t + \phi)$

So can write

$$\frac{d^2x}{dt^2} = -\frac{\gamma}{\alpha} x$$

Now combine all 3

- driving force still 0 so $F(t) = 0$

it ~~...~~ does not look like it will all cancle

so ansatz (edu guess)

$$\alpha \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + \gamma x = 0$$

$$x(t) = x_0 e^{-t/\tau} \cos(\omega t + \phi)$$

x_0, ϕ constants but don't determine dynamics

x_0 = how big your ans is

ϕ = where you start along curve

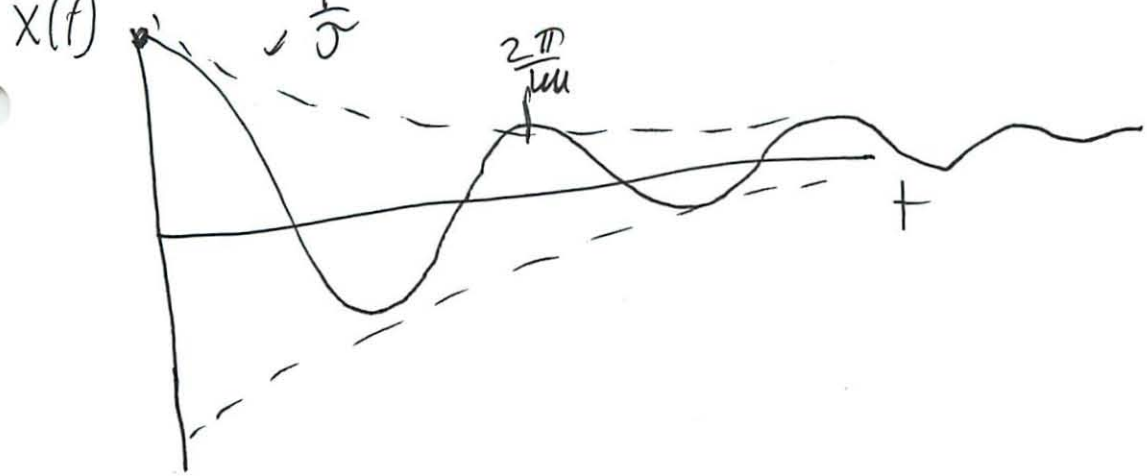
* want to solve for

τ, ω constants but <u>do</u> determine dynamics

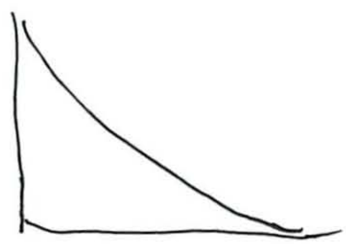
τ = decay time

ω = like a freq

(4)



So this sandwiches together



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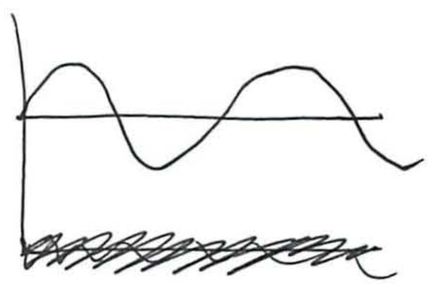


$\beta \frac{dx}{dt} + \gamma x = 0$

RC

$\alpha \frac{d^2x}{dt^2} + \gamma x = 0$

LC

$x(t) = x_0 e^{-t/\tau} \cos(\omega t + \phi)$ take deriv

$$\frac{dx}{dt} = x_0 \left[-\frac{1}{\tau} e^{-t/\tau} \cos(\omega t + \phi) - \omega e^{-t/\tau} \sin(\omega t + \phi) \right]$$

$$\frac{d^2x}{dt^2} = x_0 \left[\frac{1}{\tau^2} e^{-t/\tau} \cos(\omega t + \phi) + \frac{\omega}{\tau} e^{-t/\tau} \sin(\omega t + \phi) + \frac{\omega}{\tau} e^{-t/\tau} \sin(\omega t + \phi) - \omega^2 e^{-t/\tau} \cos(\omega t + \phi) \right]$$

this should all simplify

(5)

Sub into diff eq to check

α (arrow pointing to $\ddot{x}$)

$$\ddot{x}\cdot X_0\left[\frac{1}{\tau^2}e^{-t/\tau}\cos(\omega t+\phi)+\frac{2\omega}{\tau}e^{-t/\tau}\sin(\omega t+\phi)-\omega^2 e^{-t/\tau}\cos(\omega t+\phi)\right]+\beta X_0\left[-\frac{1}{\tau}e^{-t/\tau}\cos(\omega t+\phi-\omega e^{-t/\tau}\sin(\omega t+\phi)\right]+\gamma X_0\left[e^{-t/\tau}\cos(\omega t+\phi)\right]=0$$

Every term has $X_0 e^{-t/\tau}$

- cancle
- gather sin + cos

$$\left[\frac{\alpha}{\tau^2}-\alpha\omega^2-\frac{\beta}{\tau}+\gamma\right]\cos(\omega t+\phi)+\left[\frac{2\alpha\omega}{\tau}-\beta\omega\right]\sin(\omega t+\phi)=0$$

$\omega t+\phi=0 \rightarrow [\quad]\cdot\cos(\,)=0$

$\omega t+\phi=\frac{\pi}{2} \rightarrow [\quad]\cdot\sin(\,)=0$

So know

$$\tau_1=\frac{\beta}{\gamma}\qquad \tau_2=\frac{\alpha}{\beta}\qquad \omega_0=\sqrt{\frac{\gamma}{\alpha}}$$

(6)

This gives 2 equations which each must vanish

$$\left[\frac{\alpha}{\mathcal{J}^2} - \alpha \mu^2 - \frac{\beta}{\mathcal{J}} + \gamma\right] \cos(\mu t + \phi) = 0$$

$$\left[\frac{2\alpha\mu}{\mathcal{J}} - \beta\mu\right] \sin(\mu t + \phi) = 0$$

easier to solve

$$\frac{2\alpha\mu}{\mathcal{J}} - \beta\mu = 0$$

$$\boxed{\mathcal{J} = \frac{2\alpha}{\beta}}$$ Condition for combined eq

$$\frac{\alpha}{\left(\frac{2\alpha}{\beta}\right)^2} - \alpha\mu^2 - \frac{\beta}{\left(\frac{2\alpha}{\beta}\right)} + \gamma = 0$$

$$\frac{\beta^2}{4\alpha} - \alpha\mu^2 - \frac{\beta^2}{2\alpha} + \gamma = 0$$

$$\boxed{\mu^2 = \frac{\gamma}{\alpha} - \frac{\beta^2}{4\alpha^2}}$$

$$\mathcal{J} = 2\mathcal{J}_2$$

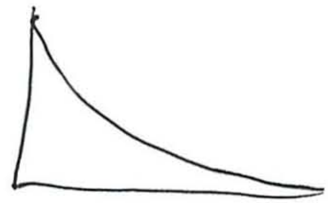
$$\mu^2 = \mu_0^2 \frac{1}{4\mathcal{J}_2^2}$$

← the constants that determine how solution will oscillates

(7)

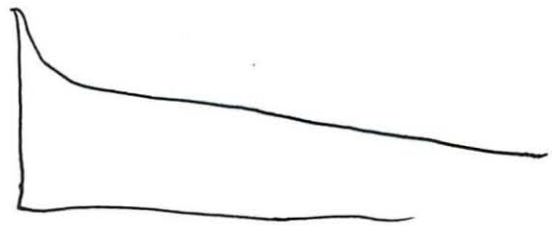
When $\omega^2 = 0$ it is critical dampening
- don't need to know

$$x(t) = (A + Bt)e^{-t/\tau}$$



$\omega^2 < 0$ = over dampening

$$x(t) = Ae^{-t/\tau_1} + Be^{-t/\tau_2}$$



critical dampening is fastest decay possible
- car shocks want and are tuned for

don't need to know the cool physics here

This was building blocks for AC circuits
+ driven

$$\omega^2 = \underbrace{\frac{1}{LC}}_{\omega_0^2} - \frac{R^2}{4L^2}$$

Now putting in driving term $F(t)$

$$F(t) = F_0 \cos(\omega t)$$

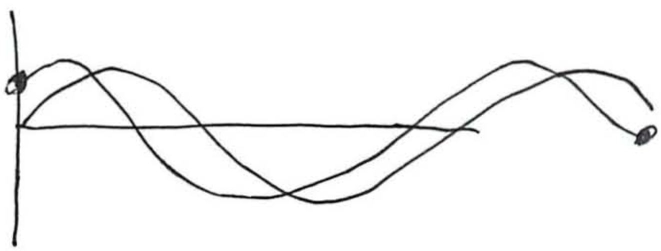
- a driving force
- AC source
- 60 Hz

⑧

What is $x(t)$ going to look like after a while
- some inital transiance when you first flip the switch
- let it die down for a second before you measure

1. Make an Ansantz (guess)

$$x(t) = \underline{X_0} \cos(\omega t + \underline{\phi})$$



) represents Voltage / charge on capicator / deriv is current

X_0 and ϕ are now dynamic

↳ ω determined by driving force and are a given (different from bottom pg 3)

Now stick this in differential equation

$$x(t) = X_0 \cos(\omega t + \phi)$$

$$\frac{dx}{dt} = -\omega X_0 \sin(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = -\omega^2 X_0 \cos(\omega t + \phi)$$

(9)

$$-\omega^2 \alpha X_0 \cos(\omega t+\phi) - \omega\beta X_0 \sin(\omega t+\phi) + \gamma X_0 \cos(\omega t+\phi) = F_0 \cos(\omega t)$$

Remember

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A+B) = \sin A \cos B - \cos A \sin B$$

$$-\omega^2\alpha X_0(\cos\omega t\cos\phi - \sin\omega t\sin\phi) - \omega\beta X_0(\sin\omega t\cos\phi + \cos\omega t\sin\phi) + \gamma X_0(\cos\omega t \cdot \cos\phi - \sin\omega t\sin\phi) = F_0\cos(\omega t)$$

$$[-\omega^2\alpha X_0\cos\phi - \omega\beta X_0\sin\phi + \gamma X_0\cos\phi - F_0]\cos(\omega t) + [\omega^2\alpha\sin\phi - \omega\beta X_0\cos\phi - \gamma X_0\sin\phi]\sin(\omega t) = 0$$

This gives 2 equations which each must vanish

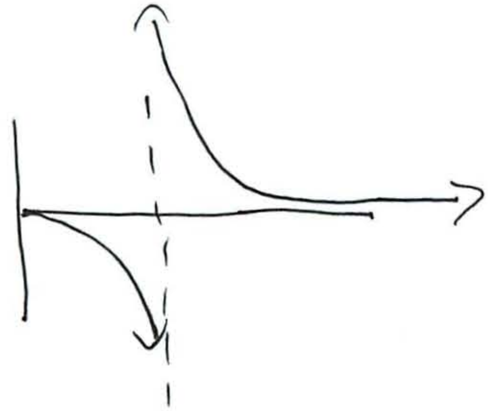
$$[\omega^2\alpha X_0\sin\phi - \omega\beta X_0\cos\phi - \gamma X_0\sin\phi]\sin(\omega t) = 0$$

$$\omega^2\alpha\sin\phi - \omega\beta\cos\phi - \gamma\sin\phi = 0$$

$$(\omega^2\alpha - \gamma)\tan\phi - \omega\beta = 0$$

$$\tan\phi = \frac{\omega\beta}{(\omega^2\alpha - \gamma)} = \frac{\omega\,\beta/\alpha}{\omega^2 - \gamma/\alpha} = \frac{\omega}{\tau_2}\cdot\frac{1}{\omega^2 - \omega_0^2}$$

(10)



ω_0 = natural freq

$\omega \to 0 \quad \phi \to 0$

$\omega \to \omega_0^{low} \quad \phi \to -\pi/2 \quad \leftarrow \omega < \omega_0$ ⊖ phase shift — capacitor like

$\omega \to \omega_0^{high} \quad \phi \to \pi/2 \quad \leftarrow \omega > \omega_0$ ⊕ phase shift — inductor like

$\omega \to \infty \quad \phi \to 0$

Now need to solve eqs

$$\left[-\omega^2 \alpha X_0 \cos\phi - \omega\beta X_0 \sin\phi + \gamma X_0 \cos\phi - F_0\right] \cos(\omega t) = 0$$

algebra a 4 bit lax

$$\left(-\omega^2 \alpha \cos\phi - \omega\beta \sin\phi + \gamma \cos\phi\right) X_0 = F_0$$

$$\left(-\omega^2 - \omega \cdot \frac{\beta}{\alpha} \tan\phi + \frac{\gamma}{\alpha}\right) X_0 = \frac{F_0}{\alpha \cos\phi}$$

$$\left[(\omega_0^2 - \omega^2) - \frac{\omega}{\tau_2} \tan\phi\right] X_0 = \frac{F_0}{\alpha} \sec\phi$$

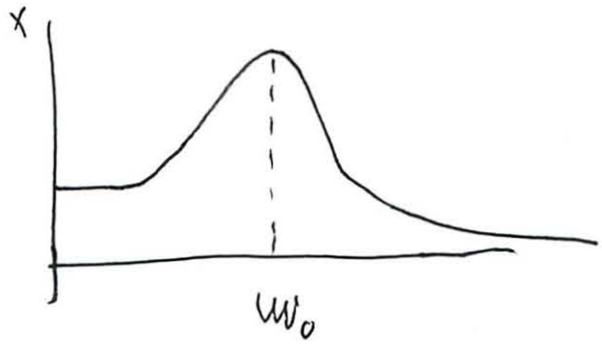
$$X_0 = \frac{F_0}{\alpha} \cdot \frac{\sqrt{1+\tan^2\phi}}{\left[(\omega_0^2 - \omega^2) - \frac{\omega}{\tau_2} \tan\phi\right]}$$

(11)

$\omega \to 0$ $\quad X_0 \to \frac{F_0}{\alpha \omega_0^2}$ going to DC, no more AC source

$\omega \to \omega_0$ $\quad X_0 \to \frac{F_0}{\alpha} \cdot \frac{\tau_2}{\omega_0}$

$\omega \to \infty$ $\quad X_0 \to 0$



↑ natural freq

large resonant response

Physics

$$\tau_1 = \frac{\beta}{\gamma} = RC$$

$$\tau_2 = \frac{\alpha}{\beta} = \frac{L}{R}$$

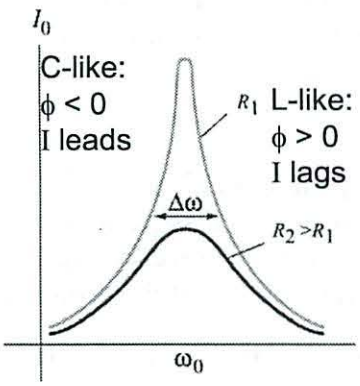
$$\omega_0 = \sqrt{\frac{\gamma}{\alpha}} = \frac{1}{\sqrt{LC}}$$

Topics: Driven LRC Circuits
Related Reading: Course Notes: Sections 12.8-12.9

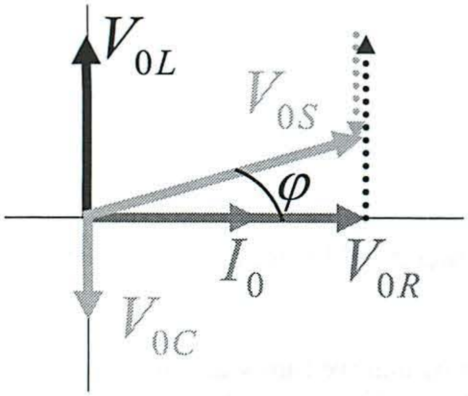
Topic Introduction

Today's problem solving focuses on the driven RLC circuit, which we discussed last class.

Terminology: Resistance, Reactance, Impedance
Before starting I would like to remind you of some terms that we throw around nearly interchangeably, although they aren't. When discussing resistors we talk about their *resistance R*, which gives the relationship between voltage across them and current through them. For capacitors and inductors we do the same, introducing the term *reactance X*. That is, $V_0=I_0X$, just like V=IR. What is the difference? In resistors the current is in phase with the voltage across them. In capacitors and inductors the current is $\pi/2$ out of phase with the voltage across them (current leads in a capacitor, lags in an inductor). This is why I can only write the relationship for the amplitudes $V_0 = I_0X$ and **not** for the time dependent values V=IX. When talking about combinations of resistors, inductors and capacitors, we use the *impedance Z*: $V_0 = I_0Z$. For a general Z the phase is neither 0 (as for R) or $\pi/2$ (as for X).



Resonance
Recall that when you drive an RLC circuit, that the current in the circuit depends on the frequency of the drive. Two typical response curves (I vs. drive ω) are shown at left, showing that at resonance ($\omega = \omega_0$) the current is a maximum, and that as the drive is shifted away from the resonance frequency, the magnitude of the current decreases. In addition to the magnitude of the current, the phase shift between the drive and the current also changes. At low frequencies, the capacitor dominates the circuit (it fills up more readily, meaning it has a higher impedance), so the circuit looks "capacitance-like" – the current leads the drive voltage. At high frequencies the inductor dominates the circuit (the rapid changes means it is fighting hard all the time, and has a high impedance), so the circuit looks "inductor-like" – the current lags the drive voltage. Notice that the resistor has the effect of reducing the overall amplitude of the current, and that its effect is particularly acute on resonance. This is because on resonance the impedance of the circuit is dominated by the resistance, whereas off resonance the impedance is dominated by either capacitance (at low frequencies) or inductance (at high frequencies).



Seeing it Mathematically – Phasors

It turns out that a nice way of looking at these relationships is thru phasor diagrams. A phasor is just a vector whose magnitude is the amplitude of either the voltage or current through a given circuit element and whose angle corresponds to the phase of that voltage or current. In thinking about time dependence of a signal, we allow the phasors to rotate about the origin (in a counterclockwise fashion) with time, and only look at their component along the y-axis. This component oscillates, just like the current and voltages in the circuit, even though the total amplitude of the signal (the length of the vector) stays the same.

We use phasors because they allow us to add voltages across different circuit elements even though those voltages are not in phase with each other (so you can't just add them as numbers). For example, the phasor diagram above illustrates the relationship of voltages in a series LRC circuit. The current *I* is assigned to be at "0 phase" (along the x-axis). The phase of the voltage across the resistor is the same. The voltage across the inductor L leads (is ahead of *I*) and the voltage across the capacitor C lags (is behind *I*). If you add up (using vector arithmetic) the voltages across R, L & C (the red and dashed blue & green lines respectively) you must arrive at the voltage across the power supply. This then gives you a rapid way of understanding the phase between the drive (the power supply voltage V_S) and the response (the current) – here labeled ϕ.

Power

Power dissipation in AC circuits is very similar to power dissipation in DC circuits – only the resistors dissipate any power. The big difference is that now the power dissipated, like everything else, oscillates in time. We thus discuss the idea of *average* power dissipation. To average a function that oscillates in time, we integrate it over a period of the oscillation, and divide by that period: $<P>=\frac{1}{T}\int_0^T P(t)\,dt$ (if you don't see why this is the case, draw some arbitrary function and ask yourself what the average height is – it's the area under the curve divided by the length). Conveniently, the average of $\sin^2(\omega t)$ (or $\cos^2(\omega t)$) is ½. Thus although the *instantaneous* power dissipated by a resistor is $P(t)=I(t)^2 R$, the *average* power is given by $<P>=\frac{1}{2}I_0^2 R = I_{rms}^2 R$, where "RMS" stands for "root mean square" (the square root of the time average of the function squared).

Important Equations

Impedance of R, L, C: $R = R$ (in phase), $X_C = \frac{1}{\omega C}$ (I leads), $X_L = \omega L$ (I lags)

Impedance of Series RLC Circuit: $Z=\sqrt{R^2+\left(X_L-X_C\right)^2}$

Phase in Series RLC Circuit: $\varphi = \tan^{-1}\left(\frac{X_L-X_C}{R}\right)$

Look at phasor diagram to see this! Pythagorean Theorem

Fri Added Lecture

4/16

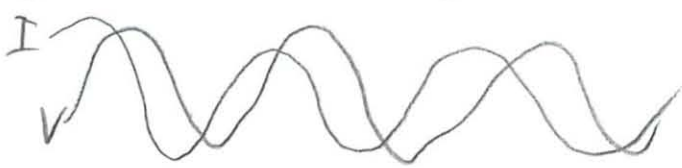
AC circuits

Resonance

- when drive freq = natural freq
- how radio tuners work
  - tune resonance = freq of station you want

Reactance

- just like resistance



So want $\frac{V_0}{I_0}$ = reactance/impedance

↑ magnitudes

just like $\frac{V}{I}$ = resistance

$X_L = \omega L$

inductance

bigger L = works harder

- harder it is for current to flow through

higher ω (freq) = works harder

$X_C = \frac{1}{\omega C}$

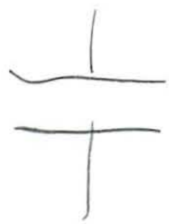
big C = eaiser to charge

- backwards

never gets fully charged up

Phase

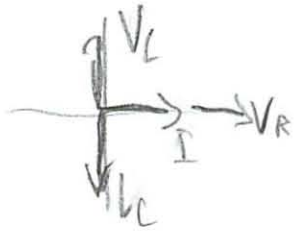
- so far in course, everything happened at same time



I leads
V lags

↓I lags
V leads

Phasors



lenght = magnitude
angle = Phase

Skip to sample problem 1

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Problem Solving 8: Driven RLC Circuits

OBJECTIVES

1. To explore the relationship between driven current and driving *emf* in three simple circuits that contain: (1) only resistance; (2) only inductance; and (3) only capacitance.

2. To examine these same relationships in the general case where R, L, and C are all present, and to do two sample problems on the LRC circuits.

REFERENCE: Sections 12.1 – 12.4, 8.02 Course Notes.

General Properties of Driven LRC Circuits

An LRC circuit is the electrical analog of a mass on a spring. We distinguish two behaviors. In the first, we consider its "free" oscillations that occur when we "kick" the circuit (charge the capacitor or send a constant current through) and then stand back and watch it oscillate. If we do this we will see a natural frequency of oscillation that decays in a finite time.

A second behavior emerges if we "drive" the LRC circuit with a source of *emf* with some (arbitrary) amplitude and frequency. If we drive the circuit with an *emf* $V(t) = V_0 \sin \omega t$, where ω is any frequency we desire (we get to pick this) and V_0 is any amplitude we desire, then the "driven" response of the system is given by

$$I(t) = I_0 \sin(\omega t - \phi)$$

where

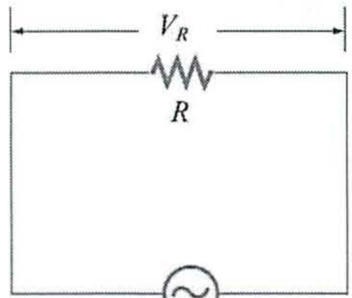
$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}, \quad \tan\phi = \left(\frac{\omega L - \dfrac{1}{\omega C}}{R}\right) \tag{8.1}$$

Note the "driven" response is at the (arbitrary) frequency of the driver, and ***not*** at the natural frequency of the system. However the system will show maximum response to the driving *emf* when the driving frequency ***is*** at the natural frequency of oscillation of the system, i.e. when $\omega = 1/\sqrt{LC}$. We can compute the average power consumed by the circuit by calculating the time average of $I(t)V(t)$ (see Section 12.4, 8.02 Course Notes):

$$\langle P(t)\rangle = \langle I(t)V(t)\rangle = \frac{1}{2} I_0 V_0 \cos\phi \tag{8.2}$$

Example 1: Driven circuit with resistance only

We begin with a circuit which contains only resistance. The circuit diagram is shown below.



The circuit equation is

$$I_R(t)R - V(t) = 0 .$$

Question 1: What is the amplitude I_{R0} and phase ϕ of the current $I_R(t) = I_{R0}\sin(\omega t - \phi)$?

Answer: (*answer this and subsequent questions on the tear-off sheet at the end*)

Question 2: What values of L and C do you choose in the general equation (8.1) to reproduce the result you obtained in your answer above? HINT: This is as easy (and as strange) as you probably first think.

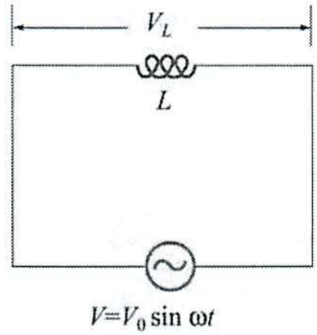
Answer:

Question 3: What is the *time-averaged* power $\langle P_R(t)\rangle = \langle I_R(t)V_R(t)\rangle$ dissipated in this circuit? You need to know that the time average of $\sin^2 \omega t$ is $\langle \sin^2 \omega t\rangle = 1/2$.

Answer:

Example 2: Driven circuit with inductance only

Now suppose the voltage source $V(t) = V_0 \sin(\omega t)$ is connected in a circuit with only self-inductance. The circuit diagram is



The circuit equation is

$$V(t) = L\frac{dI}{dt}$$

Question 4: Solve the above equation for the current as a function of time. If we write this current in the form $I_L(t) = I_{L0}\sin(\omega t - \phi)$, what is the amplitude I_{L0} and phase ϕ of the current? You may need to use the trigonometric identity that $\sin(\omega t - \phi) = \sin\omega t\cos\phi - \sin\phi\cos\omega t$.

Answer:

Question 5: What values of R and C do you choose in the general equation (8.1) to reproduce the result you obtained in the question above?

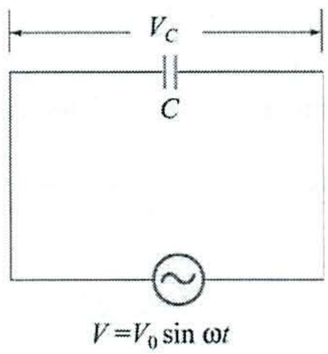
Answer:

Question 6: What is the *time-averaged* power $\langle P_L(t)\rangle = \langle I_L(t)V_L(t)\rangle$ dissipated in this circuit? You will need to know that the time-average of $\sin\omega t\cos\omega t$ is $\langle \sin\omega t\cos\omega t\rangle = 0$.

Answer:

Example 3: Driven circuit with capacitance only

The ac voltage source $V(t) = V_0 \sin(\omega t)$ is connected in a circuit with capacitance only. The circuit diagram is



The circuit equation for this circuit is

$$\frac{Q}{C} - V(t) = 0$$

If we take the time derivative of this equation we get

$$\frac{I_C}{C} - \frac{d}{dt}V(t) = \frac{I_C}{C} - \omega V_0 \cos \omega t = 0$$

Question 7: Solve the above equation for the current as a function of time. If we write this current in the form $I_C(t) = I_{C0} \sin(\omega t - \phi)$, what is the amplitude I_{C0} and phase ϕ of the current? You may need to use the trigonometric identity that $\sin(\omega t - \phi) = \sin \omega t \cos \phi - \sin \phi \cos \omega t$.

Answer:

Question 8: What is the *time-averaged* power $\langle P_C(t) \rangle = \langle I_C(t) V_C(t) \rangle$ dissipated in this circuit? You will need to know that the time-average of $\sin \omega t \cos \omega t$ is $\langle \sin \omega t \cos \omega t \rangle = 0$.

Answer:

Sample Problem 1

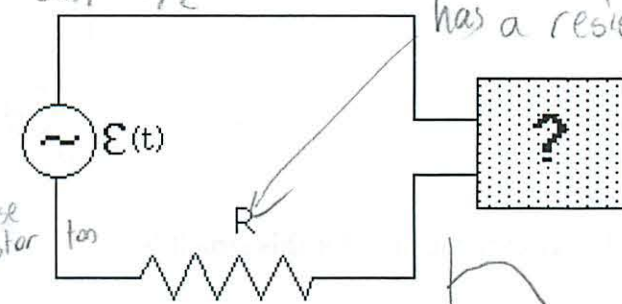
The circuit shown below contains an AC generator which provides a source of sinusoidally varying emf $\varepsilon(t) = \varepsilon_0 \sin\omega t$, a resistor with resistance R, and a "black box", which contains ***either*** an inductor ***or*** a capacitor, ***but not both***. The amplitude of the driving emf, ε_0, is $100\sqrt{2}$ Volts, and the angular frequency ω is 10 rad/sec. We measure the current in the circuit and find that it is given as a function of time by the expression: $I(t) = (10\text{ Amps})\sin(\omega t + \pi/4)$ [Note: $\pi/4$ radians = 45°, $\tan(\pi/4) = +1$].

RL circuit

↳ not π/2

Question 9: Does this current lead or lag the emf $\varepsilon(t)=\varepsilon_0 \sin(\omega t)$

Answer:



has a resistor

Current leads it by π/4

ωt − c

So

because resistor

Question 10: What is the unknown circuit element in the black box--an inductor or a capacitor?

Answer:

Capicator

I sin(ωt A) = 0

sin(ωt + π/4) = 0

needs ⊖

gets pushed to left

So it leads

Question 11: What is the numerical value of the resistance R? Indicate units.

Answer:

magnitudes = $\frac{V_0}{I_0} = \frac{100\sqrt{2}}{10} = 10\sqrt{2}\ \Omega$

Question 12: What is the numerical value of the capacitance *or* of ~~the inductance~~, as the case may be? Indicate units.

Answer:

$X_C = \frac{1}{\omega C} = 10\sqrt{2}$

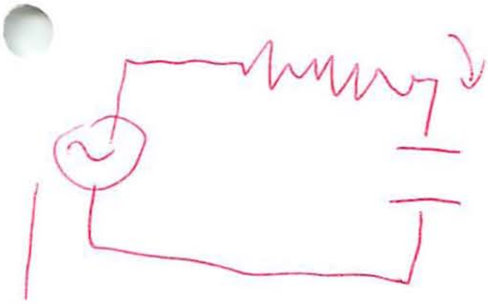
$\frac{1}{10C} = 10\sqrt{2}$

$\frac{1}{C} = 100\sqrt{2}$

$C = \frac{1}{100\sqrt{2}}$

Sample Problem 1

4/16



$I = I_0 \sin(\omega t + \pi/4)$

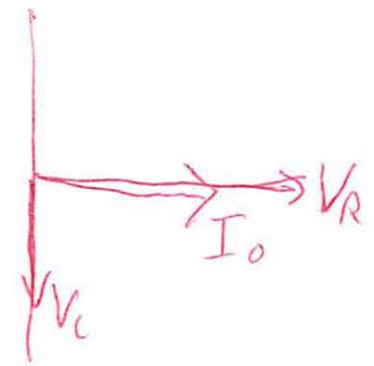
$\mathcal{E} = \sin \omega t$

$\mathcal{E} = I_0 Z$

↑ capitance included

can just calc impediance

but more conceptally



current leads by 45°

$\mathcal{E}$ lags by 45° $= V_C + V_R$

So $|V_R| = |V_C|$

$V_R = \frac{I_0}{R}$

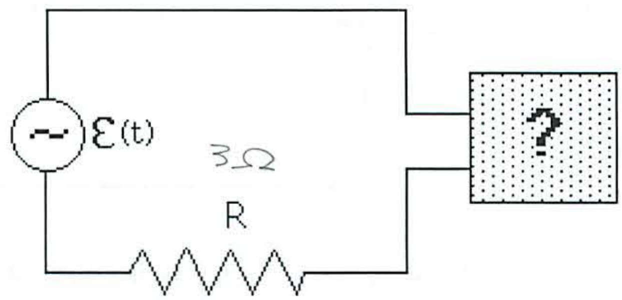
$V_{C_0} = X_C = \frac{1}{\omega C}$

at higher ω → inductor dominates
at lower ω → capicitor dominates
at intermediate ω → resonance

Sample Problem 2

The circuit shown below contains an AC generator which provides a source of sinusoidally varying emf $\varepsilon(t) = \varepsilon_0 \sin(\omega t)$, a resistor with resistance $R = 3\ \Omega$, and a "black box", which contains ***either*** an inductor ***or*** a capacitor, ***or both***. The amplitude of the driving emf, ε_0, is 6 Volt. We measure the current in the circuit at an angular frequency $\omega = 1$ radians/sec and find that it is exactly in phase with the driving emf. We measure the current in the circuit at an angular frequency $\omega = 2$ radians/sec and find that it is out of phase with the driving emf by exactly $\pi/4$ radians.

Question 13: What does the black box contain – an inductor or a capacitor, or both? Explain your reasoning. Does current lead or lag at $\omega = 2$ radians/sec?



Answer:

at ω = 1 it is in phase
just an inductor?
At ω = 2 rad/sec inductor dominates so current lags voltage.
b/c changing ω changes phase
at higher freq
Capacitor dominates at low freq

Question 14: What is the numerical value of the capacitance *or* of the inductance, *or of both*, as the case may be? Indicate units. Your answer(s) will involve simple fractions only, you will not need a calculator to find the value(s).

Answer:

$X_L = \omega L \qquad X_C = \frac{1}{\omega C}$

$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}$

$\tan\phi = \frac{X_L - X_C}{R}$

$L = 3/2\ H$

Resonance Formula

$I_0 = \frac{V_0}{\sqrt{18}} \qquad 9 + (3)^2 = \sqrt{18}$

$\tan(\pi/4) = 1(3\Omega) = (X_L - X_C) = 3$

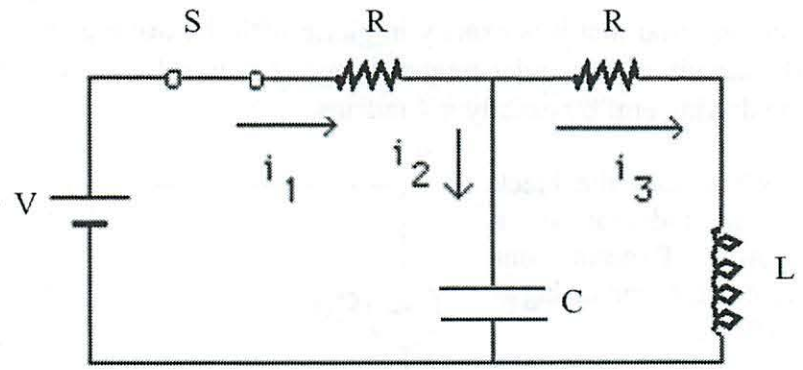
$I_0 = \frac{6}{\sqrt{18}}$

Question 15: What is numerical value of the *time-averaged* power dissipated in this circuit when $\omega = 1$ radians/sec? Indicate units, that is the time-average of $I(t)V(t)$. You will need to know that the time-average of $\sin^2 \omega t$ is ½.

Answer:

Sample Exam Question (If time, try to do this by yourself, closed notes)

A circuit consists of a battery with *emf V,* an inductor *L*, a capacitor *C*, and two resistors, each with resistance *R*, as shown in the sketch. The capacitor is initially uncharged and there is no current flowing anywhere in the circuit. The switch S has been open for a long time, and is then closed, as shown in the diagram.



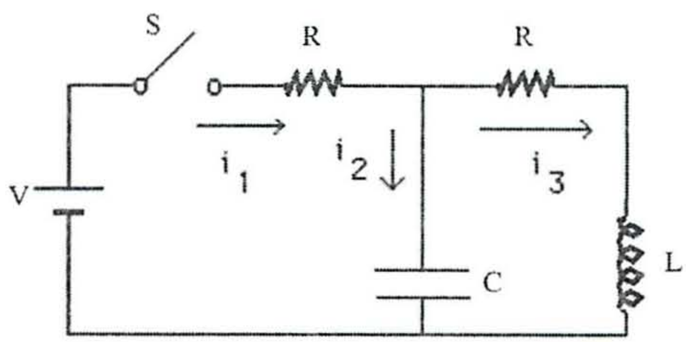
(a) Using Faraday's Law, what is the sum of the potential drops around the outer loop (the loop including both the battery and the inductor) if we move clockwise around the loop?

(b) ***Just after*** the switch *S* is closed, what are the currents i_1, i_2, and i_3 in terms of the given quantities? ***Assume that the left loop of the circuit has zero inductance.*** You do ***not*** have to solve any differential equations to answer this question.

(c) A long, long time after switch *S* is closed, what are the currents i_1, i_2, and i_3? You do ***not*** have to solve any differential equations to answer this question.

(d) A long, long time after switch S is closed, what is the charge on the capacitor? You do ***not*** have to solve any differential equations to answer this question.

(e) The switch S is now opened again. ***Just after*** the switch is opened again, what are the currents i_1, i_2, and i_3 in terms of the given quantities? ***Assume that the left loop of the circuit has zero inductance.***



Office Hrs

4/16

Exam 2 weeks from yesterday

Sample Problem 2

- $\omega = 1$ in phase
- $\omega = 2$ out of phase $\pi/4$
- $\mathcal{E}_0 = 6V$
- $R = 3\,\Omega$

What's in the box

in phase → only a resistor

or both inductor + capicator that cancle = resonance

Resonant current $= \frac{\mathcal{E}}{R} = \frac{6\text{ Volts}}{3\,\Omega} = 2\text{ Amps}$

impediance is only from resistor

lowest point in fine

natural freq $(\omega_0) = 1\,\omega$

Now we ↑ ω past natural freq

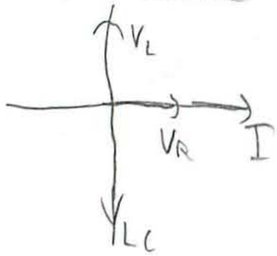
- the inductor works harder

If L dominates, who leads

- Voltage$_{PS}$ leads Current

$\omega = 1$

Resonance



$V_L = V_C$ = resonance

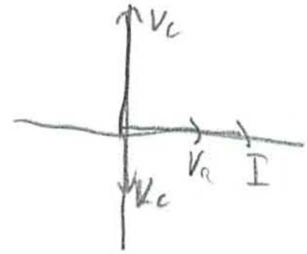
cancle

only V_R affects

$\omega = 2$

V_L gets bigger

V_C gets smaller

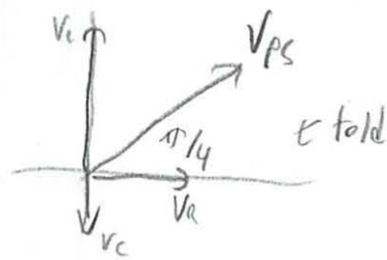


I decreases

V_R decreases too

So what is vector total $= V_{PS}$

*power supply



Now need to solve

Lets assume we did not know $\frac{\pi}{4}$

know

$$\omega_{Res} = \frac{1}{\sqrt{LC}} = 1 \text{ rad/sec} \quad ①$$

$$\tan\frac{\pi}{4} = \frac{opp}{adj} = \frac{V_L - V_C}{V_R} = \frac{X_L - X_C}{R} = 1 \quad ②$$

divided out the I

now have 2 eq w/ 2 unknows (①/②)

- R is given

$\Omega = 2$ rad/sec given

always draw phasor diagram

$X_L - X_C = R$

$\omega L - \frac{1}{\omega C} = R$

$LC\left(\frac{1}{s^2}\right) = 1$

? LCC

? have some constants

$\frac{1}{C} = \frac{L}{s^2}$

$\omega L - \frac{L}{\omega s^2}$

$L\left(\omega - \frac{1/sec^2}{\omega}\right) = R$

2 Henries = L

then solve for C

$C = \frac{1}{2}$

hint: don't plug in # till end
keep units in there

aside from algebra - good exam q.

Problem Set 10

Just LC circuit

energy is conserved

$U_C = \frac{1}{2}CV^2$ $\quad U_L = \frac{1}{2}LI^2$

Doing # 4

Exam Topics

Ampere's
Faridays
Inductance

RL
LC

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Tear off this page and turn it in at the end of class !!!!

Note: Writing in the name of a student who is not present is a COD offense.

Problem Solving 8: Driven RLC Circuits

Group 11C L01 **(e.g. L02 6A Please Fill Out)**

Names Michael Plasmeier

Jennifer Quintana

Melanie Alba

4

Example 1: Driven circuit with resistance only

Question 1: What is the amplitude I_{R0} and phase ϕ of the current $I_R(t) = I_{R0}\sin(\omega t - \phi)$?

Answer: I_{R0} :__________ ϕ :__________

Question 2: What values of L and C do you choose in the general equation (8.1) to reproduce the result you obtained in your answer above?

Answer: L:__________ C:__________

Question 3: What is the *time-averaged* power $\langle P_R(t)\rangle = \langle I_R(t)V_R(t)\rangle$ dissipated?

Answer: $\langle P_R(t)\rangle =$ __________

Example 2: Driven circuit with inductance only

Question 4: What is the amplitude I_{L0} and phase ϕ of the current $I_L(t) = I_{L0}\sin(\omega t - \phi)$?

Answer: I_{L0} :__________ ϕ :__________

Question 5: What values of R and C do you choose in the general equation (8.1) to reproduce the result you obtained in the question above?

Answer: R: __________ C:__________

Question 6: What is the *time-averaged* power $\langle P_L(t)\rangle = \langle I_L(t)V_L(t)\rangle$ dissipated?

Answer: $\langle P_L(t)\rangle =$ ______________________

Example 3: Driven circuit with capacitance only

Question 7: What is the amplitude I_{C0} and phase ϕ of the current $I_C(t) = I_{C0}\sin(\omega t - \phi)$?

Answer: I_{C0}: ______________________ ϕ: ______________

Question 8: What is the *time-averaged* power $\langle P_C(t)\rangle = \langle I_C(t)V_C(t)\rangle$ dissipated?

Answer: $\langle P_C(t)\rangle =$ ______________________

Sample Problem 1:

Question 9: Does this current lead or lag the emf $\varepsilon(t) = \varepsilon_0 \sin\omega t$

Answer: leads by $\pi/4$

Question 10: What is the unknown circuit element in the black box--an inductor or a capacitor?

Answer: Capicator

Question 11: What is the numerical value of the resistance R? Indicate units.

Answer: $\frac{V_0}{I_0} = \frac{100\sqrt{2}}{10} = 10$ ~~[illegible]~~ Ω

Question 12: What is the numerical value of the capacitance *or* of the inductance? Indicate units.

Answer: $X_C = \frac{1}{\omega C} = 10\sqrt{2} \rightarrow C = \frac{1}{100}$ ~~[illegible]~~

Sample Problem 2:

Question 13: What does the black box contain--an inductor or a capacitor, or both? Explain your reasoning. Does the current lead or lag at $\omega = 2$ radians/sec?

Answer: lags b/c inductor

Question 14: What is the numerical value of the capacitance *or* of the inductance, *or of both*, as the case may be? Indicate units. Your answer(s) will involve simple fractions only, you will not need a calculator to find the value(s).

Answer: *L:* 3/2 2 *C:* \ .5

Question 15: What is numerical value of the *time-averaged* power dissipated in this circuit when ω =1 radians/sec? Indicate units.

Answer: ______________________________

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Problem Solving 8 *Solutions*: Driven RLC Circuits

OBJECTIVES

1. To explore the relationship between driven current and driving *emf* in three simple circuits that contain: (1) only resistance; (2) only inductance; and (3) only capacitance.

2. To examine these same relationships in the general case where *R*, *L*, and *C* are all present, and to do two sample problems on the *LRC* circuits.

REFERENCE: Sections 12.1 – 12.4, 8.02 Course Notes.

General Properties of Driven LRC Circuits

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A second behavior emerges if we "drive" the *LRC* circuit with a source of *emf* with some (arbitrary) amplitude and frequency. If we drive the circuit with an *emf* $V(t) = V_0 \sin \omega t$, where ω is any frequency we desire (we get to pick this) and V_0 is any amplitude we desire, then the "driven" response of the system is given by

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where

$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}, \quad \tan\phi = \left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) \tag{8.1}$$

Note the "driven" response is at the (arbitrary) frequency of the driver, and ***not*** at the natural frequency of the system. However the system will show maximum response to the driving *emf* when the driving frequency ***is*** at the natural frequency of oscillation of the system, i.e. when $\omega = 1/\sqrt{LC}$. We can compute the average power consumed by the circuit by calculating the time average of $I(t)V(t)$ (see Section 12.4, 8.02 Course Notes):

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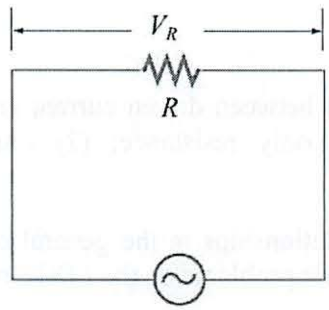


The circuit equation is

$$I_R(t)R - V(t) = 0 .$$

Question 1: What is the amplitude I_{R0} and phase ϕ of the current $I_R(t) = I_{R0}\sin(\omega t - \phi)$?

Answer: *(answer this and subsequent questions on the tear-off sheet at the end)*

$$I_{R0} = \frac{V_0}{R}, \quad \phi = 0$$

Question 2: What values of L and C do you choose in the general equation (8.1) to reproduce the result you obtained in your answer above?

Answer:

$L = 0$ and $C = \infty$ (this value of C makes q/C zero)

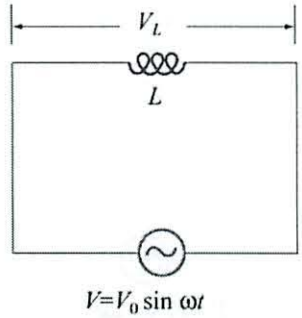
Question 3: What is the *time-averaged* power $\langle P_R(t)\rangle = \langle I_R(t)V_R(t)\rangle$ dissipated in this circuit? You will need to know that the time average of $\sin^2 \omega t$ is $\langle \sin^2 \omega t \rangle = 1/2$.

Answer:

$$\langle P_R(t)\rangle = \langle I_R(t)V_R(t)\rangle = \frac{V_0^2}{R}\langle \sin^2 \omega t\rangle = \frac{V_0^2}{2R}$$

Example 2: Driven circuit with inductance only

Now suppose the voltage source $V(t) = V_0 \sin(\omega t)$ is connected in a circuit with only self-inductance. The circuit diagram is



The circuit equation is

$$V(t) = L\frac{dI}{dt}$$

Question 4: Solve the above equation for the current as a function of time. If we write this current in the form $I_L(t) = I_{L0}\sin(\omega t - \phi)$, what is the amplitude I_{L0} and phase ϕ of the current? You may need to use the trigonometric identity that $\sin(\omega t - \phi) = \sin\omega t\cos\phi - \sin\phi\cos\omega t$.

Answer:

$I_{L0} = \dfrac{V_0}{\omega L}$ and $\phi = \pi/2$

Question 5: What values of R and C do you choose in the general equation (8.1) to reproduce the result you obtained in the question above?

Answer:

$R = 0$ and $C = \infty$ (this value of C makes q/C zero)

Question 6: What is the *time-averaged* power $\langle P_L(t)\rangle = \langle I_L(t)V_L(t)\rangle$ dissipated in this circuit? You will need to know that the time-average of $\sin\omega t\cos\omega t$ is $\langle \sin\omega t\cos\omega t\rangle = 0$.

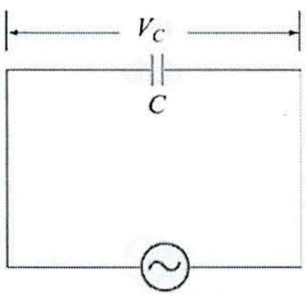
Answer:

$$I_L(t) = I_{L0}\sin(\omega t - \pi/2) = -(V_0/\omega L)\cos\omega t$$

$$\langle P_L(t)\rangle = \langle I_L(t)V_L(t)\rangle = -\frac{V_0^2}{\omega L}\langle \cos\omega t\sin\omega t\rangle = 0$$

Example 3: Driven circuit with capacitance only

The ac voltage source $V(t) = V_0 \sin(\omega t)$ is connected in a circuit with capacitance only. The circuit diagram is



The circuit equation for this circuit is

$$\frac{Q}{C} - V(t) = 0$$

If we take the time derivative of this equation we get

$$\frac{I_C}{C} - \frac{d}{dt}V(t) = \frac{I_C}{C} - \omega V_0 \cos \omega t = 0$$

Question 7: Solve the above equation for the current as a function of time. If we write this current in the form $I_C(t) = I_{C0} \sin(\omega t - \phi)$, what is the amplitude I_{C0} and phase ϕ of the current? You may need to use the trigonometric identity that $\sin(\omega t - \phi) = \sin \omega t \cos \phi - \sin \phi \cos \omega t$.

Answer:

$I_{C0} = \omega C V_0$ and $\phi = -\pi / 2$

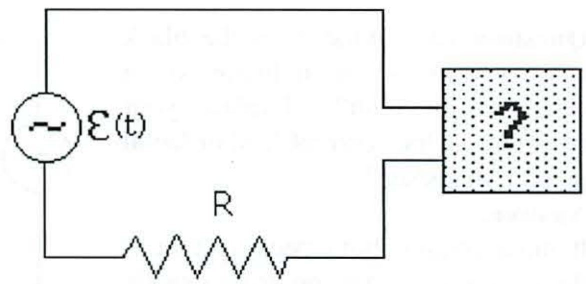
Question 8: What is the *time-averaged* power $\langle P_C(t) \rangle = \langle I_C(t) V_C(t) \rangle$ dissipated in this circuit? You will need to know that the time-average of $\sin \omega t \cos \omega t$ is $\langle \sin \omega t \cos \omega t \rangle = 0$.

Answer:

$$I_C(t) = I_{C0} \sin(\omega t + \pi / 2) = V_0 \omega C \cos \omega t$$
$$\langle P_C(t) \rangle = \langle I_C(t) V_C(t) \rangle = V_0^2 \omega C \langle \cos \omega t \sin \omega t \rangle = 0$$

Sample Problem 1

The circuit shown below contains an AC generator which provides a source of sinusoidally varying emf $\varepsilon(t) = \varepsilon_0 \sin\omega t$, a resistor with resistance R, and a "black box", which contains ***either*** an inductor ***or*** a capacitor, ***but not both***. The amplitude of the driving emf, ε_0, is $100\sqrt{2}$ Volts, and the angular frequency ω is 10 rad/sec. We measure the current in the circuit and find that it is given as a function of time by the expression: $I(t) = (10 \text{ Amps}) \sin(\omega t + \pi/4)$ [Note: $\pi/4$ radians = 45°, $\tan(\pi/4) = +1$].



Question 9: Does this current lead or lag the emf $\varepsilon(t)=\varepsilon_0 \sin(\omega t)$
Answer:

The current leads the AC generator, that is, it peaks before the driving voltage.

Question 10: What is the unknown circuit element in the black box--an inductor or a capacitor?
Answer:

The unknown circuit element must be a capacitor, because the current is leading (need a current to charge and hence get voltage on the capacitor).

Question 11: What is the numerical value of the resistance R? Indicate units.
Answer:

In this series RC circuit we have:

$$\tan\phi = \tan\left(\frac{\pi}{4}\right) = 1 = \frac{X_C}{R} \Rightarrow X_C = R$$

$$I_0 = \frac{\varepsilon_0}{\left[R^2 + X_C{}^2\right]^{1/2}} = \frac{\varepsilon_0}{\left[R^2 + R^2\right]^{1/2}} = \frac{100\sqrt{2}\text{ Volts}}{\sqrt{2R^2}} = 10\text{ Amps}$$

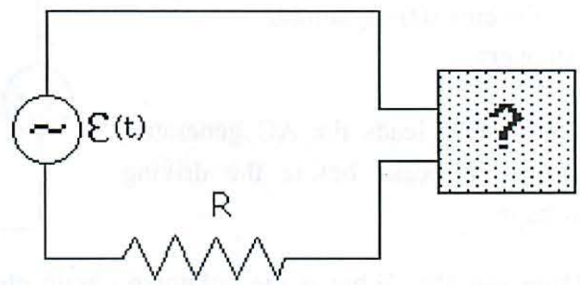
$$\Rightarrow R = 10\text{ Ohms}$$

Question 12: What is the numerical value of the capacitance *or* of the inductance, as the case may be?. Indicate units.
Answer:

$$\text{The capacitance is } C = \frac{1}{\omega R} = \frac{1}{(10\text{ radians/sec})(10\,\Omega)} = \frac{1}{100}\text{ Farads} = 10\text{ mF}$$

Sample Problem 2

The circuit shown below contains an AC generator which provides a source of sinusoidally varying emf $\varepsilon(t) = \varepsilon_0 \sin(\omega t)$, a resistor with resistance $R = 3\ \Omega$, and a "black box", which contains ***either*** an inductor ***or*** a capacitor, ***or both***. The amplitude of the driving emf, ε_0, is 6 Volt. We measure the current in the circuit at an angular frequency ω =1 radians/sec and find that it is exactly in phase with the driving emf. We measure the current in the circuit at an angular frequency $\omega = 2$ radians/sec and find that it is out of phase with the driving emf by exactly $\pi/4$ radians.



Question 13: What does the black box contain – an inductor or a capacitor, or both? Explain your reasoning. Does current lead or lag at $\omega = 2$ radians/sec?

Answer:

It must contain both because that is the only way to get the drive exactly in phase with the current (unless there are neither, but that wasn't an option).

Question 14: What is the numerical value of the capacitance *or* of the inductance, *or of both,* as the case may be? Indicate units. Your answer(s) will involve simple fractions only, you will not need a calculator to find the value(s).

Answer:

There are two unknowns (L & C) so we need two equations:

$$\text{At } \omega_1 = 1\text{s}^{-1}: \tan\phi = \tan 0 = 0 = \frac{\omega_1 L - \frac{1}{\omega_1 C}}{R} \quad \Rightarrow L = \frac{1}{\omega_1^2 C}$$

$$\text{At } \omega_2 = 2\text{s}^{-1}: \tan\phi = \tan\frac{\pi}{4} = 1 = \frac{\omega_2 L - \frac{1}{\omega_2 C}}{R} \Rightarrow \omega_2 L - \frac{1}{\omega_2 C} = R$$

$$\text{Solving: } R = \omega_2 L - \frac{1}{\omega_2 C} = \omega_2 \frac{1}{\omega_1^2 C} - \frac{1}{\omega_2 C} \Rightarrow C = \frac{1}{R}\left(\frac{\omega_2}{\omega_1^2} - \frac{1}{\omega_2}\right) = \frac{1}{(3\ \Omega)}\left(2\text{ s} - \frac{1}{2}\text{ s}\right) = \frac{1}{2}\text{ F}\ .$$

$$L = \frac{1}{\omega_1^2 C} = \frac{1}{\left(1\text{s}^{-2}\right)\left(1/2\text{ F}\right)} = 2\text{ H}$$

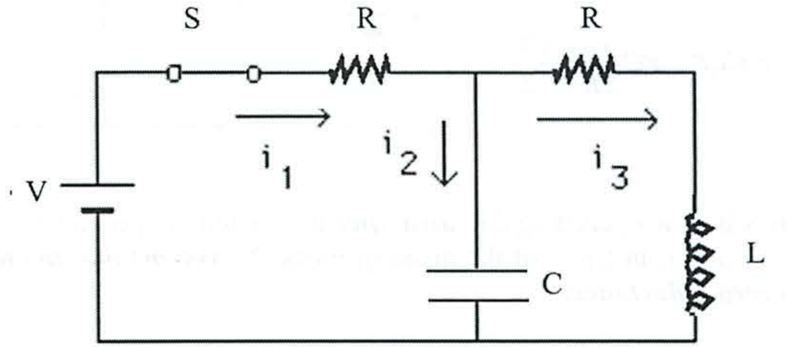
Question 15: What is numerical value of the *time-averaged* power dissipated in this circuit when ω =1 radians/sec? Indicate units, that is the time-average of $I(t)V(t)$. You will need to know that the time-average of $\sin^2 \omega t$ is ½.

Answer:

$$\text{On resonance: } \overline{P} = \frac{1}{2} I_0^2 R = \frac{\varepsilon_0^2}{2R} = \frac{(6\text{ V})^2}{2(3\ \Omega)} = 6\text{ W}$$

Sample Exam Question (If time, try to do this by yourself, closed notes)

A circuit consists of a battery with *emf V,* an inductor *L*, a capacitor *C*, and two resistors, each with resistance *R*, as shown in the sketch. The capacitor is initially uncharged and there is no current flowing anywhere in the circuit. The switch S has been open for a long time, and is then closed, as shown in the diagram.



(a) Using Kirchhoff's Loop Rule as modified for inductors, what is the sum of the potential drops around the outer loop (the loop including both the battery and the inductor) if we move clockwise around the loop?

$$V - i_1 R - i_3 R - L\frac{di_3}{dt} = 0$$

(b) ***Just after*** the switch *S* is closed, what are the currents i_1, i_2, and i_3 in terms of the given quantities? ***Assume that the left loop of the circuit has zero inductance.*** You do ***not*** have to solve any differential equations to answer this question.

$i_3 = 0$ Inductor prevents change from no current

$i_1 = i_2 = V/R$ Capacitor looks like a wire

(c) A long, long time after switch *S* is closed, what are the currents i_1, i_2, and i_3? You do ***not*** have to solve any differential equations to answer this question.

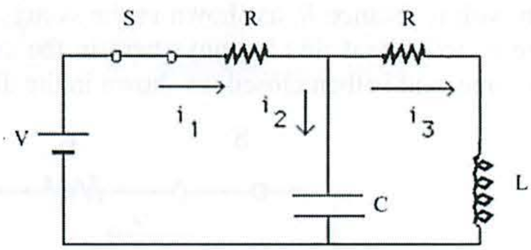
$i_2 = 0$ Capacitor is full

$i_1 = i_3 = V/2R$ Inductor looks like a wire

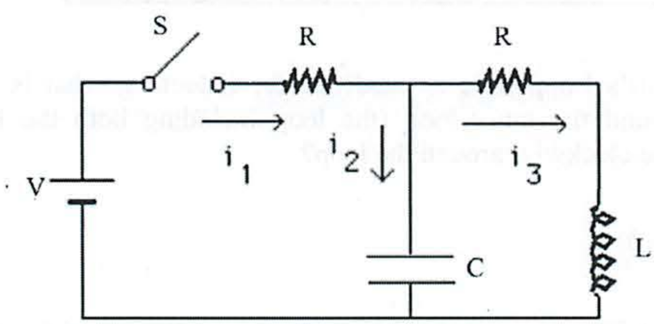
(d) A long, long time after switch S is closed, what is the charge on the capacitor? You do ***not*** have to solve any differential equations to answer this question.

The capacitor is in parallel with the resistor and inductor. The inductor isn't doing anything at this point, so

$$Q = CV_C = CV_R = Ci_3R = RC\frac{V}{2R} = \frac{CV}{2}$$



(e) The switch S is now opened again. ***Just after*** the switch is opened again, what are the currents i_1, i_2, and i_3 in terms of the given quantities? ***Assume that the left loop of the circuit has zero inductance.***



Just after the switch is opened the inductor will try to keep the current the same as it was just before the switch was opened, so

$i_1 = 0$ It's an open circuit

$i_3 = -i_2 = V/2R$ Inductor looks like a wire

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

8.02 **Spring 2010**

Problem Set 10

Due: Tuesday, April 20 at 9 pm.

Hand in your problem set in your section slot in the boxes outside the door of 32-082. Make sure you clearly write your name and section on your problem set.

Text: Liao, Dourmashkin, Belcher; Introduction to E & M MIT 8.02 Course Notes.

Week Eleven AC Circuits

Class 25 W11D1 M/T Apr 12/13	Undriven RLC Circuits; Expt. 8: RL Circuits and Undriven RLC Circuits
Reading:	Course Notes: Sections 11.5-11.11
Experiment:	Expt. 8: RL Circuits and Undriven RLC Circuits
Class 26 W11D2 W/R Apr 14/15	Driven RLC Circuits
Reading:	Course Notes: Sections 12.1-12.7
Class 27 W11D3 F Apr 16	PS08: RLC Circuits
Reading:	Course Notes: Sections 12.8-12.9

Week Twelve Self Inductance & Magnetic Energy

M Apr 19-T Apr 20 Patriot's Day Holiday

Class 28 W12D2 W/R Apr 21/22	Expt. 9: Driven RLC Circuits, Maxwell's Equations and Displacement Current; Poynting Vector & Energy Flow.
Reading:	Course Notes: Sections 13.1-13.3, 13.12.3-13.12.4
Experiment:	Expt. 9: Driven RLC Circuits

Drop Date Thurs Apr 22

Class 29 W12D3 F Apr 23	PS09: Poynting Vector and Energy Flow in a Capacitor
Reading:	Course Notes: Sections 13.1-13.3, 13.12.3-13.12.4

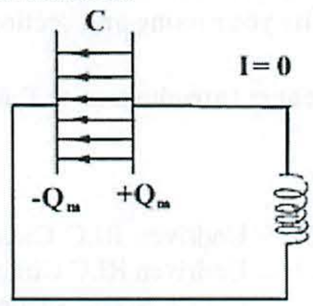
Problem 1: Show that

$$A\cos\omega t + B\sin\omega t = Q_{\mathrm{m}}\cos(\omega t + \phi),$$

where

$$Q_{\mathrm{m}} = \left(A^2 + B^2\right)^{1/2}, \text{ and } \phi = \tan^{-1}\left(-B/A\right).$$

Problem 2: In an LC circuit, the electric and magnetic fields are shown in the figure. Which of the following is true? **Explain your answer** At the moment depicted in the figure, the energy in the circuit is stored in



1. the electric field and is decreasing

2. the electric field and is constant.

3. the magnetic field and is decreasing.

4. the magnetic field and is constant.

5. in both the electric and magnetic field and is constant.

6. in both the electric and magnetic field and is decreasing.

Problem 3: In a freely oscillating *LC* circuit, (no driving voltage), suppose the maximum charge on the capacitor is $Q_{\max}$. Assume the circuit has zero resistance.

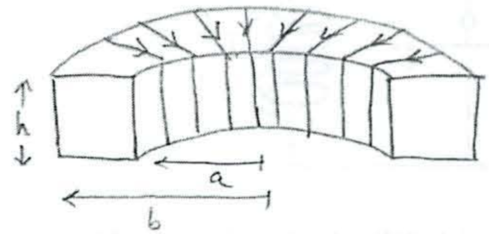
a) In terms of the maximum charge on the capacitor, what value of charge is present on the capacitor when the energy in the magnetic field is three times the energy in the electric field.

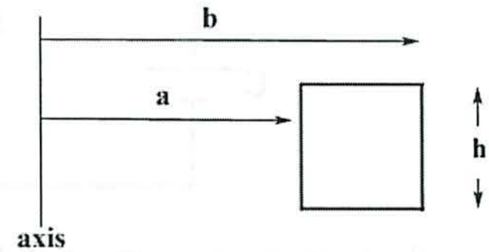
b) How much time has elapsed from when the capacitor is fully charged for this condition to arise?

c) If the resistance is non-zero, will the natural frequency of oscillation compared to the natural frequency of the ideal *LC* circuit (with zero resistance)

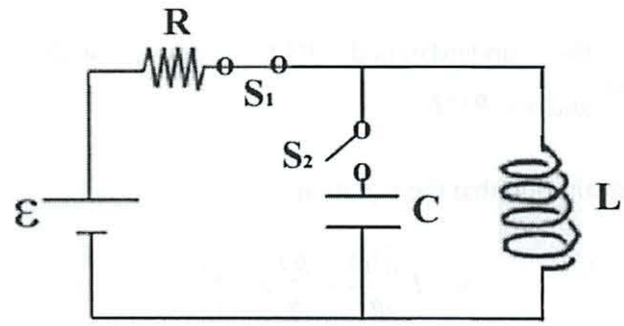
i) increase
ii) stay the same
iii) decrease

Problem 4: A toroidoil coil has N turns, and an inner radius a, outer radius b, and height h. The coil has a rectangular cross section shown in the figures below.





The coil is connected via a switch, S_1, to an ideal voltage source with electromotive force $\mathcal{E}$. The circuit has total resistance R. Assume all the self-inductance L in the circuit is due to the coil. At time $t = 0$ S_1 is closed and S_2 remains open.



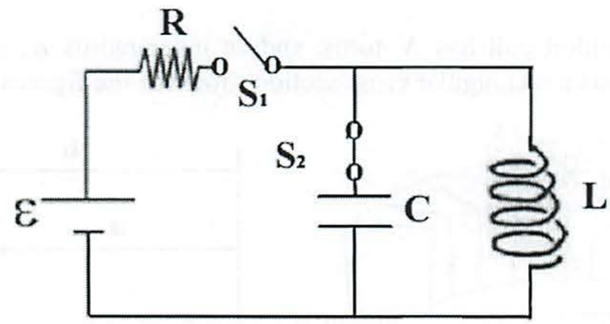
a) When a current I is flowing in the circuit, find an expression for the magnitude of the magnetic field inside the coil as a function of distance r from the axis of the coil.

b) What is the self-inductance L of the coil?

c) What is the current in the circuit a very long time $(t >> L/R)$ after S_1 is closed?

d) How much energy is stored in the magnetic field of the coil a very long time $(t >> L/R)$ after S_1 is closed?

For the next two parts, assume that a very long time $(t >> L/R)$ after the switch S_1 was closed, the voltage source is disconnected from the circuit by opening S_1, and by simultaneously closing S_2 the toroid is connected to a capacitor of capacitance C. Assume there is negligible resistance in this new circuit.



e) What is the maximum amount of charge that will appear on the capacitor?

f) How long will it take for the capacitor to first reach a maximal charge after S_2 has been closed?

Problem 5: For the underdamped RLC circuit, with $R^2 < 4L/C$, let $\gamma = \left(1/LC - R^2/4L^2\right)^{1/2}$ and $\alpha = R/2L$.

(a) Show by direct substitution that the equation

$$0 = L\frac{d^2Q}{dt^2} + \frac{dQ}{dt}R + \frac{Q}{C}$$

has a solution of the form

$$Q(t) = Ae^{-\alpha t}\cos(\gamma t + \phi)$$

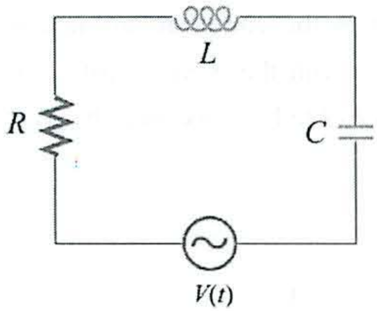
(b) Denote the current by

$$I(t) = \frac{dQ(t)}{dt} = Fe^{-\alpha t}\cos(\gamma t + \phi + \beta)$$

Find the constants F and β in terms of R, L and C as needed.

(c) Calculate the energy loss due to joule heating after one cycle of oscillation. For simplicity assume that $\phi = 0$.

Problem 6: Review Experiment 8: RL and Undriven RLC Circuits ; Read Experiment 9: Driven RLC Circuits



Consider the circuit at left, consisting of an AC function generator ($V(t) = V_0 \sin(\omega t)$, with $V_0 = 5$ V), an inductor $L = 8.5$ mH, resistor $R = 5\ \Omega$, capacitor $C = 100\ \mu$F and switch S.

The circuit has been running in equilibrium for a long time. We are now going to shut off the function generator (instantaneously replace it with a wire).

(a) Assuming that our driving frequency ω is not necessarily on resonance, what is the frequency with which the system will ring down (in other words, that current will oscillate at after turning off the function generator)? Feel free to use an approximation if you wish, just make sure you know you are.

(b) What (numerical) frequency f should we drive at to maximize the peak magnetic energy in the inductor?

(c) In this case, if we time the shut off to occur when the magnetic energy in the inductor peaks, after how long will the electric energy in the capacitor peak?

(d) Approximately how much energy will the resistor have dissipated during that time?

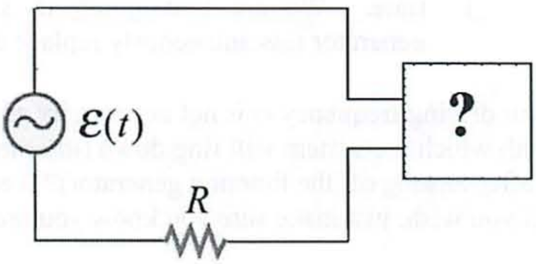
Problem 7: A series RLC circuit with $R = 10.0\ \Omega$, $L = 400$ mH and $C = 2.0\ \mu$F is connected to an AC voltage source $V(t) = V_0 \sin \omega t$ which has a maximum amplitude $V_0 = 100$ V.

(a) What is the resonant frequency ω_0?

(b) Find the rms current at resonance.

(c) Let the driving frequency be $\omega = 4000$ rad/s. Assume the current response is given by $I(t) = I_0 \sin(\omega t - \phi)$. Calculate the amplitude of the current and the phase shift between the current and the driving voltage.

Problem 8: The circuit shown below contains an AC generator which provides a source of sinusoidally varying emf $\mathcal{E} = \mathcal{E}_0 \sin \omega t$, a resistor with resistance $R = 6\,\Omega$, and a "black box", which contains ***either*** an inductor ***or*** a capacitor, ***or both***. The amplitude of the driving emf is $\mathcal{E}_0 = 6$ volt. We measure the current in the circuit at an angular frequency $\omega = 2\ \text{rad}\cdot\text{s}^{-1}$ and find that it is exactly in phase with the driving emf. We measure the current in the circuit at an angular frequency $\omega = 1\ \text{rad}\cdot\text{s}^{-1}$ and find that it is out of phase from the driving emf by exactly $\pi/4$ radians.



a) What does the black box contain – an inductor or a capacitor, or both? Explain your reasoning. Does current lead or lag at $\omega = 1\ \text{rad}\cdot\text{s}^{-1}$?

b) What is the numerical value of the capacitance *or* of the inductance, *or of both*, as the case may be? Indicate units. Your answer(s) will involve simple fractions only, you will not need a calculator to find the value(s).

c) What is ratio of the amplitudes of the current $\dfrac{I_0(\omega = 2\ \text{rad}\cdot\text{s}^{-1})}{I_0(\omega = 1\ \text{rad}\cdot\text{s}^{-1})}$?

Template:SI electromagnetism units

From Wikipedia, the free encyclopedia

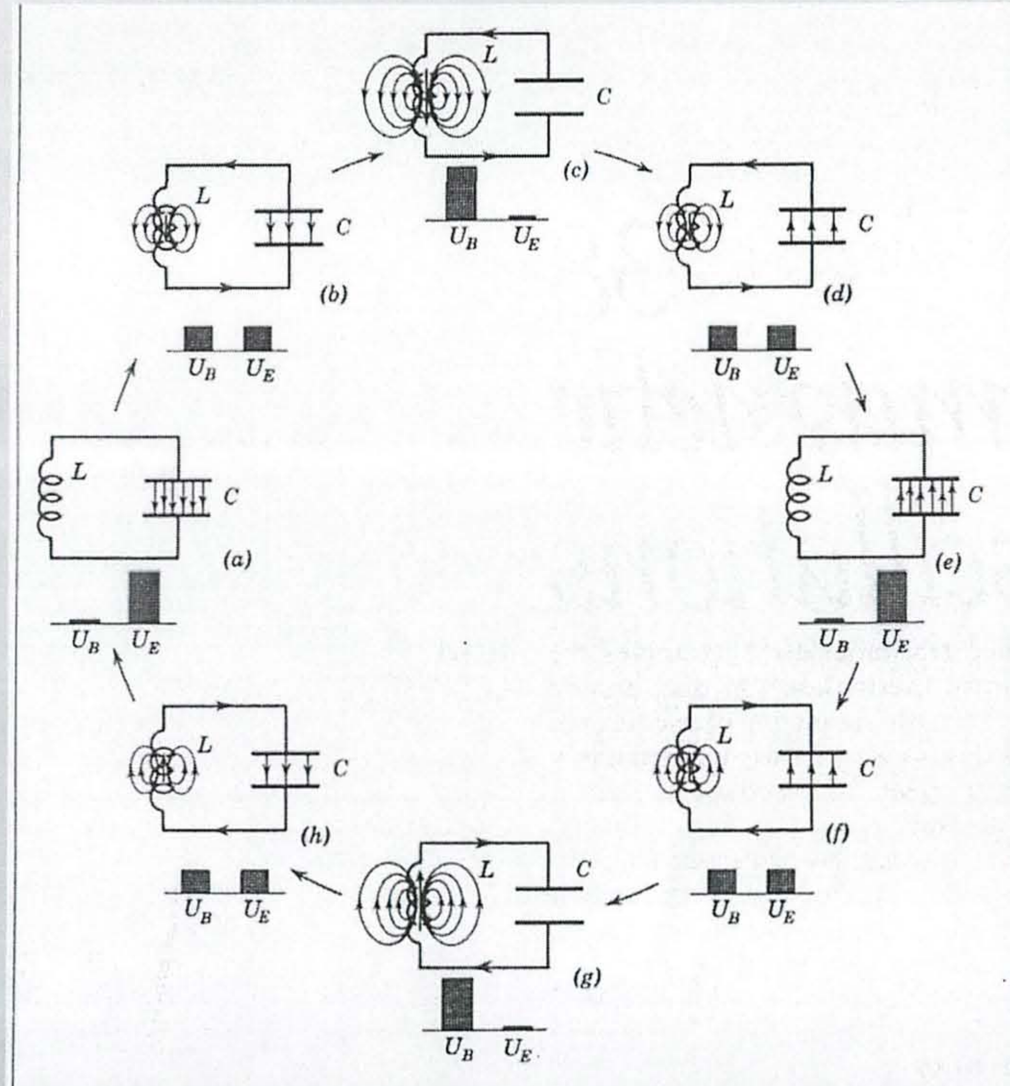
SI electromagnetism units				
Symbol[1]	Name of Quantity	Derived Units	Unit	Base Units
I	Electric current	ampere (SI base unit)	A	A (= W/V = C/s)
Q	Electric charge	coulomb	C	A·s
U, ΔV, $\Delta\varphi$; E	Potential difference; Electromotive force	volt	V	$\text{J/C} = \text{kg}\cdot\text{m}^2\cdot\text{s}^{-3}\cdot\text{A}^{-1}$
R; Z; X	Electric resistance; Impedance; Reactance	ohm	Ω	$\text{V/A} = \text{kg}\cdot\text{m}^2\cdot\text{s}^{-3}\cdot\text{A}^{-2}$
ρ	Resistivity	ohm metre	Ω·m	$\text{kg}\cdot\text{m}^3\cdot\text{s}^{-3}\cdot\text{A}^{-2}$
P	Electric power	watt	W	$\text{V}\cdot\text{A} = \text{kg}\cdot\text{m}^2\cdot\text{s}^{-3}$
C	Capacitance	farad	F	$\text{C/V} = \text{kg}^{-1}\cdot\text{m}^{-2}\cdot\text{A}^2\cdot\text{s}^4$
$\mathbf{E}$	Electric field strength	volt per metre	V/m	$\text{N/C} = \text{kg}\cdot\text{m}\cdot\text{A}^{-1}\cdot\text{s}^{-3}$
$\mathbf{D}$	Electric displacement field	coulomb per square metre	C/m^2	$\text{A}\cdot\text{s}\cdot\text{m}^{-2}$
ε	Permittivity	farad per metre	F/m	$\text{kg}^{-1}\cdot\text{m}^{-3}\cdot\text{A}^2\cdot\text{s}^4$
χ_e	Electric susceptibility	(dimensionless)	-	-
G; Y; B	Conductance; Admittance; Susceptance	siemens	S	$\Omega^{-1} = \text{kg}^{-1}\cdot\text{m}^{-2}\cdot\text{s}^3\cdot\text{A}^2$
κ, γ, σ	Conductivity	siemens per metre	S/m	$\text{kg}^{-1}\cdot\text{m}^{-3}\cdot\text{s}^3\cdot\text{A}^2$
$\mathbf{B}$	Magnetic flux density, Magnetic induction	tesla	T	$\text{Wb/m}^2 = \text{kg}\cdot\text{s}^{-2}\cdot\text{A}^{-1} = \text{N}\cdot\text{A}^{-1}\cdot\text{m}^{-1}$
Φ	Magnetic flux	weber	Wb	$\text{V}\cdot\text{s} = \text{kg}\cdot\text{m}^2\cdot\text{s}^{-2}\cdot\text{A}^{-1}$
$\mathbf{H}$	Magnetic field strength	ampere per metre	A/m	$\text{A}\cdot\text{m}^{-1}$
L, M	Inductance	henry	H	$\text{Wb/A} = \text{V}\cdot\text{s/A} = \text{kg}\cdot\text{m}^2\cdot\text{s}^{-2}\cdot\text{A}^{-2}$
μ	Permeability	henry per metre	H/m	$\text{kg}\cdot\text{m}\cdot\text{s}^{-2}\cdot\text{A}^{-2}$
χ	Magnetic susceptibility	(dimensionless)	-	-

References

This reference list does not appear in the article.

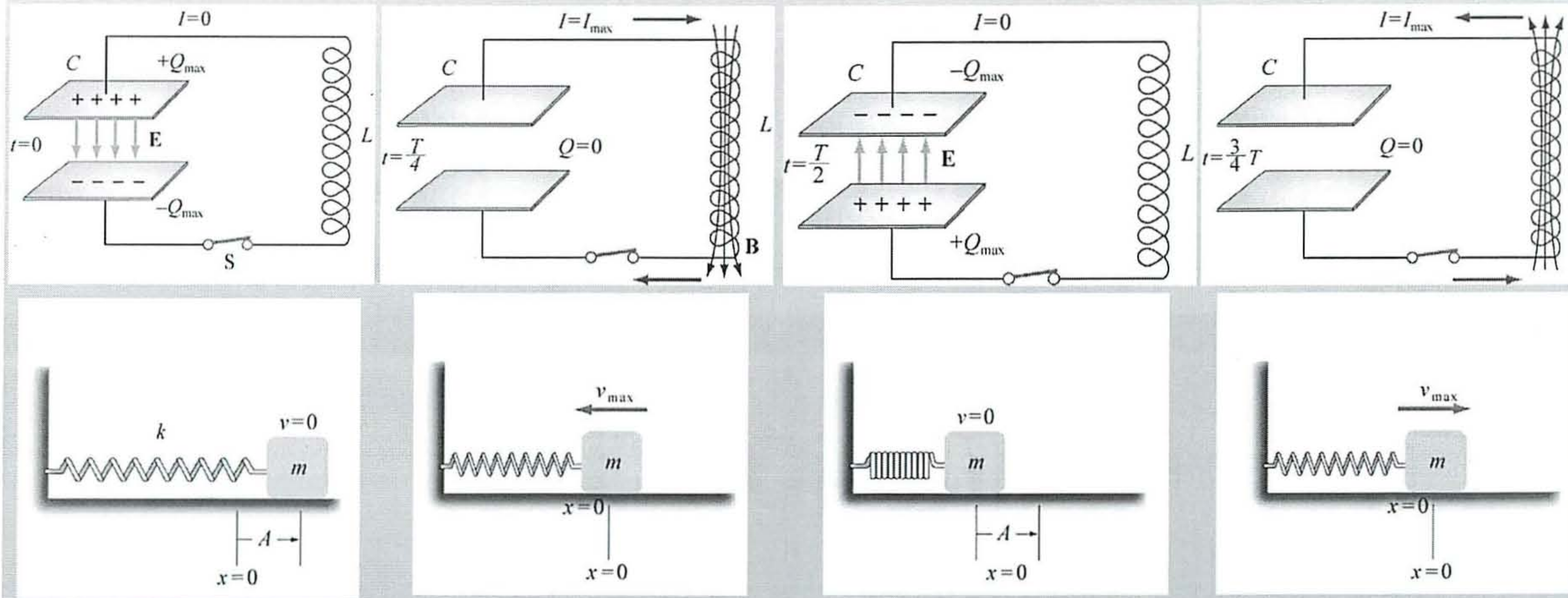
1. ^ International Union of Pure and Applied Chemistry (1993). *Quantities, Units and Symbols in Physical Chemistry*, 2nd edition, Oxford: Blackwell Science. ISBN 0-632-03583-8. pp. 14–15. Electronic version. (http://www.iupac.org/publications/books/gbook/green_book_2ed.pdf)

Summary: The Ideal LC Circuit



LC Circuit

It undergoes simple harmonic motion, just like a mass on a spring, with trade-off between charge on capacitor (Spring) and current in inductor (Mass)



8.02

P-Set 10

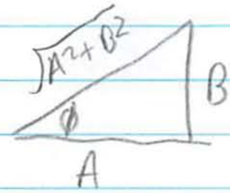
Michael Plasmeier L01 11C   −10 (90)   4/17

1. Show that $A\cos\omega t + B\sin\omega t = Q_m\cos(\omega t+\phi)$

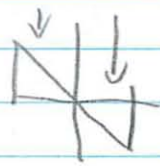
(see redo)

$Q_m = (A^2+B^2)^{1/2}$ $\phi = \tan^{-1}\left(\frac{-B}{A}\right)$

↑ so is that amplitude $\sqrt{A^2+B^2}$ ↑ Phase shift



in 2nd or 4th quad



∴ So this is basically a math problem
∴ look at SHM from 8.01

$\sqrt{A^2+B^2}\cos\left(\omega t + \tan^{-1}\left(\frac{-B}{A}\right)\right)$

SHM 8.01

So $A\cos\omega t + B\sin\omega t$ → general solution

$A = x_0$ $B = \frac{v_0}{\omega}$

$x_{max} = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$

** Can you box your answers or something? Please?

$\tan\phi = \frac{\omega x_0}{v_0}$

$\phi = \tan^{-1}\frac{\omega x_0}{v_0}$

identity $\sin(\omega t+\phi) = \sin\omega t\cos\phi + \cos\omega t\sin\phi$

$x(t) = \sqrt{\frac{2E}{k}}\sin\omega t\cos\phi + \sqrt{\frac{2E}{k}}\cos\omega t\sin\phi$

Sub back in

$$x_0 = \sqrt{\frac{2E}{k}} \sin\phi$$

$$V_0 = \omega \sqrt{\frac{2E}{k}} \cos\phi$$

so get $x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t$

or $A \cos \omega t + B \sin \omega t$

They did a really long proof of it which I dont really get Never get math parts

1. Redo Expand out according to rule

office hrs redo own

$$Q_m \cos \omega t \cos \phi - Q_m \sin \omega t \sin \phi$$

$$A = Q_m \cos \phi \qquad B = -Q_m \sin \phi$$

$$A \cos \omega t + B \cos \omega t$$

now prove this

$$Q_m = \left((Q_m \cos\phi)^2 + (-Q_m \sin \phi)^2 \right)^{1/2}$$

$$\sqrt{Q_m^2 \cos^2\phi + + Q_m^2 \sin^2\phi}$$ ← when you ^{2} sign goes away!

$$\sqrt{+Q_m^2 (\cos^2\phi + \sin^2\phi)}$$

$$\sqrt{+(Q_m^2)}$$

$$Q_m = Q_m$$

$$\phi = \tan^{-1}\left(\frac{-Q_m \sin\phi}{Q_m \cos\phi}\right)$$

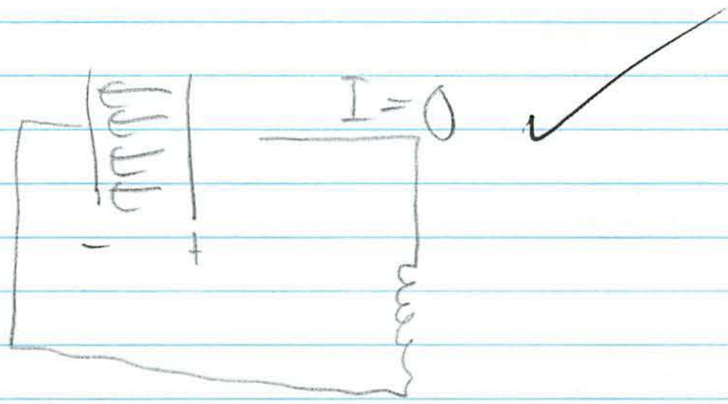
$$\phi = \tan^{-1}\left(\frac{\sin\phi}{\cos\phi}\right)$$

$$\phi = \tan^{-1}(\tan(\phi))$$

$$\phi = \phi$$

Much better!

2. In a LC circuit, what is true



all of the energy is stored in the capicator
no current flowing
constant electric field at this instance in time
no magnetic field

#2

3. In a freely oscilating LC circuit (no driving voltage)
suppose max charge on capicator = Q_{max}
0 resistance

✓

a) In terms of max charge on capicator, what value of charge is present on the capicator when energy in mag field is 3x energy electric field

$$\frac{Q}{C} + L\frac{dI}{dt} = 0$$

but that is current
charge is $\int I$
or q

~~$Q = LC\frac{dI}{dt}$~~

or do it energy

~~$U = U_E + U_B$~~
~~$= \frac{Q^2}{2C} + \frac{1}{2}LI^2 = \frac{Q_0^2}{2C}$~~

$$Q(t) = Q_0 \cos(\omega_0 t + \phi)$$

↑given ↑ $\frac{1}{\sqrt{LC}}$

bouncing b/w methods - don't know which to use

We want when $U_B = 3U_E$

~~$U = U_E + 3U_E$~~
~~$4U_E$~~

Or know what phase this occurs at

$$U_E = \frac{Q_0^2}{2C} \cos^2 \omega_0 t$$

$$U = \frac{4 Q_0}{2C} \cos^2 \omega_0 t$$

$$\frac{2 Q_0}{C} \cos^2 \omega_0 t$$

$$U = \frac{2 Q_{max}}{C} \cos^2\left(\frac{1}{\sqrt{LC}} t\right)$$

¿How to convert U to Q?

Or we just want the time t when

$$U_B = 3U_E$$

$$\left(\frac{Q_0^2}{2C}\right) \sin^2 \omega_0 t = 3 \left(\frac{Q_0^2}{2C}\right) \cos^2 \omega t$$

$$\sin^2(\omega_0 t) = 3 \cos^2(\omega t)$$

$$\frac{\sin^2(\omega_0 t)}{\cos^2(\omega_0 t)} = 3$$

$$\tan^2(\omega_0 t) = 3$$

$\tan^{-1}$ $\tan^{-1}$

hopefully figured out!

$$\omega_0 t = \tan^{-1}(\sqrt{3})$$

$$t = \frac{\pi/3}{\frac{1}{\sqrt{LC}}}$$

$$t = \frac{\pi}{3}\sqrt{LC}$$

So

$$Q(t) = Q_{max} \cos\left(\frac{1}{\sqrt{LC}} \cdot \frac{\pi\sqrt{LC}}{3} + \phi\right) = Q_{max} \cos\left(\frac{\pi}{3} + \phi\right)$$

b How much time has elapsed

Opps answered that already
must have taken long way on a

$t = \frac{\pi\sqrt{LC}}{3}$

c If the resistance is non-0, how will natural freq change

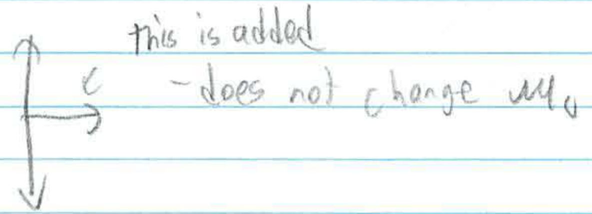
so now $\omega_0 = \frac{1}{\sqrt{LC}}$

With resistance it is a RLC circuit

ω_0 is still $\frac{1}{\sqrt{LC}}$

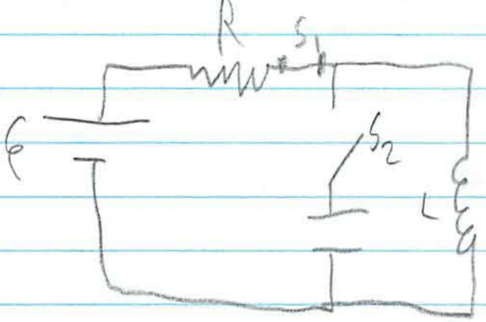
so natural freq will be the same

You can also see this on phasor. It



4. A toroidoil coil has N turns
inner radius a
outer radius b
Split into rectangular cross sections

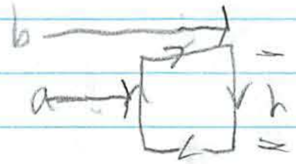
a) When current I flows, find magnetic field



Office hrs Hudson

Very common exam qu

Ampere's Law + RLC circuits

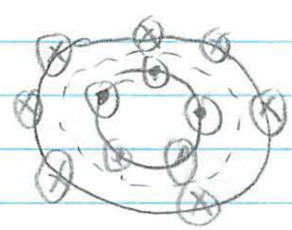


N turns

$L = \frac{\Phi_B}{L}$

Need Ampere's law on torrus

- redraw torrus so current comes out of page



1. Choose loop
   - in middle of loop
   - Field is not in center of torrus or on outside of torus

$a < r < b$

No material there - space between the 2 wires

$\oint B \cdot ds = \mu_0 I_{enc}$

$B \cdot 2\pi r = \mu_0 I N$

$B = \frac{\mu_0 I N}{2\pi r}$ ✓

CCW ← should be able to do
- just not comfortable

5) Find self inductance (L)

$$L = \frac{\Phi_B}{I}$$

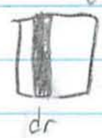
$$\Phi_B = \iint \vec{B} \cdot d\vec{A}$$

↑ have ↑ what is ?

what shapes are we adding
just look at 1 slice of torrus

↑

nothing inside again - But $\vec{B}$ field



$$\iint \frac{\mu_0 I N}{2\pi r} \cdot h \, dr$$

$\iint dA = \iint h \, dr = h(b-a) = A$

go through each cetangular slice and ask how much flux through - decreases as r gets bigger
must $\int$

$$\Phi = \frac{\mu_0 I N h}{2\pi} \ln\left(\frac{b}{a}\right)$$

How get self inductance
- divide out current

$$L = \frac{\mu_0 N h}{2\pi} \ln\left(\frac{b}{a}\right)$$

depends only on geometry
← need all of the loops

Is it I or I_{enc}?
- are N # of loops - are both creating + feeling

create + feel

$$L = \frac{\Phi_B^{\,total}}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln(b/a)$$

c) Current in circuit a long time after S_1 closed
S_2 open

$I = \frac{\varepsilon}{R}$
- after long time inductor is a wire
$\omega = 0$ = dc freq.
not ossiclating
~~steady state~~

$X_L = \omega L$
$X_C = \frac{1}{\omega C}$

d) How much energy stored in coil a very long time after switch closed

$U_L = \frac{1}{2} L I^2$

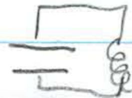
Plug in for I and L

Own

ok $I = \frac{\varepsilon}{R}$ L = value of indictor fand in part a

$$U_L = \frac{1}{2}\left(\frac{\varepsilon}{R}\right)^2 \frac{\mu_0 N^2 h}{2\pi} \ln(b/a)$$

Now assume long time after S_1 closed, S_1 is opened and S_2 is closed



No resistance
(LC circuit)

e) What is max amt. of charge on capicator?

Hudson OH

Energy conserved

Max charge when $I = 0$
When no current / energy in Inductor

Set $U_L = 0$ and solve
Can leave capitance at C

Own

$$U = U_L + U_C$$
$$\uparrow 0$$
$$U = U_C$$

or we could say the old U_L = new U_C

$$U_C = \frac{1}{2}\frac{Q^2}{C} = \frac{\mathcal{E}^2 \mu_0 N^2 h \ln(b/a)}{4\pi R}$$
$$\cdot 2C \qquad \cdot 2C$$

$$Q_{max} = \sqrt{\frac{\mathcal{E}^2 \mu_0 N h \ln(b/a) C}{2\pi R^2 \cdot 2}}$$

? very complex ans
would not have thought of it like this
many ways to approach

Hudson
011

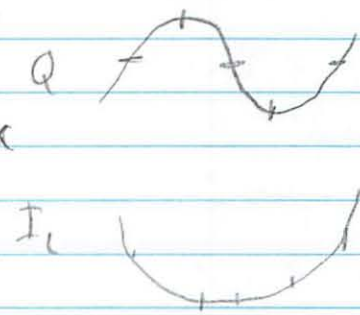
f) How long to reach max charge after S_2 closed

$\omega_0 = \frac{1}{\sqrt{LC}}$ ✓

↑ drive at this when drive
not drive this is freq it will ossilate at

find period T

0 period → q = 0 inductor max
1/4 period → max charge
1/2 period → $q_c = 0$ capicator max



Own

so it is $\frac{1}{2}T$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{1/\sqrt{LC}} = 2\pi\sqrt{LC}$

↑ and it will ossilate at $\omega = \omega_0$

$\frac{1}{2}T = \boxed{\sqrt{LC}\,\frac{\pi}{2}}$ seconds

-2

Lot of stuff to know!

Extra: for how much power lost look at

$\int_0^T \frac{I^2}{R}\,dt$

5. For the undampened RLC circuit with $R^2 < \frac{4L}{C}$

let $\gamma = \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)$ $\quad a = \frac{R}{2L}$

a) Show by direct substitution

$$0 = L\frac{d^2Q}{dt^2} + \frac{dQ}{dt}R + \frac{Q}{C}$$

↑ no driving

✓ has the solution $Q(t) = A e^{-\alpha t}\cos(\gamma t + \phi)$

Did in the Math review for phisics something like this - only way I would know how to do
Plug it in and see if it checks - very long math!

Compared to review $\alpha = \frac{1}{\gamma}$
$A = x_0$
$\omega = \alpha$

$$Q(t) = A e^{-\alpha t}\cos(\gamma t + \phi)$$

$$\frac{dQ}{dt} = A\left[\alpha e^{-\alpha t}\cos(\gamma t + \phi) - \gamma e^{\alpha t}\sin(\gamma t + \phi)\right]$$

$$\frac{d^2Q}{dt^2} = A\left[\alpha^2 e^{-\alpha t}\cos(\gamma t + \phi) + \alpha\gamma e^{-\alpha t}\sin(\gamma t + \phi)\right.$$
$$\left. + \alpha\gamma e^{-\alpha t}\sin(\gamma t + \phi) - \gamma^2 e^{-\alpha t}\cos(\gamma t + \phi)\right]$$

Sub in diff eq to check

$$L \frac{d^2Q}{dt^2} + \frac{dQ}{dt} R + \frac{Q}{C} = 0$$

$$LA\Big[\alpha^2 e^{-\alpha t}\cos(\gamma t + \phi) + \alpha \gamma e^{-\alpha t}\sin(\gamma t + \phi)$$

$$+ \alpha \gamma e^{-\alpha t}\sin(\gamma t + \phi) - \gamma^2 e^{-\alpha t}\cos(\gamma t + \phi) +$$

$$RA\Big[-\alpha e^{-\alpha t}\cos(\gamma t + \phi) - \gamma e^{-\alpha t}\sin(\gamma t + \phi)\Big] +$$

$$\frac{1}{C}\Big[A e^{-\alpha t}\cos(\gamma t + \phi)\Big]$$

and it will all cacle to 0

Might have to set $\alpha = \frac{R}{2L}$ $\gamma = \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{1/2}$

I don't really want to rewrite all this ...

$$LA\left[\frac{R^2}{4L^2} e^{-\frac{R}{2L}t} \cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\, t + \phi\right) + \frac{R}{2L} \ldots\right.$$

you get the picture
but it all cancles out to 0

$$\left[L\alpha^2 - L\lambda^2 - R\alpha + \frac{1}{C}\right]\cos(\gamma t + \phi)$$
$$+ \left[2L\gamma\alpha - R\gamma\right]\sin(\gamma t + \phi) = 0$$

b) Denote current I

$$I(t) = \frac{dQ(t)}{dt} = F e^{-\alpha t} \cos(\gamma t + \phi + \beta)$$

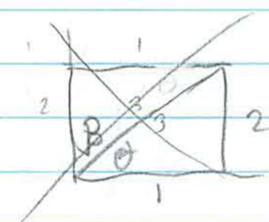
What are F and β? in terms of RLC

well $\frac{dQ}{dt} = A\left[-\alpha e^{-\alpha t}\cos(\gamma t+\phi) - \gamma e^{-\alpha t}\sin(\gamma t+\phi)\right]$

So factor out $-A e^{-\alpha t}$ as F

$$F\left[\alpha \cos(\gamma t+\phi) - \gamma \sin(\gamma t+\phi)\right]$$

But how to get that last part β?
- something about getting rid of sin?



c Calculate heating loss after 1 cycle. Assume $\phi = 0$

Damping factor $\omega' = \sqrt{\omega_0^2 - \gamma^2}$

$w_0 = \frac{1}{\sqrt{LC}}$

Or do it energy wise $U =$

Or they give $Q(t) = Q_0 e^{-\gamma t} \cos(\omega' t + \phi)$

So they just used different letters and so in this problem damping is $\gamma = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

γ is the dampening factor $\alpha = \frac{R}{2L}$

Energy is lost through the resistor

~~Δ~~ $U = I^2 R$

or $U = \frac{Q_0^2}{2C}$

$\frac{dU}{dt} =$ ~~constant~~

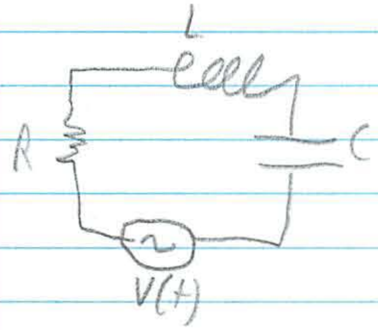
except

~~$\frac{dU}{dt} = I^2 R = V$~~

$V = IR$

very confused

G. Experiment 9; Driven RLC circuit



$V(t) = V_0 \sin(\omega t)$
$V_0 = 5V$ ← max

$L = 85\ mH$
$R = 5\ \Omega$
$C = 100\ \mu F$

Running in =librium for long time
Shut off function generator + put wire

a) Assuming ω is not nessarily on resonance
(~~Thought it always went traveled to resonance~~ non driven only)
what is the freq the system will ring down to
(in other words that current will go when turn off function generator)

? so right after turning off
will it "ring down" to natural freq?
Coure notes do not use words "ring down"

$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{8.5 \cdot 100}}$ = ~~34299 Hz~~
? what units is Hz in?
Cycle per second
opps! $f = \frac{\omega}{2\pi}$ ← rad/sec

but do I need to convert to Henries and Farads
or / keep mili Henries + microFarads I will do

~~.0054 Hz ← wrong too slow~~

~~But if do milli/micro get 5458 Hz~~
~~- that seems wrong too~~

officehrs — Should get big # $\frac{1}{\text{very small}}$

own

$$\frac{1}{\frac{\sqrt{8.5\cdot 10^{-3}\cdot 100\cdot 10^{-6}}}{2\pi}}$$ ← type in exponents, less mistakes

172.6 Hz
- much better

Think I got sig figs wrong
just type in exponentials
10^{-3} = milli
10^{-6} = micro

b) What freq should run at to maximize peak magnetic energy in the inductor?

So inductor dominates at high freq since it will fight rapid changes and current will lag.

Peak magnetic energy U_B is when all energy in inductor - which is at largest freq

$$U_L = \frac{1}{2} L I^2 = U_B$$

But we also want to maximize I - which will do auto

Look at driven chapter / AC

12-33

$$U_{L\,max} = \frac{1}{2} L I_0^2 = \frac{L V_0^2}{2R^2}$$

Basically just find I - But it asks for what freq

It will want to run at natural freq?
Max I_0 is at resonance freq

$$\omega = \omega_0 = \frac{1}{\sqrt{LC}}$$

c) In this case if we time the shut off to occur when the magnetic energy in the inductor peaks after how long will energy in capicator peak

$\frac{1}{2} T$

$T = \frac{1}{f}$

$\frac{1}{2} \frac{1}{172.6}$

.00289 seconds

d) Approx how much energy will resistor dissapate

- This question again!

~~$dU = I^2 R = qV$~~

~~$dV = IR$~~

? still confused

$V = IR$

Energy $= U = \Delta V$

$$\frac{dV}{dt} = \frac{dI}{dt} R$$

? do I need to calc from kirchoff - confused

$$I = \frac{V}{R} = \frac{V_0}{R} \sin \omega t = I_0 \sin \omega t$$

$$V_c = I X_c \qquad V_L = I X_L$$

Work ↓ Energy ↓ Power

$$\text{Power} = IV \cos \phi$$

↑ power factor

$$\phi = \tan^{-1}\left(\frac{X_L - X_c}{R}\right)$$

↑ keep reactance positive

Should have read reactance section closer ?

office hrs

Find power dissapated over $\frac{1}{4}$ period

$$P = \int_0^{1/4 T} \frac{V^2}{R} dt$$

still being driven

↑ 1/4 period

own

Reread class 27 summary

- ossilates in time, so average power disappated

$$\langle P \rangle = \frac{1}{T} \int_0^T P(t)\, dt$$
$$= \frac{1}{2} I_0^2 R \quad \text{for } \sin^2 \omega t \text{ or } \cos^2 \omega t$$
$$= I_{RMS}^2 R$$

? avg power over a period

U = power · time

alt method

Lets use above

$T = \frac{1}{172.6} = .005793$ sec $\frac{1}{4}T = .001448$ sec

$$\int_0^{.001448} \frac{V_0^2}{R} dt$$

$$\frac{5V^2}{5\Omega} t \Big]_0^{.001448}$$

$$\frac{5^2 \cdot .001448}{5}$$

$= .007242$ watts

7. A series RLC circuit

$R = 10\,\Omega$
$L = 400\,mH$
$C = 2.0\,\mu F$
AC voltage source
$V(t) = V_0 \sin \omega t$
$V_0 = 100$

a) What is resonant freq ω_0?

$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{400\,mH \cdot 2\,\mu F}} = \frac{1}{\sqrt{.000002 \cdot .4}} = 1118\,Hz$

type in $\frac{1}{\sqrt{400 \cdot 10^{-3} \cdot 2 \cdot 10^{-6}}}$ — got same thing ✓

office hrs

own

b) Find the rms current at resonance

$I_{rms} = \frac{I_0}{\sqrt{2}}$

$I_0 = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}$ ← voltage source $V = IR$
← new R complicated

from book

- where did they get this again

$I_{rms} = \frac{\frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}}{\sqrt{2}} = \frac{\frac{100}{\sqrt{10^2 + (\omega \cdot .4 - \frac{1}{\omega 2})^2}}}{\sqrt{2}}$

$X_L = \omega L$ — but what is freq
$X_C = \frac{1}{\omega C}$ — answer to a

$$= \frac{\frac{100}{\sqrt{10^2 + \left(1118 \cdot .4 - \frac{1}{1118 \cdot .000002}\right)^2}}}{\sqrt{2}}$$

$= 7.07$ amps

These are like = at resonance

c) Let driving freq $\omega = 4000$ rad/sec

Assume the current response is given by

$I(t) = I_0 \sin(\omega t - \phi)$

$P = IV = I^2R$

- so want I_0 and ϕ

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

$X_L = \omega L$

$X_C = \frac{1}{\omega C}$

$$\phi = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

$$\tan^{-1}\left(\frac{1118 \cdot .4 - \frac{1}{1118 \cdot .000002}}{10}\right)$$

$\phi = -.0027$ radians

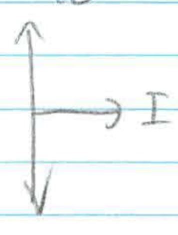
or basically 0

X_L almost $= X_C$

at resonance

↑ which is what the rule is

resonance $\rightarrow \phi = 0$



$$I_0 = -Q_0 \omega = \frac{-V_0}{\sqrt{R^2 + (X_L - X_C)^2}}$$

= found this in part b

= 7.07 amps

8.

$\mathcal{E} = \mathcal{E}_0 \sin \omega t$
$R = 6\ \Omega$
?: inductor, capicator, or both
$\mathcal{E}_0 = 6V$
$\omega = 2$ rad/sec
↳ in phase with emf
$\omega = 1$ rad/sec
out of phase $\frac{\pi}{4}$ radians

Like problem solving 8 sample problem 2

a) What is in te box?

~~both~~ must be there for it to be in phase

at 1rad/sec the capicator will be leading
So current leads voltage

b) What is L and C?

At $\omega_1 = 2$ rad/sec $= \tan \phi = \tan 0 = 0 =$

$$= \frac{\omega_1 L - \frac{1}{\omega_1 C}}{R} \qquad L = \frac{1}{\omega_1^2 C}$$

?impediance?

At $\omega_2 = 1$ rad/sec $= \tan \frac{\pi}{4} = 1 =$

$$= \frac{\omega_2 L - \frac{1}{\omega_2 C}}{R}$$

$$\omega_2 L - \frac{1}{\omega_2 C} = R$$

Solve for R

$$\omega_2 L - \frac{1}{\omega_2 C} = \omega_2 \frac{1}{\omega_1^2 C} - \frac{1}{\omega_2 C}$$

$$C = \frac{1}{R}\left(\frac{\omega_2}{\omega_1^2} - \frac{1}{\omega_2}\right)$$

$\omega_1 = 2$
$\omega_2 = 1$

$$C = \frac{1}{6\Omega}\left(\frac{1}{2^2} - \frac{1}{1}\right) = \frac{1}{6} \cdot \frac{-3}{4} = \frac{-3}{24} = +\frac{1}{8} F$$

-1

$$L = \frac{1}{\omega_1^2 C} = \frac{1}{2 \cdot (\frac{1}{8})} = 4$$

-2

c) What is the ratio of the amplitudes of the current

$$\frac{I_0(\omega = 2\ rad/sec)}{I_0(\omega = 1\ rad/sec)}$$

$$I_0 = \frac{V_0}{\sqrt{R^2 (X_L - X_C)^2}}$$

$\omega = 2$ $\quad \frac{6}{\sqrt{6^2 + \left(2 \cdot \frac{1}{8} - \frac{1}{2 \cdot 4}\right)^2}}$ $\qquad \omega = 1$ $\quad \frac{6}{\sqrt{6^2 + \left(1 \cdot \frac{1}{8} - \frac{1}{1 \cdot 4}\right)^2}}$

$I = .999783$ $\qquad I = .999783$

$$\frac{.999783}{.999783} = 1$$

That's the current, what's the ratio?... −2

Thought it would be same

; is it always the same –
yeah series circuit all currents same

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

8.02 **Spring 2010**

Problem Set 10 Solutions

Problem 1: Show that

$$A\cos\omega t + B\sin\omega t = Q_{\mathrm{m}}\cos(\omega t+\phi),$$

where

$$Q_{\mathrm{m}} = \left(A^2+B^2\right)^{1/2}, \text{ and } \phi = \tan^{-1}\left(-B/A\right).$$

Solution: Use the identity

$$Q_{\mathrm{m}}\cos(\omega t+\phi) = Q_{\mathrm{m}}\cos(\omega t)\cos(\phi) - Q_{\mathrm{m}}\sin(\omega t)\sin(\phi).$$

Thus

$$A\cos(\omega t) + B\sin(\omega t) = Q_{\mathrm{m}}\cos(\omega t)\cos(\phi) - Q_{\mathrm{m}}\sin(\omega t)\sin(\phi).$$

Comparing coefficients we see that

$$A = Q_{\mathrm{m}}\cos\phi.$$

$$B = -Q_{\mathrm{m}}\sin\phi.$$

Therefore

$$\left(A^2+B^2\right) = Q_{\mathrm{m}}^{\ 2}\left(\cos^2\phi+\sin^2\phi\right).$$

So

$$Q_{\mathrm{m}} = \left(A^2+B^2\right)^{1/2}.$$

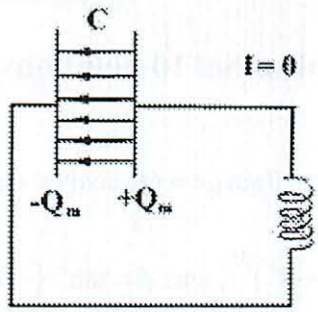
Also

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{-B/Q_{\mathrm{m}}}{A/Q_{\mathrm{m}}} = -\frac{B}{A}.$$

Hence

$$\phi = \tan^{-1}(-B/A).$$

Problem 2: In an LC circuit, the electric and magnetic fields are shown in the figure. Which of the following is true? **Explain your answer** At the moment depicted in the figure, the energy in the circuit is stored in



1. the electric field and is decreasing
2. the electric field and is constant.
3. the magnetic field and is decreasing.
4. the magnetic field and is constant.
5. in both the electric and magnetic field and is constant.
6. in both the electric and magnetic field and is decreasing.

Explain your answer. 2. The total energy in the LC circuit is constant since there is no resistance and hence no dissipation of electromagnetic energy into thermal energy. At the moment depicted in the figure the capacitor is completely charged, with maximum electric field strength between the plates, and electric potential energy stored in the electric field. Immediately after the instant depicted in the figure, the capacitor starts to discharge, the electric field strength and electric field energy decreases. At the moment depicted in the figure no current is flowing, so the magnetic field strength is zero and magnetic field energy is zero. Immediately after the instant depicted in the figure the current flows clockwise and the magnetic field strength and field energy increases. But the decrease in the electric field energy is equal to the increase in the magnetic field energy.

Problem 3: In a freely oscillating LC circuit, (no driving voltage), suppose the maximum charge on the capacitor is $Q_{\max}$. Assume the circuit has zero resistance.

a) In terms of the maximum charge on the capacitor, what value of charge is present on the capacitor when the energy in the magnetic field is three times the energy in the electric field.

Solution: The total energy is constant hence

$$U = \frac{Q_{\max}{}^2}{2C} = U_{elec} + U_{mag} = \frac{Q^2}{2C} + U_{mag}$$

Suppose $U_{mag} = 3U_{elec}$. Then the electromagnetic energy in the system is

$$\frac{Q_{\max}{}^2}{2C} = U_{elec} + U_{mag} = 4U_{elec} = 4\frac{Q^2}{2C}$$

We can solve the above equation for the charge on the capacitor and find that

$$Q = \frac{Q_{\max}}{2}.$$

b) How much time has elapsed from when the capacitor is fully charged for this condition to arise?

The initial conditions for the charge and the current at time $t = 0$ are

$$Q(t = 0) = Q_{\max}$$
$$I(t = 0) = 0$$

Therefore the charge on the capacitor varies in time according to

$$Q(t) = Q_{\max} \cos \omega_0 t$$

where $\omega_0 = \sqrt{1/LC}$. When

$$Q(t) = Q_{\max} / 2 = Q_{\max} \cos \omega_0 t$$

we can solve for the time,

$$t = \frac{1}{\omega_0} \cos^{-1}(1/2) = \sqrt{LC}(\pi/3)$$

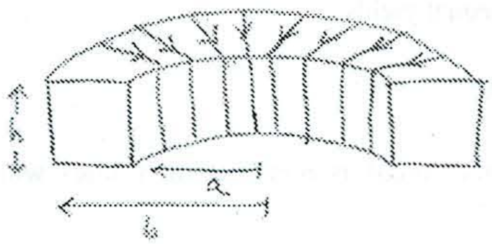
c) If the resistance is non-zero, will the natural frequency of oscillation compared to the natural frequency of the ideal *LC* circuit (with zero resistance)

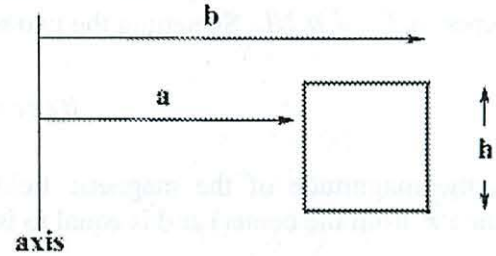
i) increase
ii) stay the same
iii) decrease

Solution: The frequency of oscillation for an underdamped RLC circuit is less that the frequency of oscillation for an ideal LC according to

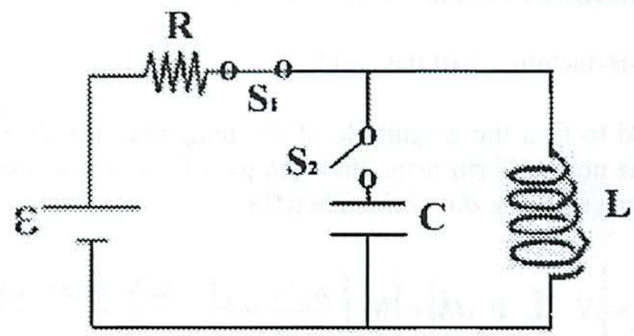
$$f_{RLC} = \frac{1}{2\pi}\sqrt{(\omega_{LC})^2 - \left(\frac{R}{2L}\right)^2} = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC}\right)^2 - \left(\frac{R}{2L}\right)^2}.$$

Problem 4: A toroidoil coil has N turns, and an inner radius a, outer radius b, and height h. The coil has a rectangular cross section shown in the figures below.



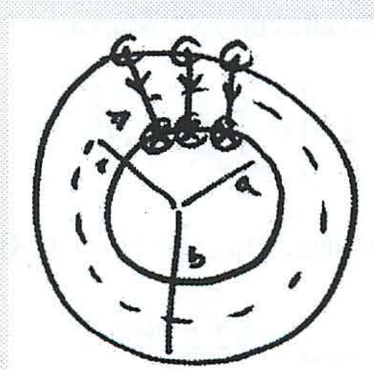


The coil is connected via a switch, S_1, to an ideal voltage source with electromotive force $\mathcal{E}$. The circuit has total resistance R. Assume all the self-inductance L in the circuit is due to the coil. At time $t=0$ S_1 is closed and S_2 remains open.



a) When a current I is flowing in the circuit, find an expression for the magnitude of the magnetic field inside the coil as a function of distance r from the axis of the coil.

Solution: The magnetic field is zero for $r<a$ and $r>b$. Choose a circle of radius r with $a<r<b$ for your Amperian loop (see figure below).



Then the left hand side of Ampere's Law, $\oint_{circle} \vec{\mathbf{B}} \cdot d\vec{\mathbf{r}} = \mu_0 I_{enc}$, becomes $\oint_{circle} \vec{\mathbf{B}} \cdot d\vec{\mathbf{r}} = B2\pi r$.
Since all N turns cuts through theAmperian circle, the right hand side of Ampere's law becomes $\mu_0 I_{enc} = \mu_0 NI$. So setting the two sides equal yields

$$B2\pi r = \mu_0 NI .$$

Thus the magnitude of the magnetic field in the toroid is non-uniform (varies with distance r from the center) and is equal to is

$$B = \begin{cases} 0, & r < a \text{ and } r > b \\ \dfrac{\mu_0 N I}{2\pi r}, & a < r < b \end{cases}$$

The field points in the clockwise direction when viewed from above

b) What is the self-inductance L of the coil?

Solution: We first need to find the magnitude of the magnetic flux through the toroid. We need to integrate the non-uniform magnetic field over the cross sectional area of one turn so we use for the area element $da = hdr$. Then the

$$\left| \int_{toroid} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \right| = \left| N \int_{one\ turn} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \right| = \left| N \int_{r=a}^{r=b} \frac{\mu_0 NI}{2\pi r} hdr \right| = \frac{\mu_0 N^2 h \ln(b/a)}{2\pi} I .$$

The magnetic flux through the toroid is proportional to the current,

$$N \int_{one\,turn} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = LI .$$

The constant of proportionality is called the self-inductance,

$$L = \left| N \int_{one\,turn} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \Big/ I \right| .$$

The unit of self-inductance is the henry, [H], $[H] = [T \cdot m^2] / [A]$ and is given by

$$L_{toroid} = \frac{\mu_0 N^2 h \ln(b/a)}{2\pi}$$

c) What is the current in the circuit a very long time ($t >> L/R$) after S_1 is closed?

Solution: A very long time after the switch S_1 was closed, the current is steady so the inductor acts like a short and the current in the circuit is

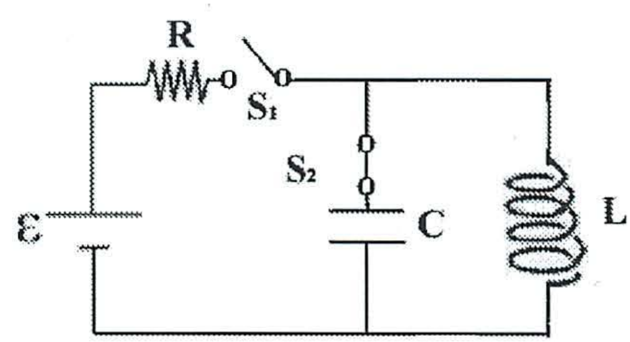
$$I = \varepsilon / R$$

d) How much energy is stored in the magnetic field of the coil a very long time ($t >> L/R$) after S_1 is closed?

Solution: The energy stored in the magnetic field is equal to

$$U_{mag} = \frac{1}{2}LI^2 = \frac{\mu_0 N^2 h \ln(b/a)}{4\pi R^2}\varepsilon^2 .$$

For the next two parts, assume that a very long time ($t >> L/R$) after the switch S_1 was closed, the voltage source is disconnected from the circuit by opening S_1, and by simultaneously closing S_2 the toroid is connected to a capacitor of capacitance C. Assume there is negligible resistance in this new circuit.



e) What is the maximum amount of charge that will appear on the capacitor?

Solution: When the switch S_2 is closed the current in the circuit is $I = \varepsilon / R$. The maximum amount of charge occurs when all the magnetic energy is converted to electrical energy

$$U_{elec} = \frac{Q_{max}^{\;2}}{2C} = U_{mag,0} = \frac{1}{2}LI^2 = \frac{\mu_0 N^2 h \ln(b/a)}{4\pi R^2}\varepsilon^2 .$$

We can solve the above equation for the maximal charge on the capacitor

$$Q_{\max} = \sqrt{2CU_{mag,0}} = \sqrt{CLI_0^{\ 2}} = \sqrt{\frac{2C\mu_0 N^2 h \ln(b/a)}{4\pi}}\ \varepsilon / R.$$

f) How long will it take for the capacitor to first reach a maximal charge after S_2 has been closed?

Solution: It will take one quarter cycle, or

$$t = \frac{1}{4}T = \frac{1}{4}\frac{2\pi}{\omega_0} = \frac{\pi}{2}\sqrt{LC} = \frac{\pi}{2}\sqrt{\frac{C\mu_0 N^2 h \ln(b/a)}{2\pi}}$$

Problem 5: For the underdamped RLC circuit, $R^2 < 4L/C$, let $\gamma = \left(1/LC - R^2/4L^2\right)^{1/2}$ and $\alpha = R/2L$. (a) Show by direct substitution that the equation

$$0 = L\frac{d^2Q}{dt^2} + \frac{dQ}{dt}R + \frac{Q}{C}$$

has a solution of the form

$$Q(t) = Ae^{-\alpha t}\cos(\gamma t + \phi)$$

(b) Denote the current by

$$I(t) = \frac{dQ(t)}{dt} = Fe^{-\alpha t}\cos(\gamma t + \phi + \beta)$$

Find the constants F and β in terms of R, L and C as needed.

Solution: (a) The first approach is by direct substitution. Calculate the first and second derivatives

$$\frac{dQ}{dt} = -\alpha Ae^{-\alpha t}\cos(\gamma t + \phi) - \gamma Ae^{-\alpha t}\sin(\gamma t + \phi)$$

$$\frac{d^2Q}{dt^2} = \alpha^2 Ae^{-\alpha t}\cos(\gamma t + \phi) + \alpha\gamma Ae^{-\alpha t}\sin(\gamma t + \phi) + \gamma\alpha Ae^{-\alpha t}\sin(\gamma t + \phi) - \gamma^2 Ae^{-\alpha t}\cos(\gamma t + \phi)$$

Then the differential equation becomes

$$0 = L\frac{d^2Q}{dt^2} + \frac{dQ}{dt}R + \frac{Q}{C}$$

becomes

$$\begin{aligned}
0 &= L\left(\alpha^2 Ae^{-\alpha t}\cos(\gamma t + \phi) + \alpha\gamma Ae^{-\alpha t}\sin(\gamma t + \phi) + \gamma\alpha Ae^{-\alpha t}\sin(\gamma t + \phi) - \gamma^2 Ae^{-\alpha t}\cos(\gamma t + \phi)\right) \\
&+ R\left(-\alpha Ae^{-\alpha t}\cos(\gamma t + \phi) - \gamma Ae^{-\alpha t}\sin(\gamma t + \phi)\right) \\
&+ \frac{1}{C}\left(Ae^{-\alpha t}\cos(\gamma t + \phi)\right)
\end{aligned}$$

This simplifies to

$$RHS = \left(L\left(\alpha^2 - \gamma^2\right) - \alpha R + \frac{1}{C} \right) Ae^{-\alpha t}\cos(\gamma t + \phi) + \left(\left(2L\alpha - R\right)\gamma\right) Ae^{-\alpha t}\sin(\gamma t + \phi)$$

Recall that

$$\alpha^2 - \gamma^2 = \left(R/2L\right)^2 - \left(1/LC - R^2/4L^2\right) = \frac{R^2}{2L} - \frac{1}{C}$$

Thus

$$L\left(\alpha^2 - \gamma^2\right) - \alpha R + \frac{1}{C} = \left(\frac{R^2}{2L} - \frac{1}{C} \right) - \frac{R^2}{2L} + \frac{1}{C} = 0$$

Also

$$\left(2L\alpha - R\right) = 2L\frac{R}{2L} - R = 0$$

So both coefficients on the RHS vanish hence

$$RHS = \left(L\left(\alpha^2 - \gamma^2\right) - \alpha R + \frac{1}{C} \right) Ae^{-\alpha t}\cos(\gamma t + \phi) + \left(\left(2L\alpha - R\right)\gamma\right) Ae^{-\alpha t}\sin(\gamma t + \phi) = 0$$

(b) From part (a)

$$I = \frac{dQ}{dt} = -\alpha Ae^{-\alpha t}\cos(\gamma t + \phi) - \gamma Ae^{-\alpha t}\sin(\gamma t + \phi) = e^{-\alpha t}\left(C\cos(\gamma t + \phi) + D\sin(\gamma t + \phi)\right)$$

where

$$C = -\alpha A \text{ and } D = -\gamma A$$

Using the results from problem 1

$$I = e^{-\alpha t}\left(C\cos(\gamma t + \phi) + D\sin(\gamma t + \phi)\right) = Fe^{-\alpha t}\cos(\gamma t + \phi + \beta)$$

where

$$F = \left(C^2 + D^2\right)^{1/2} = \left(\left(-\alpha A\right)^2 + \left(-\gamma A\right)^2\right)^{1/2} = A\left(\alpha^2 + \gamma^2\right)^{1/2}$$
$$= A\left(\left(R/2L\right)^2 + \left(1/LC - R^2/4L^2\right)\right)^{1/2} = \frac{A}{\sqrt{LC}}$$

and

$$\beta = \tan^{-1}(-D/C) = \tan^{-1}(-\gamma/\alpha) = \tan^{-1}\left(-\frac{\left(1/LC - R^2/4L^2\right)^{1/2}}{R/2L} \right)$$

Note: A second approach to part (a) explicitly solves the equation by first conjecturing that the solution is of the form

$$Q = Ae^{zt}$$

where z is a number (possibly complex). Then

$$\frac{dQ}{dt} = zAe^{zt}, \qquad \frac{d^2Q}{dt^2} = z^2 Ae^{zt}$$

so the circuit equation becomes

$$0 = \left(Lz^2 + zR + + \frac{1}{C} \right) Ae^{zt}$$

The condition for the solution is that the characteristic polynomial

$$Lz^2 + zR + + \frac{1}{C} = 0$$

This equation has solutions of the form

$$z = \frac{-R \pm \left(R^2 - 4L/C\right)^{1/2}}{2L}$$

When $R^2 < 4L/C$, we have two solutions for z, however the solutions are complex. Let $\gamma = \left(1/LC - R^2/4L^2\right)^{1/2}$ and $\alpha = R/2L$. Recall that the imaginary number $i = \sqrt{-1}$. Then $z_1 = -\alpha + i\gamma t$ and $z_2 = -\alpha - i\gamma t$. So the charge becomes

$$Q = A_1 e^{-\alpha + i\gamma t} + A_2 e^{-\alpha - i\gamma t} = \left(A_1 e^{i\gamma t} + A_2 e^{-i\gamma t}\right) e^{-\alpha t},$$

where A_1 and A_2 are constants.

We shall transform this expression into a more familiar equation involving sine and cosine functions with using the Euler formula,

$$e^{\pm i\gamma t} = \cos\gamma t \pm i\sin\gamma t\,.$$

We can rewrite our solution as

$$Q = \left(A_1\left(\cos\gamma t + i\sin\gamma t\right) + A_2\left(\cos\gamma t - i\sin\gamma t\right)\right)e^{-\alpha t}$$

A little rearrangement yields

$$Q = \left(\left(A_1 + A_2\right)\cos\gamma t + i\left(A_1 - A_2\right)\sin\gamma t\right)e^{-\alpha t}$$

Define two new constants

$$C = A_1 + A_2 \text{ and } D = i\left(A_1 - A_2\right).$$

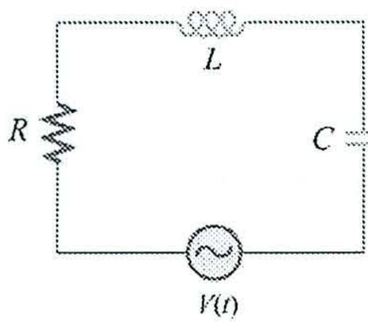
Then our solution looks like

$$Q = \left(C\cos\gamma t + D\sin\gamma t\right)e^{-\alpha t}.$$

Now use the identity from problem 1: $C\cos\omega t + D\sin\omega t = A\cos\left(\omega t + \phi\right)$ where $A = \left(C^2 + D^2\right)^{1./2}$ and $\phi = \tan^{-1}(-D/C)$. Thus

$$Q(t) = Ae^{-\alpha t}\cos(\gamma t + \phi)$$

Problem 6: Review Experiment 8: RL and Undriven RLC Circuits **; Read** Experiment 9: Driven RLC Circuits



Consider the circuit at left, consisting of an AC function generator ($V(t) = V_0 \sin(\omega t)$, with $V_0 = 5$ V), an inductor $L = 8.5$ mH, resistor $R = 5\ \Omega$, capacitor $C = 100\ \mu$F and switch S.

The circuit has been running in equilibrium for a long time. We are now going to shut off the function generator (instantaneously replace it with a wire).

a) Assuming that our driving frequency ω is not necessarily on resonance, what is the frequency with which the system will ring down (in other words, that current will oscillate at after turning off the function generator)? Feel free to use an approximation if you wish, just make sure you know you are.

The ringdown frequency is independent of the drive frequency. It will always ring down at the natural frequency. Actually, when there is resistance in the circuit the natural frequency is modified slightly (the approximation I mention in the question is ignoring the effect of the resistance). So the ring down frequency is:

$$\omega_0 = 1/\sqrt{LC} \approx 1100\ \text{s}^{-1}$$

(b) What (numerical) frequency f should we drive at to maximize the peak magnetic energy in the inductor?

The peak magnetic energy depends on the peak current, which is maximized when we drive on resonance, in other words, we should drive at $f_0 = \omega_0/2\pi \approx 175$ Hz.

(c) In this case, if we time the shut off to occur when the magnetic energy in the inductor peaks, after how long will the electric energy in the capacitor peak?

The electric energy is max when the current goes to zero, which is a quarter period after the current is a maximum. So after $t = T/4 = 1/(4f_0) = 1.4$ ms

(d) Approximately how much energy will the resistor have dissipated during that time?

The current will be $I = (V_0/R)\sin\omega_0 t$ so the average power dissipation (recalling that the average of $\sin^2(x) = ½$) will be $\overline{P} = V_0^2/2R = 2.5$ Watts. So over 1.4 ms, about 3.6 mJ.

Problem 7: A series *RLC* circuit with $R = 10.0\ \Omega$, $L = 400$ mH and $C = 2.0\ \mu\text{F}$ is connected to an AC voltage source $V(t) = V_0 \sin\omega t$ which has a maximum amplitude $V_0 = 100$ V.

(a) What is the resonant frequency ω_0?

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(400\,\text{mH})(2.0\,\mu\text{F})}} = \frac{1}{\sqrt{8.0\times10^{-7}\,\text{s}}} = 1.1\times10^{3}\ \text{rad/s}$$

(b) Find the rms current at resonance.

At resonance, $Z = R$. Therefore,

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{(V_0/\sqrt{2})}{R} = \frac{(100V/\sqrt{2})}{10.0\Omega} = 7.07\text{A}$$

(c) Let the driving frequency be $\omega = 4000$ rad/s. Assume the current response is given by $I(t) = I_0 \sin(\omega t - \phi)$. Calculate the amplitude of the current and the phase shift between the current and the driving voltage.

$$X_C = \frac{1}{\omega C} = \frac{1}{(4000\,\text{rad/s})(2.0\,\mu\text{F})} = 125\,\Omega$$

$$X_L = \omega L = (4000\,\text{rad/s})(400\,\text{mH}) = 1600\,\Omega$$

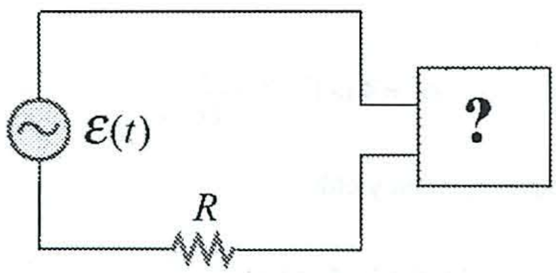
$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(10.0\Omega)^2 + (1600\Omega - 125\Omega)^2} = 1475\,\Omega$$

So

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{100\text{ V}}{\sqrt{(10.0\Omega)^2 + (1600\Omega - 125\Omega)^2}} = \frac{100\text{ V}}{1475\,\Omega} = 6.8\times10^{-2}\text{ A}$$

$$\phi = \tan\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{1600 - 125}{10.0}\right) = \tan^{-1}(147.5) = 89.6°$$

Problem 8: The circuit shown below contains an AC generator which provides a source of sinusoidally varying emf $\mathcal{E} = \mathcal{E}_0 \sin \omega t$, a resistor with resistance $R = 6\,\Omega$, and a "black box", which contains ***either*** an inductor ***or*** a capacitor, ***or both***. The amplitude of the driving emf is $\mathcal{E}_0 = 6$ volt. We measure the current in the circuit at an angular frequency $\omega = 2 \text{ rad} \cdot \text{s}^{-1}$ and find that it is exactly in phase with the driving emf. We measure the current in the circuit at an angular frequency $\omega = 1 \text{ rad} \cdot \text{s}^{-1}$ and find that it is out of phase from the driving emf by exactly $\pi / 4$ radians.



a) What does the black box contain – an inductor or a capacitor, or both? Explain your reasoning. Does current lead or lag at $\omega = 1 \text{ rad} \cdot \text{s}^{-1}$?

Solution: Since the current is exactly in phase with the inductor at $\omega = 2 \text{ rad} \cdot \text{s}^{-1}$, the circuit is resonance and therefore the box must contain both an inductor and capacitor in series. For $\omega = 1 \text{ rad} \cdot \text{s}^{-1}$, the driving angular frequency is below resonance, therefore the circuit is acting capacitively, which means the current leads the driving emf.

b) What is the numerical value of the capacitance *or* of the inductance, *or of both*, as the case may be? Indicate units. Your answer(s) will involve simple fractions only, you will not need a calculator to find the value(s).

When $\omega = 1 \text{ rad} \cdot \text{s}^{-1}$, the phase angle $\phi = -\pi / 4$. So $\tan \phi = \tan(-\pi / 4) = -1$. Since

$$\tan \phi = \frac{X_L - X_C}{X_R} = \frac{\omega L - \dfrac{1}{\omega C}}{R} = -1.$$

We have that for $\omega = 1 \text{ rad} \cdot \text{s}^{-1}$

$$L - \frac{1}{C} = -R.$$

We can divide through by L in the above equation yielding

$$1\ \text{rad}\cdot\text{s}^{-1} - \frac{1}{LC(1\ \text{rad}\cdot\text{s}^{-1})} = -\frac{R}{L}$$

We also have the resonance condition at $\omega = 2\ \text{rad}\cdot\text{s}^{-1}$,

$$\omega = 2\ \text{rad}\cdot\text{s}^{-1} = \frac{1}{\sqrt{LC}}.$$

Thus

$$\omega^2 = 4\ \text{rad}^2\cdot\text{s}^{-2} = \frac{1}{LC}$$

Substituting that in the above equation yields

$$1\ \text{rad}\cdot\text{s}^{-1} - 4\ \text{rad}\cdot\text{s}^{-1} = -\frac{R}{L}.$$

Solving for L then yields

$$L = \frac{R}{3\ \text{rad}\cdot\text{s}^{-1}} = \frac{6\ \Omega}{3\ \text{rad}\cdot\text{s}^{-1}} = 2\ \text{H}.$$

From the resonance condition

$$4\ \text{rad}^2\cdot\text{s}^{-2}C = \frac{1}{L(4\ \text{rad}^2\cdot\text{s}^{-2})} = \frac{1}{(2\ \Omega)(4\ \text{rad}^2\cdot\text{s}^{-2})} = \frac{1}{8}\ \text{F}.$$

c) What is ratio of the amplitudes of the current $\dfrac{I_0(\omega = 2\ \text{rad}\cdot\text{s}^{-1})}{I_0(\omega = 1\ \text{rad}\cdot\text{s}^{-1})}$?

At resonance $\omega = 2\ \text{rad}\cdot\text{s}^{-1}$:

$$I_0(\omega = 2\ \text{rad}\cdot\text{s}^{-1}) = \frac{\varepsilon_0}{R} = \frac{6\ \text{V}}{6\ \Omega} = 1\ \text{A}.$$

Below resonance

$$I_0(\omega = 1\ \text{rad}\cdot\text{s}^{-1}) = \frac{\varepsilon_0}{\left(R^2 + (X_L - X_C)^2\right)^{1/2}} = \frac{\varepsilon_0}{\left(R^2 + (\omega L - \frac{1}{\omega C})^2\right)^{1/2}}.$$

From the phase condition $\omega L - \frac{1}{\omega C} = -R$, the amplitude at $\omega = 1\ \text{rad}\cdot\text{s}^{-1}$ becomes

$$I_0(\omega = 1\ \text{rad}\cdot\text{s}^{-1}) = \frac{\varepsilon_0}{\left(R^2 + R^2\right)^{1/2}} = \frac{\varepsilon_0}{\sqrt{2}R} = \frac{1}{\sqrt{2}}\ \text{A}.$$

Therefore the ratio of the amplitudes of the current is

$$\frac{I_0(\omega = 2\ \text{rad}\cdot\text{s}^{-1})}{I_0(\omega = 1\ \text{rad}\cdot\text{s}^{-1})} = \frac{\varepsilon_0 / R}{\varepsilon_0 / \sqrt{2}R} = \sqrt{2}.$$

Topic: Driven RLC Circuit Lab, Displacement Current, Maxwell's Eqs., Energy Flow and Poynting Vectors
Related Reading: Course Notes: Sections 13.1-13.3, 13.12.3-13.12.4
Experiments: (9) Driven LRC Circuits

Topic Introduction

Today we continue thinking about self inductance, in which the changing flux from a circuit induces an EMF in itself, and address the question of magnetic energy. We then turn to the important question of how energy moves around – power flow and the Poynting vector.

Fixing Ampere's Law: Displacement Current
When thinking about power flow with the Poynting Vector, it is clear that both an electric and magnetic field must be present in a region of space if power is going to flow into or out of that region. This works fine for (dis)charging a solenoid, for example, where you know from Ampere's law that the current in the solenoid makes a B field, and that the changing B field, by Faraday's Law, will create an E field. In a parallel plate capacitor, on the other hand, although we know that the charge on the capacitor creates an E field (Gauss's Law), since no current is flowing THROUGH the capacitor (in the region between the plates), it would seem at first that by Ampere's law there should be no B field in there. This can't be though – we know that we can bring energy into/out of a capacitor.

The problem is that Ampere's law, as you have seen it thus far, is incomplete. Magnetic fields can be generated by currents, but they can also be generated by changing electric fields (hence in a charging capacitor, the changing electric field between the plates generates a magnetic field which then allows calculation of a Poynting vector and the rate of energy flow into the capacitor). Since changing electric fields generate magnetic fields, just like the flow of current, we define the "displacement current" $I_d = \varepsilon_0 \frac{d\Phi_E}{dt}$. Ampere's Law then becomes slightly modified to account for this new "current": $\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \left(I_{\text{penetrate}} + I_d \right)$

Maxwell's Equations
Now that we have all of Maxwell's equations, let's review:

$$(1) \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{in}}{\varepsilon_0} \qquad (2) \oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

$$(3) \oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt} \qquad (4) \oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{penetrate} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

(1) Gauss's Law states that electric charge creates diverging electric fields.
(2) Magnetic Gauss's Law states that there are no magnetic charges (monopoles).
(3) Faraday's Law states that changing magnetic fields induce electric fields (which curl around the changing flux).
(4) Ampere-Maxwell's Law states that magnetic fields are created both by currents and by changing electric fields, and that in each case the field curls around its creator.

These equations are the cornerstone of the theory of electricity and magnetism. Together with the Lorentz Force $\left(\vec{\mathbf{F}} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})\right)$ they pretty much describe all of E&M, and from them we can derive mathematically the major equations you learned this semester (like Coulomb's Law and Biot-Savart). People even put them on T-shirts. They are important and you should try hard to keep them in mind.

Energy and the Poynting Vector

We have now described the energy stored in capacitors and inductors in terms of the voltage across and current through them respectively. We have also described that energy in terms of the E & B fields contained inside them. We have, however, only described the power, the change in energy, in terms of current and voltage: $P = VI$. As you might imagine, we can also think of this power in terms of the fields. We define the Poynting Vector, $\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}$, which describes how energy is travelling per unit area per unit time. If a capacitor or inductor is "charging" (increasing in stored energy) then the Poynting vector will point into the element. When discharging S points outward. Integrating the Poynting Vector over the surface of an object allows us to determine how much energy in entering or leaving.

Important Equations

Energy stored in Inductor: $U = \frac{1}{2}LI^2$

Energy Density in B Field: $u_B = \dfrac{B^2}{2\mu_0}$

Poynting Vector: $\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}$

Displacement Current: $I_d = \varepsilon_0 \dfrac{d\Phi_E}{dt}$

Maxwell Eqs.

(1) $\displaystyle\oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{in}}{\varepsilon_0}$ (2) $\displaystyle\oint\!\!\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$

(3) $\displaystyle\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}$ (4) $\displaystyle\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{penetrate} + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$

Experiment 9: Driven LRC Circuits

Preparation: Read pre-lab

In this lab you will drive a series RLC circuit around its natural frequency to confirm that it is the resonance frequency (or at least close to it) and to determine the properties of the circuit both on and off resonance.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
8.02

Experiment 9: Driven RLC Circuits

OBJECTIVES

1. To explore the time dependent behavior of driven RLC Circuits
2. To understand the idea of resonance, and to determine the behavior of current and voltage in a driven RLC circuit above, below and at the resonant frequency

PRE-LAB READING

INTRODUCTION

In the last lab, Undriven RLC Circuits, you did the equivalent of getting a push on a swing and then sitting still, waiting for the swing to gradually slow down to a stop. Most children will tell you that although that might be fun, it's much more fun to get repeated pushes, or, if you have the coordination, to move your body back and forth at the correct rate and drive the swing, making it swing higher and higher.

This is an example of resonance in a mechanical system. In this lab we will explore its electrical analog – the RLC (resistor, inductor, capacitor) circuit – and better understand what happens when it is driven above, below and at the resonant frequency.

The Details: Oscillations

In this lab you will be investigating current and voltages (EMFs) in RLC circuits. Although in the previous experiment these decayed after being given a kick (Fig. 1b), today we will drive the circuit and see continuous oscillations as a function of time (Fig. 1a).

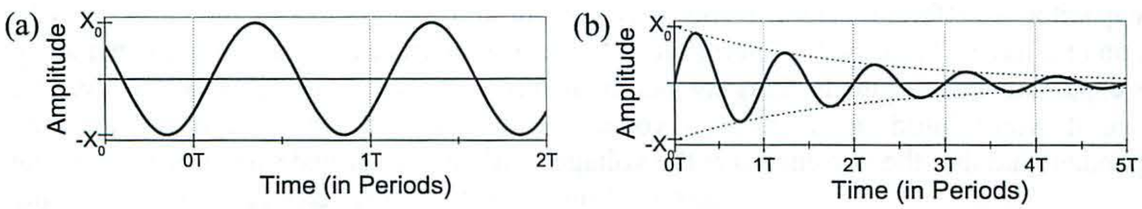


Figure 1 Oscillating Functions. (a) A purely oscillating function $x = x_0 \sin(\omega t + \varphi)$ has fixed amplitude x_0, angular frequency ω (period $T = 2\pi/\omega$ and frequency $f = \omega/2\pi$), and phase ϕ (in this case $\phi = -0.2\pi$). (b) The amplitude of a damped oscillating function decays exponentially (amplitude *envelope* indicated by dotted lines)

Driven Circuits: Resonance

will always go at resonance

In the previous lab we charged the capacitor in a series RLC circuit and then “let it go,” allowing the energy to gradually dissipate through the resistor. This time we will instead add a battery that periodically pushed current through the system. Such a battery is called an *AC* (alternating current) *function generator*, and the voltage it generates can oscillate with a given amplitude, frequency and shape (in this lab we will use a sine wave). When hooked up to an RLC circuit we get a driven RLC circuit (Fig. 2a) where the current oscillates at the same frequency as, but not necessarily in phase with, the driving voltage. The amplitude of the current depends on the driving frequency, reaching a maximum when the function generator drives at the resonant frequency, just like a swing (Fig. 2b)

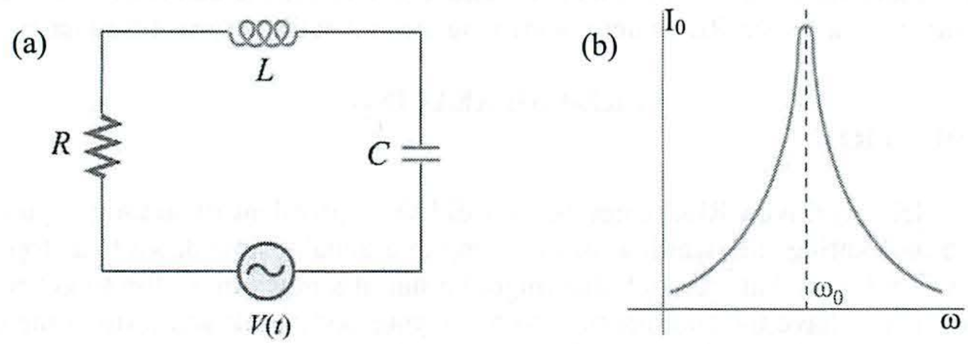


Figure 2 Driven *RLC* Circuit. (a) The circuit (b) The magnitude of the oscillating current I_0 reaches a maximum when the circuit is driven at its resonant frequency

One Element at a Time

In order to understand how this resonance happens in an RLC circuit, its easiest to build up an intuition of how each individual circuit element responds to oscillating currents. A resistor obeys Ohm’s law: $V = IR$. It doesn’t care whether the current is constant or oscillating – the amplitude of voltage doesn’t depend on the frequency and neither does the phase (the response voltage is always in phase with the current).

A capacitor is different. Here if you drive current at a low frequency the capacitor will fill up and have a large voltage across it, whereas if you drive current at a high frequency the capacitor will begin discharging before it has a chance to completely charge, and hence it won’t build up as large a voltage. We see that the voltage is frequency dependent and that the current *leads* the voltage (with an uncharged capacitor you see the current flow and then the charge/potential on the capacitor build up).

never thought of it as that

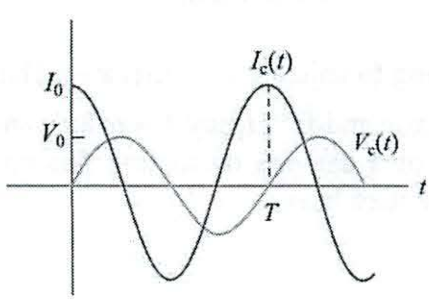


Figure 3 Current and Voltage for a Capacitor A capacitor driven with a sinusoidal current will develop a voltage that lags the current by 90° (the voltage peak comes ¼ period later than the current peak).

An inductor is similar to a capacitor but the opposite. The voltage is still frequency dependent but the inductor will have a larger voltage when the frequency is high (it doesn't like change and high frequency means lots of change). Now the current *lags* the voltage – if you try to drive a current through an inductor with no current in it, the inductor will immediately put up a fight (create an EMF) and then later allow current to flow.

When we put these elements together we will see that at low frequencies the capacitor will "dominate" (it fills up limiting the current) and current will lead whereas at high frequencies the inductor will dominate (it fights the rapid changes) and current will lag. At resonance the frequency is such that these two effects balance and the current will be largest in the circuit. Also at this frequency the current is in phase with the driving voltage (the AC function generator). this is in driven

Resistance, Reactance and Impedance - was not as clear in begining

We can make the relationship between the magnitude of the current through a circuit element and magnitude of the voltage drop across it (or EMF generated by it for an inductor) more concrete by introducing the idea of impedance. Impedance (usually denoted by *Z*) is a generalized resistance, and is composed of two parts – resistance (*R*) and reactance (*X*). All of these terms refer to a constant of proportionality between the magnitude of current through and voltage across (EMF generated by) a circuit element: $V_0 = I_0 Z$, $V = IR$, $V_0 = I_0 X$. The difference is in the phase between the current and voltage. In an element with only resistance (a resistor) the current through it is in phase with the voltage across it. In an element with only reactance (capacitor, inductor) the current leads or lags the voltage by 90º. A combination of these elements in series or parallel will lead to a circuit with impedance $Z = \sqrt{R^2 + X^2}$ and a phase that depends on the ratio of the reactance and resistance: $\tan\varphi = X/R$ (note that the phase ϕ has the correct behavior as $X \to 0$ or $R \to 0$). all are same idea

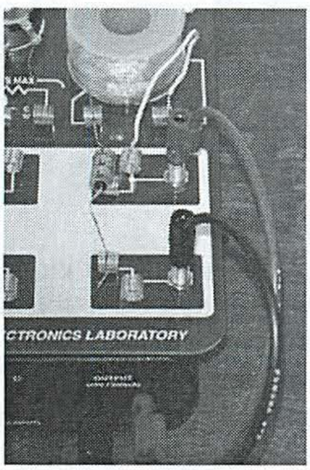
The reactance of an inductor $X_L = \omega L$ and of a capacitor $X_C = -1/\omega C$. First of all, note that these have the correct frequency dependence. An inductor has a high reactance at high frequencies (it takes a lot of effort to change the current through an inductor at high frequencies) whereas a capacitor has a high reactance at low frequencies (it "fills up" to have a large potential across it). The sign on the capacitive reactance is a convention, indicating that it leads to the current leading rather than lagging ($V(t) = V_0 \sin(\omega t + \phi)$ & $I(t) = I_0 \sin \omega t$, so phase ϕ is negative for capacitors). Some people instead write $X = X_L - X_C$ and keep all reactances positive – feel free to use whichever convention you prefer.

APPARATUS

1. Science Workshop 750 Interface

In this lab we will again use the Science Workshop 750 interface as an AC function generator, whose voltage we can set and current we can measure. We will also use it to measure the voltage across the capacitor using a voltage probe.

2. AC/DC Electronics Lab Circuit Board



We will also again use the circuit board, set up with a 100 μF capacitor in series with the coil (which serves both as the resistor and inductor in the circuit), as pictured at left.

Figure 4 Setup of the AC/DC Electronics Lab Circuit Board. In addition, we will connect a voltage probe in parallel with the capacitor (not pictured).

GENERALIZED PROCEDURE

In this lab you will measure the behavior of a series RLC circuit, driven sinusoidally by a function generator.

Part 1: Driving the RLC Circuit on Resonance
Now the circuit is driven with a sinusoidal voltage and you will adjust to frequency while monitoring plots of $I(t)$ and $V(t)$ as well as V vs. I.

Part 2: What's The Frequency?
The circuit is driven with an unknown frequency and you must determine if its above or below resonance.

Part 3: What's That Trace?
Current and voltage across the function generator and capacitor are recorded, but you must determine which trace is which.

END OF PRE-LAB READING

what is overdrive, etc

IN-LAB ACTIVITIES

EXPERIMENTAL SETUP

1. Download the LabView file from the web and save the file to your desktop. Start LabView by double clicking on this file.
2. Set up the circuit pictured in Fig. 4 of the pre-lab reading (no core in the inductor!)
3. Connect a voltage probe to channel A of the 750 and connect it across the capacitor.

MEASUREMENTS

I = 8.5mH of coil
C =

Part 1: Driving the RLC Circuit on Resonance

Now we will use the function generator to drive the circuit with a sinusoidal voltage.

1. Enter the frequency that you measured in part 1 of the previous lab as a starting point to find the resonant frequency. If you don't recall this frequency, you can just start at 150 Hz.
2. Press GO to start recording the function generator current and voltage vs. time, as well as a "phase plot" of voltage vs. current.
3. Adjust the frequency up and down to find the resonant frequency and observe what happens when driving above and below resonance.

Question 1:

What is the resonant frequency? What are two ways you can determine this?

570 Hz — current + voltage on top of each other
current vs voltage phaser make a line
- when not correct its a circle

Question 2:

What is the impedance of the circuit when driven on resonance (hint: use the phase plot)?

$= \sqrt{R^2 + (X_L - X_C)^2}$ or $\frac{\text{Voltage}}{\text{current}}$

↑ use this

Question 3:

When driving on resonance, insert the core into the inductor. Are you now driving at, above or below the new resonant frequency of the circuit? How can you tell? Why?

Current is lagging so it is more inductor like

ω_0 decreased ↓

[illegible] changed – drive freq did not change

Now you are above ω_0

(Simpler than I thought)

Part 2: What's The Frequency?

For the remainder of the lab you will make some measurements where you are given incomplete information (for example, you won't be shown the frequency or won't be told what is being plotted). From the results you must determine the missing information. If you find this difficult, play with the circuit using the "further questions" tab to get a better feeling for how the circuit behaves.

1. Remove the core from the inductor
2. Press GO to record the function generator current and voltage

Question 4:

At this frequency is the circuit capacitor- or inductor-like? Are we above or below resonance? Current lagging → inductor like
above resonance freq

Part 3: What's That Trace?

1. Press GO to record the function generator current and voltage as well as the voltage across the capacitor. Note that you are not told which trace corresponds to which value.

Question 5:

What value is recorded in each of the three traces (I, V_{FG} or V_C)? How do you know?

Disconect circuit - Green left so its V_{FG}

Reconnect circuit, Disconnect voltage sensor Red disappears so its V_C

So Blue must be I

$\frac{V_{FG}}{\sqrt{V_R^2+(V_L-V_C)^2}}$

Question 6:

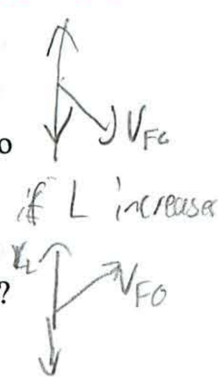
Are we above, below or on resonance? How do you know?

Below resonce because capicator like

current leading function voltage ← V_{FG} in middle of V_C & I so capicator like

Further Questions (for experiment, thought, future exam questions...)

- For a random frequency can you bring the circuit into resonance by slowly inserting the core into the coil? Are there any conditions on the frequency (e.g. does it need to be above or below the resonant frequency of the circuit with the empty coil)?
- Could you do part 3 if you were given only two traces instead of three? Would it matter which two you were given?
- What is the energy doing in the driven circuit? Is the resistor still dissipating power? If so, where is this power coming from?
- With a resistor in series with the coil and capacitor, at what frequency is the energy dissipation a maximum? How could you verify this experimentally?



Current leads V_C by 90°
which is 90° out of phase
V_{FG} is just in the middle

4/21

Class 28: Outline

Hour 1: Download LabView

Expt.9: Driven RLC Circuits

Maxwell's Equations and Displacement Current

Hour 2:

Poynting Vector and Energy Flow

Ampere's

Faraday's

Self inductance

Exam

Not Transformers
Mutual Inductance

EM Waves

Driven RLC

Poynting

Experiment 9: Driven RLC Circuit

low freq - capicator dominating
current leads

What to Learn from Lab

1) Properties of resonance? How can you tell when you are on resonance?
2) From plot I & V vs. t OR I vs. V:
   - Which is leading (I or V)?
   - L-like or C-like?
   - Above or below resonance?

1. So at freq < 1 kHz current lags
- inductor only

When say low freq is inductor like, then its only inductor

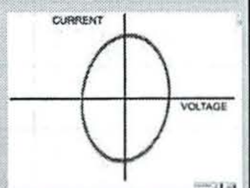
$$I_0 = \frac{V_0}{2} = \frac{V_0}{\sqrt{R^2+(X_L-X_C)^2}} \qquad X_L = \omega L \qquad X_C = \frac{1}{\omega C}$$

PRS Questions: Resonance

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

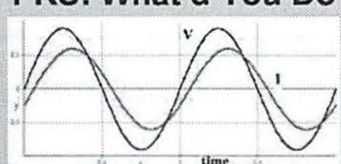
PRS: Leading or Lagging

The graph shows the current versus the voltage in a driven RLC circuit at a given driving frequency. In this plot



- 0% 1. Current lags voltage by ~90°
- 0% 2. Current leads voltage by ~90°
- 0% 3. Current and voltage are almost in phase
- 0% 4. We don't have enough information (but they aren't in phase!)
- 0% 5. I don't know

PRS: What'd You Do?



The graph shows current & voltage vs. time in a driven RLC circuit. We had been in resonance a second ago but then either put in or took out the core from the inductor. Which was it?

- 0% 1. Put in the core
- 0% 2. Took out the core
- 0% 3. I don't know

current lagging, voltagle leading
inductor like ← make it bigger
put in core ✓

above ω_0
must have lowered it
by ↑ L
by putting in core

Maxwell's Equations

$$\oint\!\!\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0} \quad \text{(Gauss's Law)}$$

$$\oint\!\!\oint_S \vec{B}\cdot d\vec{A} = 0 \quad \text{(Magnetic Gauss's Law)}$$

$$\oint_C \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's Law)}$$

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0 I_{enc} \quad \text{(Ampere's Law)}$$

Is there something missing?

Maxwell's Equations

$$\oint\!\!\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0} \quad \text{(Gauss's Law)}$$

$$\oint\!\!\oint_S \vec{B}\cdot d\vec{A} = 0 \quad \text{(Magnetic Gauss's Law)}$$

$$\oint_C \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{A} \quad \text{(Faraday's Law)}$$

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0 I_{enc} \quad \text{(Ampere's Law)}$$

$$\oint\!\!\oint_S \vec{J}\cdot d\vec{A} = -\frac{d}{dt}\iiint_V \rho\, dV = -\frac{dQ_{in}}{dt} \quad \text{(charge conservation)}$$

$$\vec{F}_q = q(\vec{E} + \vec{v}_q \times \vec{B}) \quad \text{(Lorentz Force Law)}$$

Is there something missing?

Maxwell's Equations
Class Discussion: Can you find the missing piece?

~~V_{FG} must be 2nd or 3rd?~~

think about Phasor diagram

90° out of phase

I leads V_C by 90°

V_C V_{FG} green

V_{PS} = function (V_L V_F R)

Inductor totally dominate

powersupply + current always within 90° of each other

(Exam Review

Grab loop + stop it from rotating - driven by battery

What happens to current?

Motor opposes change in flux

Wants to continue moving

Puts more current into it

- blenders melt if too much stopping

OG loop is always unhappy (dipole moment)

Commutator flips current direction

dipole wants to rotate to magnetic field - induced EMF

(faraday's law - back EMF)

fighting change - fighting the battery

turn off back EMF by holding current ↑

- back emf no longer lowering battery voltage

Current is highest at startup since no EMF

2 currents listed on package of motors

Maxwell's Equations
One Last Modification:
Displacement Current

P28-10

$P = I\Delta V = \iint \vec{S} \cdot d\vec{A}$

$u_E = \frac{1}{2}\epsilon_0 E^2$

$u_B = \frac{1}{2\mu_0} B^2$

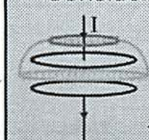
integrate over volume

exam qui prove they are =

skip slides

Ampere's Law: Capacitor

Consider a charging capacitor:



Use Ampere's Law to calculate the magnetic field just above the top plate

$$\text{Ampere's law: } \oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc}$$

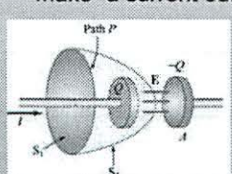
1) Red Amperian Area, $I_{enc} = I$
2) Green Amperian Area, $I = 0$

What's Going On?

P28-11

Displacement Current

We don't have current between the capacitor plates but we do have a changing E field. Can we "make" a current out of that?



$$E = \frac{Q}{\varepsilon_0 A} \Rightarrow Q = \varepsilon_0 EA = \varepsilon_0 \Phi_E$$

$$\boxed{\frac{dQ}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt} \equiv I_d}$$

This is called (for historic reasons) the **Displacement Current**

P28-12

└ "fake" current

Ampere's Law is missing something

* changing E field *

add to ampere's law

Maxwell-Ampere's Law

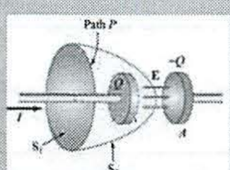
$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 (I_{encl} + I_d)$$

$$= \mu_0 I_{encl} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

both physical currents
changing electric fields

Displacement Current: Direction

The displacement current flows in the direction of the electric field if positive or opposite it if negative



$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt}$$

If we enclose all of the flux then $I_d = I$

final form

Maxwell's Equations

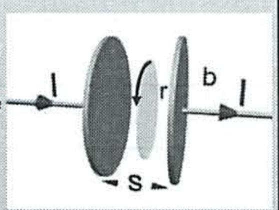
$\oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{in}}{\varepsilon_0}$	(Gauss's Law)
$\oint\!\!\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$	(Magnetic Gauss's Law)
$\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}$	(Faraday's Law)
$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$	(Ampere-Maxwell Law)

no magnetic monopoles
no motion of magnetic charges
same?
current
changing E field
lots of symmetry
remember relationships

PRS Questions: Capacitor

PRS: Capacitor

Consider a circular capacitor, with an Amperian loop (radius r) in the plane midway between the plates. When the capacitor is charging, the line integral of the magnetic field around the Amperian loop (in direction shown) is



- 0% 1. Zero (No current through loop)
- 0% 2. Positive
- 0% 3. Negative
- 0% 4. Can't tell (need to know direction of E)
- 0% 5. I don't know

:00

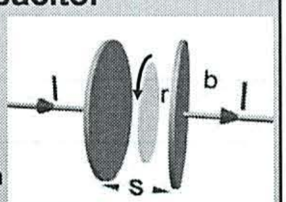
Something about move loop and then it will be here

See solution sheet

0

PRS: Capacitor

If instead of integrating around the pictured Amperian loop we were to integrate around an Amperian loop of the same radius as the plates (b) then the integral would be



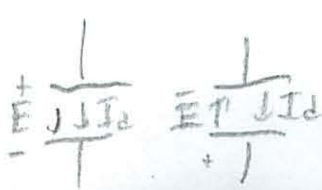
- 0% 1. The same
- 0% 2. Larger
- 0% 3. Smaller
- 0% 4. I don't know

*pretend current flows through capicator it does not

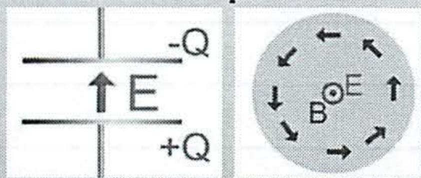
but for displacement current

$J_d = \epsilon_0 \frac{dE}{dt}$

$I_d = \epsilon_0 \frac{d\Phi_E}{dt}$ same dir as real current



PRS: Capacitor

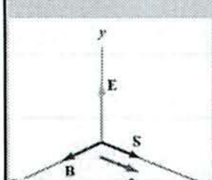


The figures above show a side and top view of a capacitor with charge Q and electric and magnetic fields E and B at time *t*. At this time the charge *Q* is:

0% 1. Increasing in time
0% 2. Constant in time.
0% 3. Decreasing in time.
0% 4. I don't know

Energy Flow

Poynting Vector



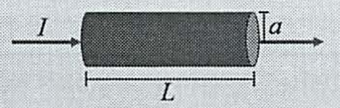
Power flow per unit area:

$$\vec{\mathbf{S}} = \frac{\vec{\mathbf{E}} \times \vec{\mathbf{B}}}{\mu_0} \quad \text{: Poynting vector}$$

$u_E = \frac{1}{2}\epsilon_0 E^2$

$u_B = \frac{1}{2\mu_0} B^2$

Group Problem: Resistor Power

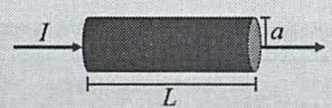


Consider the above cylindrical resistor, with current I and voltage drop ΔV. Calculate the power in terms of the electric and magnetic fields at the surface of the resistor.

There is a geometric factor. What is it?

$P = I\Delta V$

In Class Solution: Resistor Power



$$\Delta V = EL \qquad B = \frac{\mu_0 I}{2\pi a} \Rightarrow I = \frac{2\pi a B}{\mu_0}$$

$$P = \Delta V \cdot I = EL \cdot \frac{2\pi a B}{\mu_0} = (2\pi a L) \cdot \frac{EB}{\mu_0}$$

Surface area

$\oint \vec{B} \cdot ds = \mu_0 I_{enc}$

$B(2\pi a) = \mu_0 I$

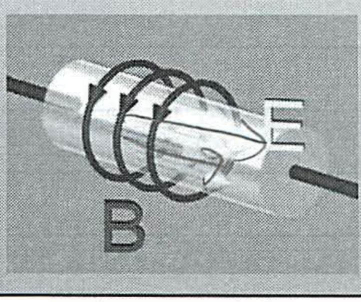
$B = \frac{\mu_0 I}{2\pi a}$

$|S| = \frac{EB}{\mu_0}$ $\frac{\text{power}}{\text{unit area}}$ e magnitude

Energy Flow: Resistor

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

On surface of resistor is INWARD



$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$ Poynting vector

* points in the direction energy flows

is it ~~radially outwards~~ direction

both at right

B is CCW

points in dir energy flow

& resistor flowing in power

- pick one place + calc at that place

at top $E \odot$ $\vec{B} \leftarrow$ $= \vec{E} \times B$ is down

radially inward

Power & Energy in Circuit Elements

$$P = \iint_{Surface} \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}}$$ Dissipates Power

$$u_E = \frac{1}{2}\,\varepsilon_0 E^2$$

$$u_B = \frac{1}{2\mu_0} B^2$$

Store Energy

POWER When (dis)charging

P28-25

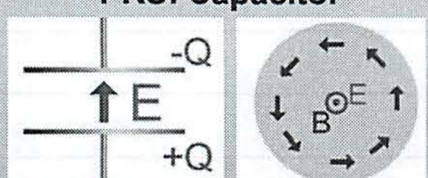
— When changing energy charging or discharging

PRS Questions: Poynting Vector

P28-26

combining everything in the course

PRS: Capacitor



The figures above show a side and top view of a capacitor with charge Q and electric and magnetic fields E and B at time t. At this time the charge Q is:

0% 1. Increasing in time
0% 2. Constant in time.
0% 3. Decreasing in time.
0% 4. I don't know

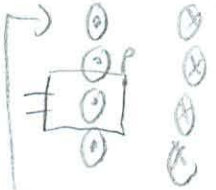
20

ampere's law

Magnetic field between plates of magnetic field
need poing vector to get energy into a capicator

Another look at Inductance

P28-28

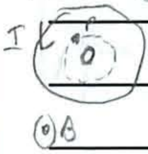


$\oint B \cdot ds = \mu_0 I_{enc}$ Ampere

$BL = \mu_0 I(t) NL$

$B = \frac{\mu_0 I N}{L}$ right hand rule = up

Draw so see flux & I into or out of page top view



$\oint E \cdot ds = -\frac{d\phi}{dt}$ Faraday

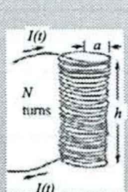
$E \cdot 2\pi r = -\frac{d}{dt}(B \cdot \pi r^2)$

$E = \frac{r}{2}\frac{dB}{dt}$

$= \frac{r N}{2H}\frac{dI}{dt}$ plug in B

E field points clockwise to drive $\vec{B}$ field into page

Group Problem: Inductor



A solenoid of radius a and length h has an increasing current $I(t)$ as pictured.
Consider a point P at radius r ($r < a$).

1. Find the magnetic field **B**(t) at P vs. time t
2. Find the electric field **E**(t) at P
3. Find the Poynting vector **S**(t) at P
4. What is the total power flux into/out of the inductor?
5. Does this make sense? How? (Hint: What's U?)

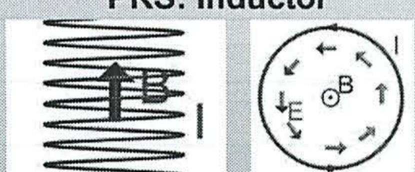
P28-29

No clue

Why electric field floating in space

Faradays Law induction
- current changing
- B field changing w/ time

PRS: Inductor



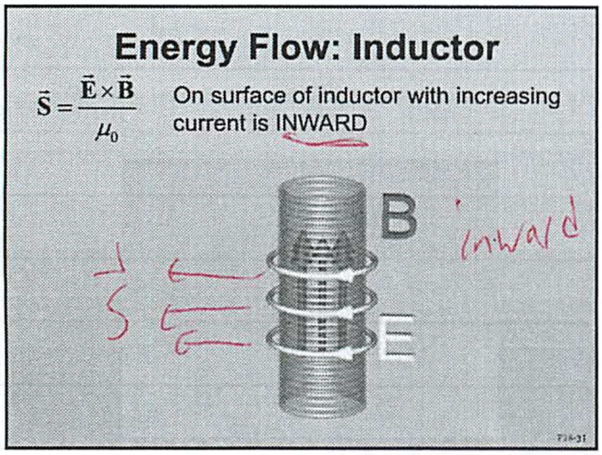
The figures above show a side and top view of a solenoid carrying current I with electric and magnetic fields E and B at time t. In the solenoid, the current I is:

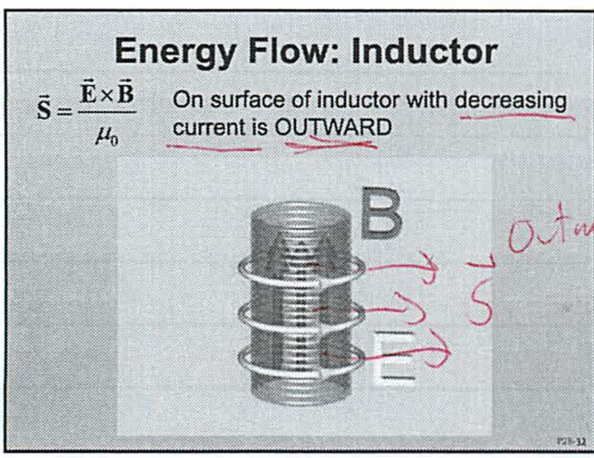
0%	1. Increasing in time
0%	2. Constant in time.
0%	3. Decreasing in time.
0%	4. I don't know

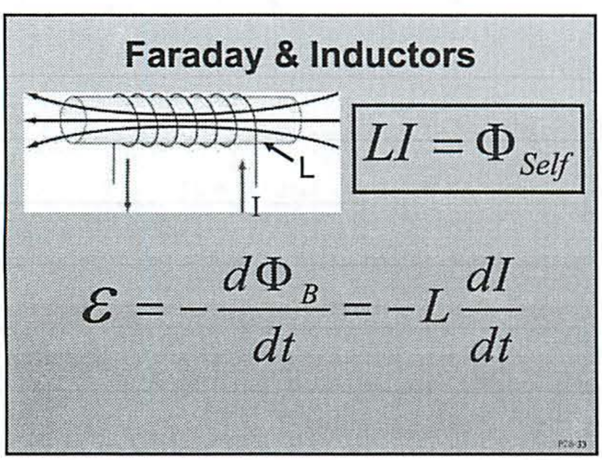
:20

current decreasing
wants to increase it
Clockwise
wants to ↑B b/c current falling
so $\vec{B}$ is out of page

$\vec{E} \times \vec{B}$ outwards
- energy leaves inductor
- when current ↓

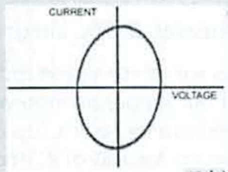

Energy Flow: Inductor
$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$
On surface of inductor with increasing current is INWARD
B
E
inward
$\vec{S}$


Energy Flow: Inductor
$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$
On surface of inductor with decreasing current is OUTWARD
B
E
Outward
$\vec{S}$


Faraday & Inductors
L
I
$LI = \Phi_{Self}$
$\mathcal{E} = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$

PRS: Leading or Lagging

The graph shows the current versus the voltage in a driven RLC circuit at a given driving frequency. In this plot



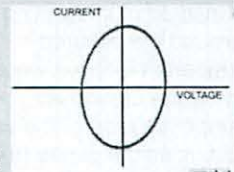
0% 1. Current lags voltage by ~90°
0% 2. Current leads voltage by ~90°
0% 3. Current and voltage are almost in phase
0% 4. We don't have enough information (but they aren't in phase!)
0% 5. I don't know

20

PRS: Answer Leading or Lagging

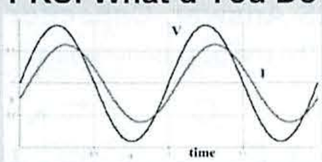
Answer: 4. Can't Tell

Without the direction you can't tell whether the current or voltage is leading or lagging. You can only tell that you aren't in phase (in fact, you are out of phase by ~90°)



P28- 2

PRS: What'd You Do?



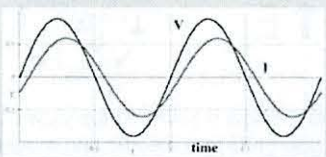
The graph shows current & voltage vs. time in a driven RLC circuit. We had been in resonance a second ago but then either put in or took out the core from the inductor. Which was it?

0% 1. Put in the core
0% 2. Took out the core
0% 3. I don't know

P28- 3

PRS Answer: What'd You Do?

Answer: 1. You put in the core

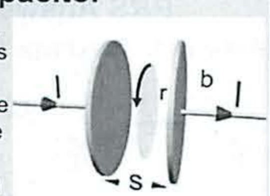


The current lags the voltage which means that the circuit is inductor-like which means that we made the inductance bigger (put in the core).

P28- 4

PRS: Capacitor

Consider a circular capacitor, with an Amperian loop (radius r) in the plane midway between the plates. When the capacitor is charging, the line integral of the magnetic field around the Amperian loop (in direction shown) is



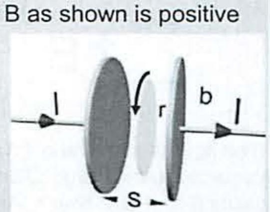
0% 1. Zero (No current through loop)
0% 2. Positive
0% 3. Negative
0% 4. Can't tell (need to know direction of E)
0% 5. I don't know

:00

PRS Answer: Capacitor

Answer: 2. The integral of B as shown is positive

Here the displacement current is the same direction as the current regardless of whether we are charging or discharging, so the B field is in the direction in which we are integrating

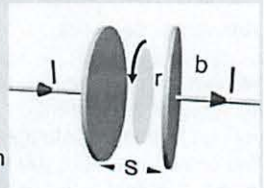


$$\oint_C \vec{\mathbf{B}} \cdot d\vec{s} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

P28- 6

PRS: Capacitor

If instead of integrating around the pictured Amperian loop we were to integrate around an Amperian loop of the same radius as the plates (b) then the integral would be

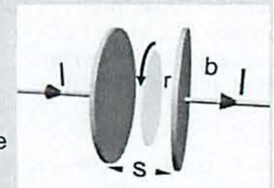


- 0% 1. The same
- 0% 2. Larger
- 0% 3. Smaller
- 0% 4. I don't know

PRS Answer: Capacitor

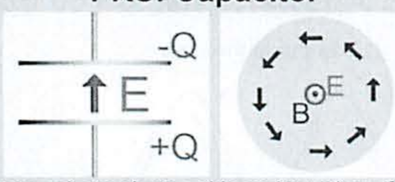
Answer: 2. The integral is larger for larger r

As we increase the radius of our Amperian loop we enclose more flux (up until we enclose all of it) and hence the magnitude of the integral will increase.



$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

PRS: Capacitor

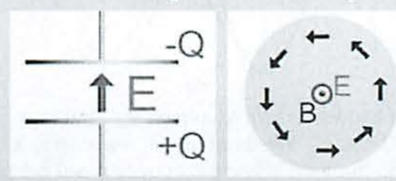


The figures above show a side and top view of a capacitor with charge Q and electric and magnetic fields E and B at time *t*. At this time the charge Q is:

- 0% 1. Increasing in time
- 0% 2. Constant in time.
- 0% 3. Decreasing in time.
- 0% 4. I don't know

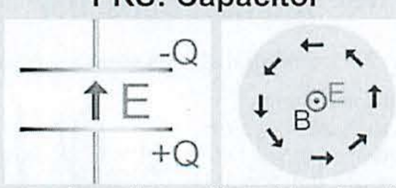
PRS Answer: Capacitor

Answer: 1. The charge Q is increasing in time



The B field is counterclockwise, which means that the current (real & displacement) must be flowing out of the page = up. So positive charge is being carried to the bottom plate, and the total charge is increasing.

PRS: Capacitor

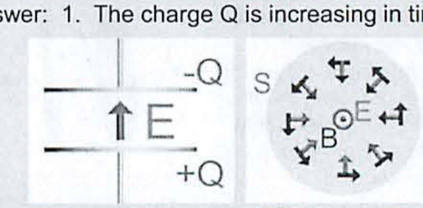


The figures above show a side and top view of a capacitor with charge Q and electric and magnetic fields E and B at time *t*. At this time the charge Q is:

- 0% 1. Increasing in time
- 0% 2. Constant in time.
- 0% 3. Decreasing in time.
- 0% 4. I don't know

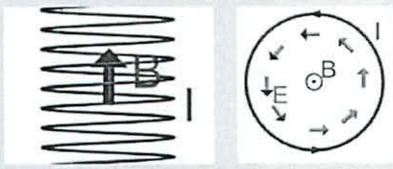
PRS Answer: Capacitor

Answer: 1. The charge Q is increasing in time



The direction of the Poynting Flux **S** (= **E** x **B**) inside the capacitor is inward. Therefore electromagnetic energy is flowing inward, and the energy in the electric field inside is increasing. Thus Q must be increasing, since E is proportional to Q.

PRS: Inductor



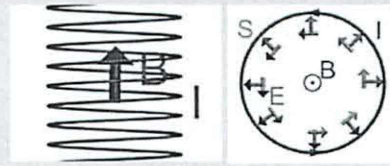
The figures above show a side and top view of a solenoid carrying current I with electric and magnetic fields E and B at time t. In the solenoid, the current I is:

- 0% 1. Increasing in time
- 0% 2. Constant in time.
- 0% 3. Decreasing in time.
- 0% 4. I don't know

:20

PRS Answer: Inductor

Answer: 3. The current *I* is decreasing in time



The Poynting Flux **S** (= **E** x **B**) inside the solenoid is outward from the center of the solenoid. Therefore EM energy is flowing outward, and the energy in the magnetic field inside is decreasing. Thus *I* must be decreasing, since B is proportional to *I*.

P28-14

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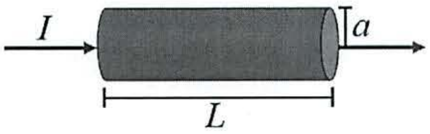
In Class W12D1_2 Solutions: Resistor Power



Problem: For the cylindrical resistor at left, with current I and voltage drop ΔV, calculate the power in terms of the electric and magnetic fields at its surface.
There is a geometric factor. What is it?

Solution:
Resistors are pretty straight forward. Assuming the electric field is uniform we simply have:

$$\Delta V = EL$$

The magnetic field we get from Ampere's Law:

$$B = \frac{\mu_0 I}{2\pi a} \Rightarrow I = \frac{2\pi a B}{\mu_0}$$

So then power is given by:

$$P = \Delta V \cdot I = EL \cdot \frac{2\pi a B}{\mu_0} = 2\pi a L \cdot \frac{EB}{\mu_0}$$

The geometric factor, $2\pi a L$, is simply the surface area of the resistor. The field dependence, EB/μ_0, is the magnitude of the Poynting vector, which has units of power per unit area. In the case of the resistor the Poynting vector points radially inwards (by example, at the top of the pictured resistor, $\vec{\mathbf{E}}$ is to the right, $\vec{\mathbf{B}}$ is out of the page, and $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$ is thus down, into the resistor. The power "consumed" by the resistor is simply the Poynting flux, the integral of the Poynting vector dotted into the surface area of the resistor.

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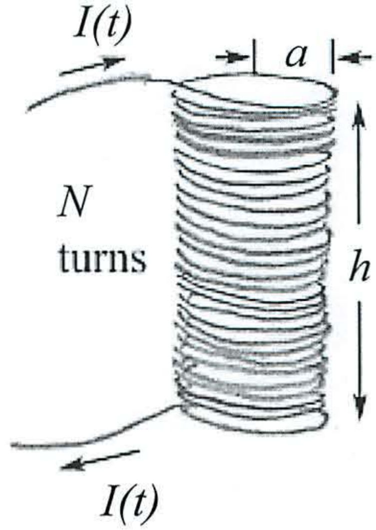
Department of Physics: 8.02

In Class W12D1_3 Solutions: Inductor



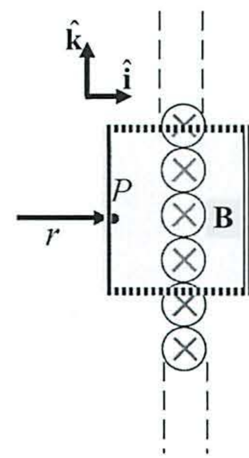
Problem: A solenoid of radius a and length h has an increasing current $I(t)$ as pictured. Consider a point P at radius r ($r < a$).

1. Find the magnetic field **B**(t) at P vs. time t (mag. & dir.)
2. Find the electric field **E**(t) at P
3. Find the Poynting vector **S**(t) at P
4. What is the total power flux into/out of the inductor?
5. Does this make sense? How? (Hint: What's U?)

Solution:

1. Find the magnetic field **B**(t) at P vs. time t (mag. & dir.)

This is just Ampere's law with a loop of length ℓ passing through point P (as shown for the cutaway picture at right)



$$\oint_{\text{contour}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_o I_{\text{through}} \Rightarrow B\ell = \mu_o NI\,\ell/h$$

$$\boxed{\vec{\mathbf{B}}(t) = \frac{\mu_o NI(t)}{h}\hat{\mathbf{k}}}$$

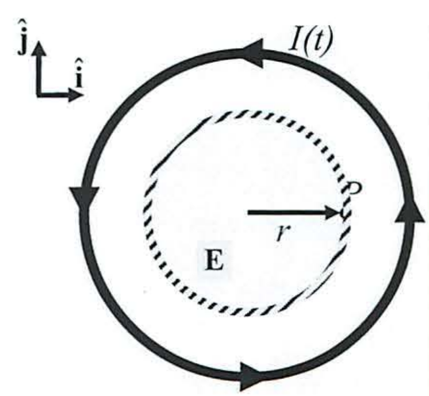
2. Find the electric field **E**(t) at P

The electric field arises from induction, so we use Faraday's Law, with the loop as shown in the top view at bottom right:

$$\oint \vec{\mathbf{E}} \cdot \mathbf{d}\vec{\mathbf{s}} = -\frac{d}{dt}\iint \vec{\mathbf{B}} \cdot \mathbf{d}\vec{\mathbf{A}} \Rightarrow E2\pi r = -\frac{dB}{dt}\pi r^2$$

$$\boxed{\vec{\mathbf{E}}(t) = -\frac{r}{2}\left|\frac{dB}{dt}\right|\hat{\mathbf{j}} = -\frac{\mu_o rN}{2h}\left|\frac{dI}{dt}\right|\hat{\mathbf{j}}}$$

3. Find the Poynting vector **S**(t) at P



$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = -\frac{1}{\mu_o}\frac{r}{2}\left|\frac{dB}{dt}\right| B\,\hat{\mathbf{i}} \text{ (i.e. radially inwards)}$$

4. What is the total power flux into/out of the capacitor?

$$P = \iint \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}} = SA = \frac{a}{2\mu_o}\left|\frac{dB}{dt}\right| B \cdot 2\pi a h = \pi a^2 h \cdot \frac{B}{\mu_o}\left|\frac{dB}{dt}\right|$$

5. Does this make sense? How? (Hint: What's U?)

Yes! $P = \pi a^2 h \cdot \frac{B}{\mu_o}\left|\frac{dB}{dt}\right| = \frac{d}{dt}\left(\pi a^2 h \cdot \frac{B^2}{2\mu_o}\right) = \frac{dU_B}{dt}$

Topics: Displacement Current, Energy Flow and the Poynting Vector
Related Reading: Course Notes: Sections 13.1-13.3, 13.12.3-13.12.4

Topic Introduction

Today you will work through analytic problems related to what you studied this week: displacement current, EM Waves and the Poynting vector.

Displacement Current

Recall that the displacement current is what we call the ability to create a magnetic field by allowing an electric field to change in time. Although calling it a "current" isn't strictly accurate (there is no flowing charge), the displacement current $I_d = \varepsilon_0 \frac{d\Phi_E}{dt}$ does act very much like a current, creating a magnetic field that curls around it. We derived the idea from thinking about a capacitor. In a capacitor, no current flows between the plates, but when the capacitor is charging or discharging (with a current I flowing onto/off of the capacitor plates) then the electric field between the plates changes, and the displacement current looks like a current I as well, uniformly distributed across the plates and flowing between them.

Energy and the Poynting Vector

The Poynting Vector $\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}$ describes how much energy passes through a given area per unit time, and points in the direction of energy flow. Although this is commonly used when thinking about electromagnetic radiation, it generically tells you about energy flow, and is particularly useful in thinking about energy in circuit components. For example, consider a cylindrical resistor. The current flows through it in the direction that the electric field points. The B field curls around. The Poynting vector thus points radially *into* the resistor – the resistor consumes energy. In today's problem solving session you will calculate the Poynting vector in a capacitor, and will find that if the capacitor is charging then **S** points in towards the center of the capacitor (energy flows into the capacitor) whereas if the capacitor is discharging **S** points outwards (it is giving up energy).

Important Equations

Displacement Current: $I_d = \varepsilon_0 \frac{d\Phi_E}{dt}$

Poynting Vector: $\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}$

Today 1st half
was more of Day 28's lecture
on that day's PP slides

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Problem Solving 9: The Displacement Current and Poynting Vector

OBJECTIVES

1. To introduce the "displacement current" term that Maxwell added to Ampere's Law
2. To find the magnetic field inside a charging cylindrical capacitor using this new term in Ampere's Law.
3. To introduce the concept of energy flow through space in the electromagnetic field.
4. To quantify that energy flow by introducing the Poynting vector.
5. To do a calculation of the rate at which energy flows into a capacitor when it is charging, and show that it accounts for the rate at which electric energy stored in the capacitor is increasing.

REFERENCE: Sections 13-1 and 13-6, 8.02 Course Notes.

The Displacement Current

In magnetostatics (the electric and magnetic fields do not change with time), Ampere's law established a relation between the line integral of the magnetic field around a closed path and the current flowing across any open surface with that closed path as a boundary of the open surface,

$$\oint_{\substack{\text{closed}\\\text{path}}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} = \mu_0 \iint_{\substack{\text{open}\\\text{surface}}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}}.$$

For reasons we have discussed in class, Maxwell argued that in time-dependent situations this equation was incomplete and that an additional term should be added:

$$\oint_{\substack{\text{closed}\\\text{loop}}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d}{dt} \iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \qquad (9.1)$$

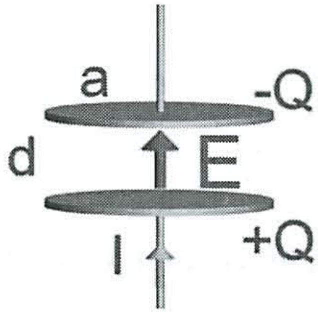
or

$$\oint_{\substack{\text{closed}\\\text{loop}}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} + \mu_0 I_d \qquad (9.2)$$

where $I_d = \varepsilon_0 \dfrac{d\Phi_E}{dt}$ is the displacement current.

An Example: The Charging Capacitor

A capacitor consists of two circular plates of radius a separated by a distance d (assume $d << a$). The center of each plate is connected to the terminals of a voltage source by a thin wire. A switch in the circuit is closed at time $t = 0$ and a current $I(t)$ flows in the circuit. The charge on the plate is related to the current according to $I(t) = \frac{dQ(t)}{dt}$. We begin by calculating the electric field between the plates. Throughout this problem you may ignore edge effects. We assume that the electric field is zero for $r > a$.

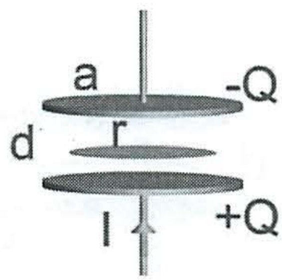


Question 1: Use Gauss' Law to find the electric field between the plates when the charge on them is Q (as pictured). The vertical direction is the $\hat{\mathbf{k}}$ direction.
Answer ***(write your answer to this and subsequent questions on the tear-sheet!)*****:**

$Q_{enc} = Q$ $\quad E \cdot \pi a^2 = \frac{Q_{enc}}{\epsilon_0}$

$\vec{E} = \frac{Q_{enc}}{\pi a^2 \epsilon_0} \hat{k}$

Now take an imaginary flat disc of radius $r < a$ inside the capacitor, as shown below.



Question 2: Using your expression for $\vec{\mathbf{E}}$ above, calculate the electric flux through this flat disc of radius $r < a$ in the plane midway between the plates. Take the surface normal to the imaginary disk to be in the $+\hat{\mathbf{k}}$ direction.

Answer: $\Phi_E = \iint_{\substack{flat \\ disk}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} =$

$\vec{E} \cdot d\vec{A}$

$\vec{E} \cdot A = \frac{Q_{enc}}{\epsilon_0}$

This electric flux is changing in time because as the plates are charging up, the electric field is increasing with time.

Question 3: Calculate the Maxwell displacement current,

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{d}{dt} \iint_{disc(r)} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

through the flat disc of radius $r < a$ in the plane midway between the plates, in terms of r, $I(t)$, and a.

Answer:

$I_d = \frac{dQ}{dt} = I$

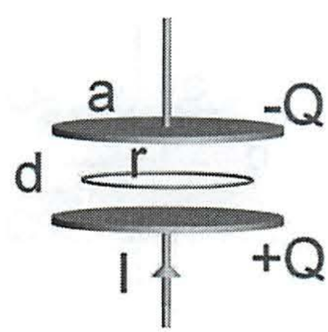
Question 4: What is the conduction current $\iint_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}}$ through the flat disc of radius $r < a$?

"Conduction" current just means the current due to the flow of real charge across the surface (e.g. electrons or ions).

Answer:

$I_{real} = 0$

Since the capacitor plates have an axial symmetry and we know that the magnetic field due to a wire runs in azimuthal circles about the wire, we assume that the magnetic field between the plates is non-zero, and also runs in azimuthal circles.



Question 5: Choose for an Amperian loop a circle of radius $r < a$ in the plane midway between the plates. Calculate the line integral of the magnetic field around the circle, $\oint_{\text{circle}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$. Express your answer in terms of $|\vec{\mathbf{B}}|$ and r. The line element $d\vec{\mathbf{s}}$ is right-handed with respect to $d\vec{\mathbf{A}}$, that is counterclockwise as seen from the top.

Answer: $\oint_{\text{circle}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} =$ $B(2\pi r)$

Question 6: Now use the results of your answers above, and apply the generalized Ampere' Law Equation (9.1) or (9.2), find the magnitude of the magnetic field at a distance $r < a$ from the axis.

Answer:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 (I_{enc} + I_d)$$

$$B(2\pi r) = \mu_0 (0 + I_d)$$

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \quad \text{ccw}$$

point in dir I + curl

Question 7: If you use your right thumb to point along the direction of the electric field, as the plates charge up, does the magnetic field point in the direction your fingers curl on your right hand or opposite the direction your fingers curl on your right hand?

Answer: in same direction
-follows right hand rule

Question 8: Would the direction of the magnetic field change if the plates were discharging? Why or why not?

Answer: Flipped "current" flows the other way

The Poynting Vector

Once a capacitor has been charged up, it contains electric energy. We know that the energy stored in the capacitor came from the battery. How does that energy get from the battery to the capacitor? Energy flows through space from the battery into the ***sides*** of the capacitor. In electromagnetism, the rate of energy flow per unit area is given by the Poynting vector

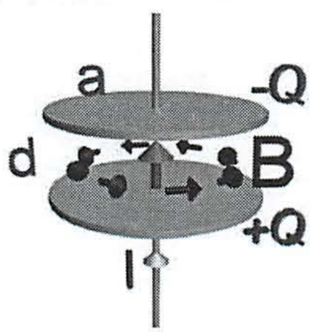
$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}} \quad \text{(units: } \frac{\text{joules}}{\text{sec square meter}}\text{)}$$

To calculate the amount of electromagnetic energy flowing through a surface, we calculate the surface integral $\iint \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}}$ (units: $\frac{\text{joules}}{\text{sec}}$ or watts).

Energy Flow in a Charging Capacitor

We show how to do a Poynting vector calculation by explicitly calculating the Poynting vector inside a charging capacitor. The electric field and magnetic fields of a charging cylindrical capacitor are (ignoring edge effects)

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{Q(t)}{\pi a^2 \varepsilon_0}\hat{\mathbf{k}} & r \le a \\ \vec{\mathbf{0}} & r > a \end{cases} \qquad \vec{\mathbf{B}} = \begin{cases} \dfrac{\mu_0 I(t)}{2\pi a}\dfrac{r}{a}\hat{\boldsymbol{\varphi}} & r < a \\ \dfrac{\mu_0 I(t)}{2\pi r}\hat{\boldsymbol{\varphi}} & r > a \end{cases}$$



→ fingers ∫ curl up

thumb out ⊙

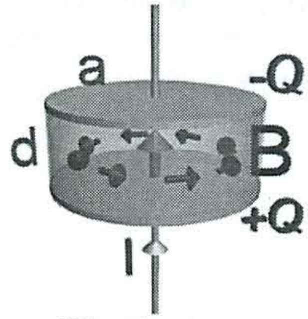
Question 9: What is the Poynting vector for $r \le a$?

$\frac{1}{\mu_0}\vec{E} \times \vec{B}$

radially out

$$\frac{\frac{Q_{enc}}{\pi a^2 \varepsilon_0} \times \frac{\mu_0 I}{2\pi a}}{\mu_0} = \frac{Q_{enc} I}{2\pi^2 a^3 \varepsilon_0}$$

Since the Poynting vector points radially into the capacitor, electromagnetic energy is flowing into the capacitor through the sides. To calculate the total energy flow into the capacitor, we evaluate the Poynting vector right at $r = a$ and integrate over the sides $r = a$.



Question 10: Calculate the flux $\iint \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}}$ of the Poynting vector evaluated at $r = a$ through an imaginary cylindrical surface of radius a and height d, i.e. over the side of the capacitor. Your answer should involve *Q, a, I,* and *d.* What are the units of this expression?

$\vec{E} \cdot \vec{A} = \frac{Q_{enc}}{\epsilon_0}$

$\vec{B}(2\pi a) = \mu_0 I$ ... $\frac{}{2\pi a}$

$\vec{E} = \frac{Q}{\epsilon_0 \pi a^2} \hat{k}$

$\left(\frac{\vec{E} \times \vec{B}}{\mu_0}\right)(A)$

$\frac{\left(\frac{Q}{\epsilon_0 \pi a^2}\right)\left(\frac{\mu_0 I}{2\pi a^2}\right)}{\mu_0} 2\pi a d = \frac{Q I d}{a^2 \epsilon_0 \pi}$

radially out

$\frac{\text{Joules}}{\text{sec}}$

watts

Question 11: The capacitance of a parallel plate capacitor is $C = \frac{\varepsilon_o Area}{d} = \frac{\varepsilon_o \pi a^2}{d}$. Rewrite your answer to Question 10 above using the capacitance *C*. Your answer should involve only *Q, I,* and *C*.

$\frac{1}{C} = \frac{d}{\epsilon_0 \pi a^2}$

$\iint S \cdot dA = \frac{QI}{C}$ radially out

watts

Question 12: The total electrostatic energy stored in the capacitor at time t is given by $\frac{1}{2}\frac{Q(t)^2}{C}$. Show that the rate at which this energy is increasing as the capacitor is charged is equal to the rate at which energy is flowing into the capacitor through the sides, as calculated in Question 11 above. That is, where this energy is coming from is from the flow of energy through the sides of the capacitor.

differentiate → $\frac{2Q(t)}{2C} \rightarrow \frac{2QI}{2C} = \frac{QI}{C}$ watts radially out

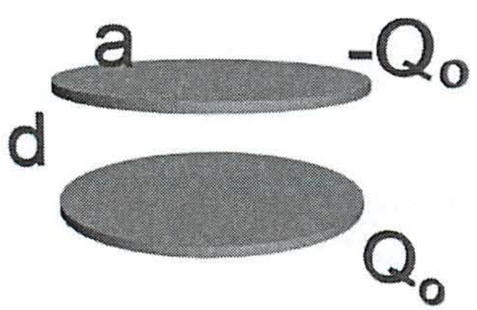
Question 13: Suppose the capacitor is discharging instead of charging, i.e. $Q(t) > 0$ but now $\frac{dQ(t)}{dt} > 0$ What changes in the picture above? Explain.

direction of current + thus Poying Vector   Energy flowing in other direction

Sample Exam Question (If time, try to do this by yourself, closed notes)

A capacitor consists of two parallel circular plates of radius a separated by a distance d (assume $a >> d$). The capacitor is initially charged to a charge Q_o. At $t = 0$, this capacitor begins to discharge because we insert a circular resistor of radius a and height d between the plates, such that the ends of the resistor make good electrical contact with the plates of the capacitor. The capacitor then discharges through this resistor for $t \geq 0$, so the charge on the capacitor becomes a function of time $Q(t)$. *Throughout this problem you may ignore edge effects.*

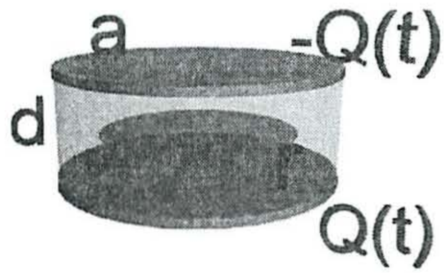
$t < 0$



$t \geq 0$

a). Use Gauss's Law to find the electric field between the plates. Is this electric field upward or downward?

b). For $t \geq 0$, consider an imaginary open surface of radius $r < a$ inside the capacitor with its normal $\mathbf{d\vec{A}}$ upward (see figure)



For $t \geq 0$, what is the current flowing through this open surface in terms of $Q(t)$ or $\frac{dQ(t)}{dt}$ and the parameters given. Define the direction of positive current to be upward, and be careful about signs.

c) For this same imaginary open surface, what is the time rate of change of the electric flux though the surface, in terms of $Q(t)$ or $\frac{dQ(t)}{dt}$ and the parameters given (hint: use your answer above for **E**).

d) What is the integral of the magnetic field around the contour bounding this open circle, using the Ampere-Maxwell Law? Be careful of signs.

e) Does your answer in (d) make sense in terms of the energy dissipation and energy flow in this problem? You must explain your answer clearly and logically to get credit.

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Tear off this page and turn it in at the end of class !!!!

Note:
Writing in the name of a student who is not present is a Committee on Discipline offense.

Problem Solving 9: Displacement Current and Poynting Vector

Group 11B (e.g. 2B Please Fill Out)

Names Jennifer Quintana
Melanie Alba
Michael Plasmeier

see solns...
(5)

Question 1: Use Gauss' Law to find the electric field between the plates as a function of time t, in terms of $Q(t)$, a, ε_0, and π.

Answer: $\vec{E} = \frac{Q_{enc}}{\pi a^2 \epsilon_0} \hat{k}$ ✓

Question 2: Using your expression for $\vec{\mathbf{E}}$ above, calculate the electric flux through the flat disc of radius $r < a$.

Answer: $\Phi_E = \frac{Q_{enc}}{\epsilon_0}$ $\iint E \cdot dA = \Phi$

Question 3: Calculate the Maxwell displacement current $I_d = \varepsilon_0 \frac{d\Phi_E}{dt}$ through the disc.

Answer: $I_d = \frac{dQ}{dt} = I$ $\frac{r^2}{a^2} I$

Question 4: What is the conduction current through the flat disc of radius $r < a$?
Answer:

$I_{Real} = 0$ ✓

Question 5: Calculate the line integral of the magnetic field around the circle, $\oint_{circle} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$.

Answer: $B(2\pi r)$ ✓

Question 6: What is the magnitude of the magnetic field at a distance $r < a$ from the axis. Your answer should be in terms of r, $I(t)$, μ_o, π, and a.

Answer: $\oint B \cdot d\vec{s} = \mu_0 (I_{enc_R} + I_d)$ $\quad \vec{B} = \frac{\mu_0 I}{2\pi r}$ ccw $\quad$ w/ "I" $= I \frac{r^2}{a^2}$

$B(2\pi r) = \mu_0 (0 + I)$

Question 7: If you use your right thumb to point along the direction of the electric field, as the plates charge up, does the magnetic field point in the direction your fingers curl on your right hand or opposite the direction your fingers curl on your right hand?

Answer: In the same direction ✓

Question 8: Would the direction of the magnetic field change if the plates were discharging? Why or why not?

Answer: Yes, because "current" flows the other way. ✓

Question 9: What is the Poynting vector for $r \leq a$?

Answer: Radially outward $\frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{\frac{Q_{enc}}{\pi a^2 \epsilon_0} \times \frac{\mu_0 I}{2\pi r}}{\mu_0} = \frac{Q_{enc} I}{2\pi^2 a^3 \epsilon_0}$ radially outward $\quad I = I\frac{r^2}{a^2}$

Question 10: Calculate the flux $\iint \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}}$ of the Poynting vector evaluated at $r = a$ through an imaginary cylindrical surface of radius a and height d, with area $A = 2\pi ab$, i.e. over the sides of the capacitor. Your answer should involve Q, a, I, d, π, and ε_o. What are the units of this expression?

Answer: $\frac{QId}{a^2 \epsilon_0 \pi}$ radially out Watts ✓

Question 11: Rewrite your answer to Question 10 above using the capacitance C.

Answer: $\frac{1}{C} = \frac{d}{\epsilon_0 \pi a^2} \quad = \frac{QI}{C}$ Radially out ✓

Question 12: The total electrostatic energy stored in the capacitor at time t is given by $Q^2(t)/2C$. Show that the rate at which this energy is increasing as the capacitor is charged is equal to the rate at which energy is flowing into the capacitor through the sides, as calculated in Question 11 above. That is, where this energy is coming from is from the flow of energy through the sides of the capacitor.

Answer: $\frac{2Q(t)}{2C}\left(\frac{dQ}{dt}\right) \Rightarrow \frac{2QI}{2C} = \frac{QI}{C} = \frac{QI}{C}$ Watts :) ✓

Question 13: Suppose the capacitor is discharging instead of charging, i.e. $Q(t) > 0$ but now $dQ(t)/dt < 0$ What changes in the picture above? Explain.

Answer: The direction of current and the Poynting vector because energy is flowing in the other direction.

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Problem Solving 9 *Solutions*: Displacement Current and Poynting Vector

OBJECTIVES

1. To introduce the "displacement current" term that Maxwell added to Ampere's Law
2. To find the magnetic field inside a charging cylindrical capacitor using this new term in Ampere's Law.
3. To introduce the concept of energy flow through space in the electromagnetic field.
4. To quantify that energy flow by introducing the Poynting vector.
5. To do a calculation of the rate at which energy flows into a capacitor when it is charging, and show that it accounts for the rate at which electric energy stored in the capacitor is increasing.

REFERENCE: Sections 13-1 and 13-6, 8.02 Course Notes.

The Displacement Current

In magnetostatics (the electric and magnetic fields do not change with time), Ampere's law established a relation between the line integral of the magnetic field around a closed path and the current flowing across any open surface with that closed path as a boundary of the open surface,

$$\oint_{\substack{\text{closed}\\\text{path}}} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} = \mu_0 \iint_{\substack{\text{open}\\\text{surface}}} \vec{\mathbf{J}}\cdot d\vec{\mathbf{A}}.$$

For reasons we have discussed in class, Maxwell argued that in time-dependent situations this equation was incomplete and that an additional term should be added:

$$\oint_{\substack{\text{closed}\\\text{loop}}} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} + \mu_0\varepsilon_0 \frac{d}{dt}\iint_S \vec{\mathbf{E}}\cdot d\vec{\mathbf{A}} = \mu_0 I_{\text{enc}} + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} \tag{9.1}$$

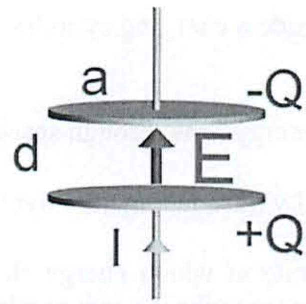
or

$$\oint_{\substack{\text{closed}\\\text{loop}}} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} + \mu_0 I_d \tag{9.2}$$

where $I_d = \varepsilon_0 \dfrac{d\Phi_E}{dt}$ is the displacement current.

An Example: The Charging Capacitor

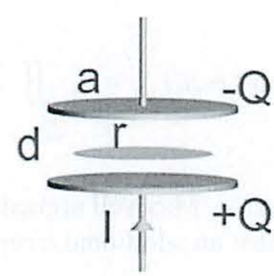
A capacitor consists of two circular plates of radius a separated by a distance d (assume $d << a$). The center of each plate is connected to the terminals of a voltage source by a thin wire. A switch in the circuit is closed at time $t = 0$ and a current $I(t)$ flows in the circuit. The charge on the plate is related to the current according to $I(t) = \frac{dQ(t)}{dt}$. We begin by calculating the electric field between the plates. Throughout this problem you may ignore edge effects. We assume that the electric field is zero for $r > a$.



Question 1: Use Gauss' Law to find the electric field between the plates when the charge on them is Q (as pictured). The vertical direction is the $\hat{\mathbf{k}}$ direction.
Answer ***(write your answer to this and subsequent questions on the tear-sheet!)***:

$$\vec{\mathbf{E}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{k}} = \frac{Q}{\pi a^2 \varepsilon_0}\hat{\mathbf{k}}$$

Now take an imaginary flat disc of radius $r < a$ inside the capacitor, as shown below.



Question 2: Using your expression for $\vec{\mathbf{E}}$ above, calculate the electric flux through this flat disc of radius $r < a$ in the plane midway between the plates. Take the surface normal to the imaginary disk to be in the $\hat{\mathbf{k}}$ direction.

$$\iint_{\substack{flat\\disk}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E\pi r^2 = \frac{Q\pi r^2}{\varepsilon_0 \pi a^2} = \frac{Q}{\varepsilon_0 a^2} r^2$$

This electric flux is changing in time because as the plates are charging up, the electric field is increasing with time.

Question 3: Calculate the Maxwell displacement current,

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{d}{dt} \iint_{disc(r)} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

through the flat disc of radius $r < a$ in the plane midway between the plates, in terms of r, $I(t)$, and a.

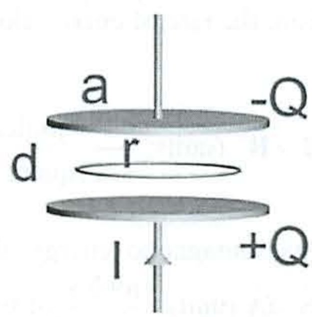
$$I_d = \varepsilon_0 \frac{d}{dt} \iint_{disc(r)} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \varepsilon_0 \frac{d}{dt}\left(\frac{Q(t)}{\varepsilon_0 a^2} r^2\right) = \frac{r^2}{a^2} I(t)$$

Question 4: What is the conduction current $\iint_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}}$ through the flat disc of radius $r < a$?

"Conduction" current just means the current due to the flow of real charge across the surface (e.g. electrons or ions).

Answer: zero.

Since the capacitor plates have an axial symmetry and we know that the magnetic field due to a wire runs in azimuthal circles about the wire, we assume that the magnetic field between the plates is non-zero, and also runs in azimuthal circles.



Question 5: Choose for an Amperian loop a circle of radius $r < a$ in the plane midway between the plates. Calculate the line integral of the magnetic field around the circle, $\oint_{circle} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$. Express your answer in terms of $|\vec{\mathbf{B}}|$ and r. The line element $d\vec{\mathbf{s}}$ is right-handed with respect to $d\vec{\mathbf{A}}$, that is counterclockwise as seen from the top.

Answer: $\oint_{circle} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\pi r)$

Question 6: Now use the results of your answers above, and apply the generalized Ampere' Law Equation (9.1) or (9.2), find the magnitude of the magnetic field at a distance $r < a$ from the axis.

$$B(2\pi r) = \mu_0 I_d = \mu_0 \frac{r^2}{a^2} I \;\Rightarrow B = \mu_0 \frac{I\,r}{2\pi a^2}$$

Question 7: If you use your right thumb to point along the direction of the electric field, as the plates charge up, does the magnetic field point in the direction your fingers curl on your right hand or opposite the direction your fingers curl on your right hand?

The magnetic field points, as the plates charge up, in the same direction as your fingers curl on your right hand.

Question 8: Would the direction of the magnetic field change if the plates were discharging? Why or why not?

The magnetic field would reverse direction because the electric flux is now decreasing in time.

The Poynting Vector

Once a capacitor has been charged up, it contains electric energy. We know that the energy stored in the capacitor came from the battery. How does that energy get from the battery to the capacitor? Energy flows through space from the battery into the ***sides*** of the capacitor. In electromagnetism, the rate of energy flow per unit area is given by the Poynting vector

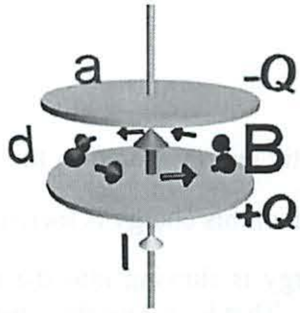
$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}}\times\vec{\mathbf{B}} \quad (\text{units: } \frac{\text{joules}}{\text{sec square meter}})$$

To calculate the amount of electromagnetic energy flowing through a surface, we calculate the surface integral $\iint \vec{\mathbf{S}}\cdot d\vec{\mathbf{A}}$ (units: $\frac{\text{joules}}{\text{sec}}$ or watts).

Energy Flow in a Charging Capacitor

We show how to do a Poynting vector calculation by explicitly calculating the Poynting vector inside a charging capacitor. The electric field and magnetic fields of a charging cylindrical capacitor are (ignoring edge effects)

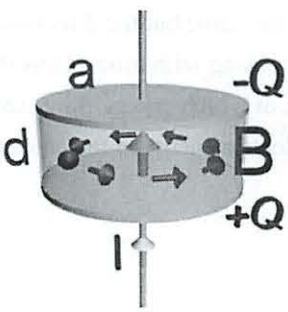
$$\vec{\mathbf{E}} = \begin{cases} \dfrac{Q(t)}{\pi a^2 \varepsilon_0}\hat{\mathbf{k}} & r \le a \\ \vec{\mathbf{0}} & r > a \end{cases} \qquad \vec{\mathbf{B}} = \begin{cases} \dfrac{\mu_0 I(t)}{2\pi a}\dfrac{r}{a}\hat{\boldsymbol{\varphi}} & r < a \\ \dfrac{\mu_0 I(t)}{2\pi r}\hat{\boldsymbol{\varphi}} & r > a \end{cases}$$



Question 9: What is the Poynting vector for $r \le a$?

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}}\times\vec{\mathbf{B}} = -\frac{QIr}{2\pi^2 a^4 \varepsilon_0}\hat{\mathbf{r}}$$

Since the Poynting vector points radially into the capacitor, electromagnetic energy is flowing into the capacitor through the sides. To calculate the total energy flow into the capacitor, we evaluate the Poynting vector right at $r = a$ and integrate over the sides $r = a$.



Question 10: Calculate the flux $\iint \vec{\mathbf{S}}\cdot d\vec{\mathbf{A}}$ of the Poynting vector evaluated at $r = a$ through an imaginary cylindrical surface of radius a and height d, i.e. over the side of the capacitor. Your answer should involve Q, a, I, and d. What are the units of this expression?

$$\iint \vec{\mathbf{S}}\cdot d\vec{\mathbf{A}} = -\frac{QId}{\pi a^2 \varepsilon_0}$$ units are joules/sec or watts

Question 11: The capacitance of a parallel plate capacitor is $C = \frac{\varepsilon_o Area}{d} = \frac{\varepsilon_o \pi a^2}{d}$. Rewrite your answer to Question 10 above using the capacitance C. Your answer should involve only Q, I, and C.

$$\iint \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}} = -\frac{QI}{C}$$

Question 12: The total electrostatic energy stored in the capacitor at time t is given by $\frac{1}{2}\frac{Q(t)^2}{C}$. Show that the rate at which this energy is increasing as the capacitor is charged is equal to the rate at which energy is flowing into the capacitor through the sides, as calculated in Question 11 above. That is, where this energy is coming from is from the flow of energy through the sides of the capacitor.

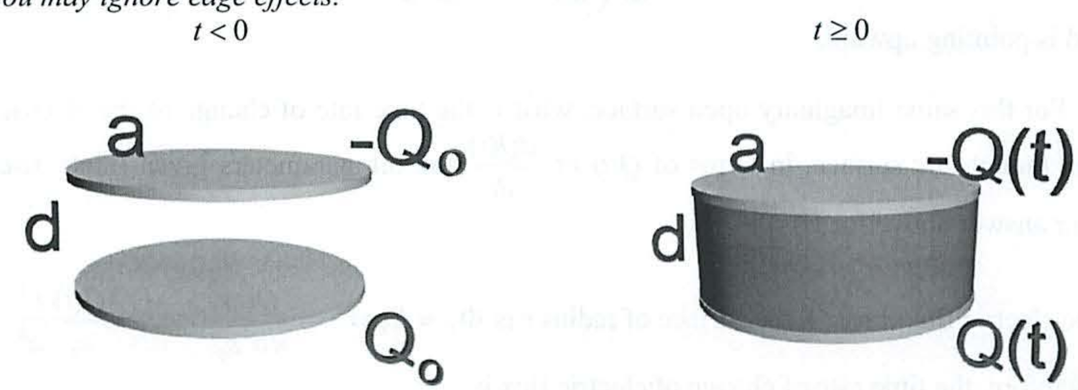
$\frac{d}{dt}\frac{Q^2}{2C} = \frac{Q}{C}\frac{dQ}{dt} = \frac{QI}{C}$ and this is the rate at which energy is flowing into the capacitor

Question 13: Suppose the capacitor is discharging instead of charging, i.e. $Q(t) > 0$ but now $\frac{dQ(t)}{dt} > 0$ What changes in the picture above? Explain.

The expression for the electric field is the same but the direction of the magnetic field reverses if the charge is decreasing instead of increasing with time. Thus the direction of the Poynting vector is outward instead of inward. The rate at which energy flows *out* of the capacitor through the sides is now equal to the rate at which the electric energy in the capacitor is decreasing.

Sample Exam Question (If time, try to do this by yourself, closed notes)

A capacitor consists of two parallel circular plates of radius a separated by a distance d (assume $a >> d$). The capacitor is initially charged to a charge Q_o. At $t = 0$, this capacitor begins to discharge because we insert a circular resistor of radius a and height d between the plates, such that the ends of the resistor make good electrical contact with the plates of the capacitor. The capacitor then discharges through this resistor for $t \geq 0$, so the charge on the capacitor becomes a function of time $Q(t)$. *Throughout this problem you may ignore edge effects.*

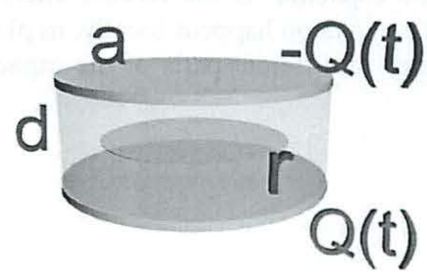


a). Use Gauss's Law to find the electric field between the plates as a function of $Q(t)$ and given parameters and ε_o. Is this electric field upward or downward?

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(\pi a^2) = \frac{Q(t)}{\varepsilon_0} \quad \Rightarrow E = \frac{Q(t)}{\pi a^2 \varepsilon_0}$$

The direction is upward, from positive to the negative plate.

b). For $t \geq 0$, consider an imaginary open surface of radius $r < a$ inside the capacitor with its normal $\mathbf{d\vec{A}}$ upward (see figure)



For $t \geq 0$, what is the current flowing through this open surface in terms of *Q(t)* or $\frac{dQ(t)}{dt}$ and the parameters given. Define the direction of positive current to be upward, and be careful about signs.

The current flowing through this surface is

$$I = -\frac{dQ}{dt}\left(\frac{\pi r^2}{\pi a^2}\right) = -\frac{dQ}{dt}\frac{r^2}{a^2}$$

and is pointing upward.

c) For this same imaginary open surface, what is the time rate of change of the electric flux though the surface, in terms of *Q(t)* or $\frac{dQ(t)}{dt}$ and the parameters given (hint: use your answer above for **E**).

The electric flux through the surface of radius *r* is $\Phi_E = E(\pi r^2) = \frac{Q(t)}{\pi a^2 \varepsilon_0}(\pi r^2) = \frac{Q(t)}{\varepsilon_0}\frac{r^2}{a^2}$

Therefore, the time rate of change of electric flux is

$$\frac{d\Phi_E}{dt} = \frac{1}{\varepsilon_0}\frac{r^2}{a^2}\frac{dQ(t)}{dt}$$

d) What is the integral of the magnetic field around the contour bounding this open circle, using the Ampere-Maxwell Law? Be careful of signs.

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0\left(-\frac{dQ}{dt}\frac{r^2}{a^2}\right) + \mu_0\varepsilon_0\left(\frac{1}{\varepsilon_0}\frac{r^2}{a^2}\frac{dQ}{dt}\right) = 0$$

e) Does your answer in (d) make sense in terms of the energy dissipation and energy flow in this problem? You must explain your answer clearly and logically to get credit.

The result in (d) implies that no magnetic field is generated in this process, due to the exact cancellation of the current and the displacement through the surface. Thus there is no energy flow inside of the capacitor as the electric energy is converted to thermal energy. This is because this conversion happens locally, in place, and there is no need of an energy flux to carry energy to different parts of the capacitor or to circuit elements outside of the capacitor.

Topics: Maxwell's Equations, EM Radiation & Energy Flow
Related Reading: Course Notes: Sections 13.3-13.4, 13.11, 13.12.1-13.12.2

Topic Introduction

Today we will put together much of the physics we have learned in the class to see how electricity and magnetism interact with each other. We begin by finalizing Maxwell's Equations, and then describe their result – electromagnetic (EM) radiation.

Maxwell's Equations
Now that we have all of Maxwell's equations, let's review:

$$(1) \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{in}}{\varepsilon_0} \qquad (2) \oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

$$(3) \oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt} \qquad (4) \oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

(1) Gauss's Law states that electric charge creates diverging electric fields.
(2) Magnetic Gauss's Law states that there are no magnetic charges (monopoles).
(3) Faraday's Law states that changing magnetic fields induce electric fields (which curl around the changing flux).
(4) Ampere-Maxwell's Law states that magnetic fields are created both by currents and by changing electric fields, and that in each case the field curls around its creator.

The last piece of this last equation is the one piece you have not seen and we will justify its addition in class. These equations are the cornerstone of the theory of electricity and magnetism. Together with the Lorentz Force $\left(\vec{\mathbf{F}} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})\right)$ they pretty much describe all of E&M, and from them we can derive mathematically the major equations you learned this semester (like Coulomb's Law and Biot-Savart). People even put them on T-shirts. They are important and you should try hard to keep them in mind.

Electromagnetic Radiation
The fact that changing magnetic fields create electric fields and that changing electric fields create magnetic fields means that oscillating electric and magnetic fields can propagate through space (each pushing forward the other). This is electromagnetic (EM) radiation. It is the single most useful discovery we discuss in this class, not only allowing us to understand natural phenomena, like light, but also to create EM radiation to carry a variety of useful information: radio, broadcast television and cell phone signals, to name a few, are all EM radiation. In order to understand the mathematics of EM radiation you need to understand how to write an equation for a traveling wave (a wave that propagates through space as a function of time). Any function that is written f(x-vt) satisfies this property. As t increases, a function of this form moves to the right (increasing x) with velocity v. You can see this as follows: At t=0 f(0) is at x=0. At a later time t=t, f(0) is at x=vt. That is, the function has moved a distance vt during a time t.
Sinusoidal traveling waves (plane waves) look like waves both as a function of position and as a function of time. If you sit at one position and watch the wave travel by you say that it has a period *T*, inversely related to its frequency *f*, and angular frequency,

$\omega\left(T = f^{-1} = 2\pi\omega^{-1}\right)$. If instead you freeze time and look at a wave as a function of position, you say that it has a wavelength λ, inversely related to its wavevector k $\left(\lambda = 2\pi k^{-1}\right)$. Using this notation, we can rewrite our function $f(x-vt) = f_0\sin(kx-\omega t)$, where $v = \omega/k$.
We typically treat both electric and magnetic fields as plane waves as they propagate through space (if you have one you must have the other). They travel at the speed of light (v=c). They also obey two more constraints. First, their magnitudes are fixed relative to each other: $E_0 = cB_0$ (check the units!) Secondly, E & B always oscillate at right angles to each other and to their direction of propagation (they are *transverse* waves). That is, if the wave is traveling in the z-direction, and the E field points in the x-direction then the B field must point along the y-direction. More generally we write $\hat{\mathbf{E}} \times \hat{\mathbf{B}} = \hat{\mathbf{p}}$, where $\hat{\mathbf{p}}$ is the direction of propagation.

Important Equations

Maxwell's Equations:

$$(1)\ \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{in}}{\varepsilon_0} \qquad (2)\ \oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

$$(3)\ \oint_C \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt} \qquad (4)\ \oint_C \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \mu_0 I_{enc} + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$

EM Plane Waves:

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = E_0 \sin\left(k\hat{\mathbf{p}} \cdot \vec{\mathbf{r}} - \omega t\right)\hat{\mathbf{E}}$$
$$\vec{\mathbf{B}}(\vec{\mathbf{r}},t) = B_0 \sin\left(k\hat{\mathbf{p}} \cdot \vec{\mathbf{r}} - \omega t\right)\hat{\mathbf{B}}$$

with $E_0 = cB_0$; $\hat{\mathbf{E}} \times \hat{\mathbf{B}} = \hat{\mathbf{p}}$; $\omega = ck$

4/26

Class 30: Outline

Hour 1:
- Maxwell's Equations
- Electromagnetic Radiation
- Plane Waves

Hour 2:
- Standing Waves
- Energy Flow

P30- 1

Maxwell's Equations

P30- 2

Maxwell's Equations

$\oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \dfrac{Q_{in}}{\varepsilon_0}$ (crossed out → 0)	(Gauss's Law)
$\oint\!\!\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$	(Magnetic Gauss's Law)
$\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\dfrac{d\Phi_B}{dt}$	(Faraday's Law)
$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \dfrac{d\Phi_E}{dt}$ ($\mu_0 I_{enc}$ crossed out → 0)	(Ampere-Maxwell Law)

+ if we ever find + magnetic monopol

What about free space (no charge or current)?

when no charges + currents

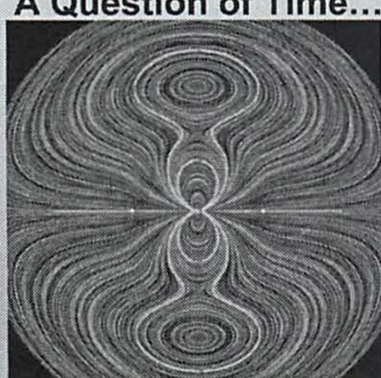
P30- 3

Electromagnetism Review

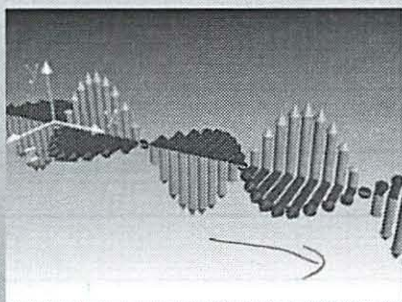
- E fields are created by:
  - (1) electric charges (Gauss's Law)
  - (2) time changing B fields Gauss's Law (Faraday's Law) Faraday's Law
- B fields are created by
  - (3a) moving electric charges (Ampere-Maxwell Law) Ampere's Law
  - (3b) time changing E fields (Ampere-Maxwell Law) Maxwell's Addition
- No magnetic charge
  - (4) Conservation of magnetic flux (Unnamed Law)
- Conservation of Charge
- E (B) fields exert forces on (moving) electric charges

Electromagnetic Radiation

A Question of Time…



Electromagnetic Radiation: Plane Waves



moves/propigates in +x dir

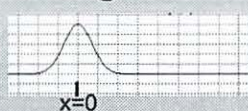
One direction E
B

* fields in phase

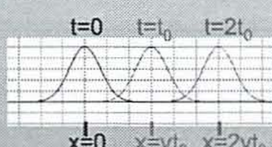
Comes at of Maxwells eq

Traveling Waves

Consider f(x) =



What is g(x,t) = f(x-vt)?



f(x-vt) is traveling wave moving to the right!

"propigating"

Waves in general

at a later time
peak moves →

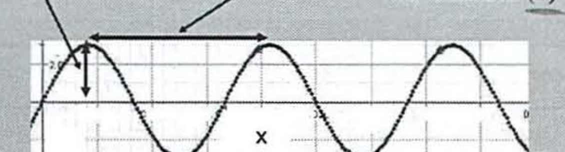
Traveling Sine Wave

What is g(x,t) = f(x+vt)? Travels to left at velocity v

$y = y_0\sin(k(x+vt)) = y_0\sin(kx+kvt)$

Look at $t = 0$: $g(x,0) = y = y_0\sin(kx)$:

Amplitude (y_0) Wavelength $(\lambda) = \dfrac{2\pi}{\text{wavenumber } (k)}$



same sign, same direction
if − travels to right

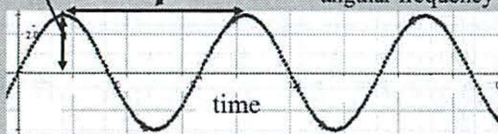
Traveling Sine Wave

$$g(x,t) = y_0 \sin k(x + vt)$$

Look at x=0: $g(0,t) = y_0 \sin(kvt) \equiv y_0 \sin(\omega t)$

Period $(T) = \dfrac{1}{\text{frequency } (f)} = \dfrac{2\pi}{\text{angular frequency } (\omega)}$

Amplitude (y_0)



Traveling Sine Wave

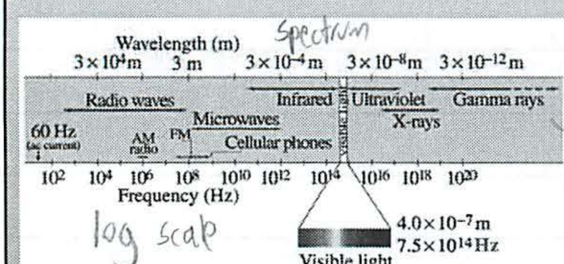
- Wavelength: λ
- Frequency : f
- Wave Number: $k = \dfrac{2\pi}{\lambda}$
- Angular Frequency: $\omega = 2\pi f$
- Period: $T = \dfrac{1}{f} = \dfrac{2\pi}{\omega}$
- Speed of Propagation: $v = \dfrac{\omega}{k} = \lambda f$
- Direction of Propagation: $+x$

$$\boxed{y = y_0 \sin(kx - \omega t)}$$

Relationships

f = # cycles/second

Electromagnetic Waves



spectrum

nuleon

log scale

Remember: $\lambda f = c$

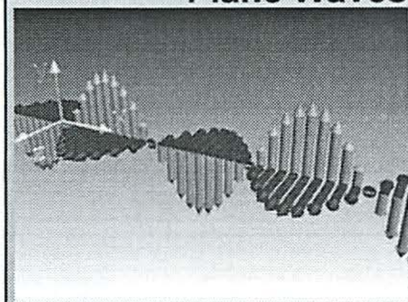
↑ always true

c = velocity of c

Hertz studied waves in detail

one can derive from stuff we already know

Electromagnetic Radiation: Plane Waves



Watch 2 Ways:
1) Sine wave traveling to right (+x)
2) Collection of out of phase oscillators (watch one position)

Don't confuse vectors with heights – they are magnitudes of E (gold) and B (blue)

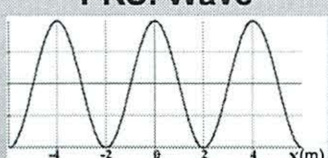
T30-13

★ Memorize Video

PRS Question: Wave

T30-14

PRS: Wave



The graph shows a plot of the function y = cos(k x). The value of k is

- 0% 1. ½ m^{-1}
- 0% 2. ¼ m^{-1}
- 0% 3. π m^{-1}
- 0% 4. $\pi/2$ m^{-1} ✓
- 0% 5. I don't know

:20

92%

$k = \frac{2\pi}{\lambda}$ $\frac{2\pi}{4m} = \frac{1}{2}\pi\ m^{-1}$

Group Work:
Do Problem 1
In this Java Applet

P30-16

$\lambda = 4$

$f = \frac{5 \text{ waves}}{10 \text{ sec}} = \frac{1}{2}$ freq

speed $= \lambda f = 2$ units?

$y_0 = 4$

$y = 4 \sin\left(\frac{2\pi}{4} x - 2\pi \frac{1}{2} t\right)$

Direction of Propagation

$\vec{\mathbf{E}} = \hat{\mathbf{E}} E_0 \sin(k(\hat{\mathbf{p}} \cdot \vec{\mathbf{r}}) - \omega t); \quad \vec{\mathbf{B}} = \hat{\mathbf{B}} B_0 \sin(k(\hat{\mathbf{p}} \cdot \vec{\mathbf{r}}) - \omega t)$

$\hat{\mathbf{E}} \times \hat{\mathbf{B}} = \hat{\mathbf{p}}$

$\hat{\mathbf{E}}$	$\hat{\mathbf{B}}$	$\hat{\mathbf{p}}$	$(\hat{\mathbf{p}} \cdot \vec{\mathbf{r}})$
$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$	z
$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$	$\hat{\mathbf{i}}$	x
$\hat{\mathbf{k}}$	$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	y
$\hat{\mathbf{j}}$	$\hat{\mathbf{i}}$	$-\hat{\mathbf{k}}$	$-z$
$\hat{\mathbf{k}}$	$\hat{\mathbf{j}}$	$-\hat{\mathbf{i}}$	$-x$
$\hat{\mathbf{i}}$	$\hat{\mathbf{k}}$	$-\hat{\mathbf{j}}$	$-y$

P30-17

carry energy

poying vector - very general chart

How do Maxwell's Eq lead to waves propigating

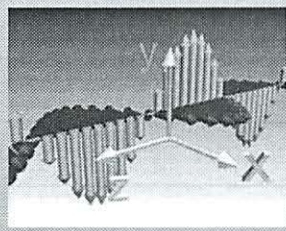
PRS Question:
Direction of Propagation

P30-18

16.07-?
6.041 = hardest

PRS: Direction of Propagation

The figure shows the E (yellow) and B (blue) fields of a plane wave. This wave is propagating in the

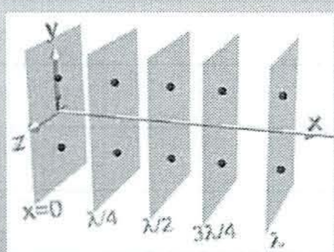


1. +x direction
2. –x direction
3. +z direction
4. –z direction
5. I don't know

20
P30-19

so know along z axis
$\vec{E} \times \vec{B}$
gives →

Group Problem: Plane Waves



1) Plot E, B at each of the ten points pictured for $t=0$

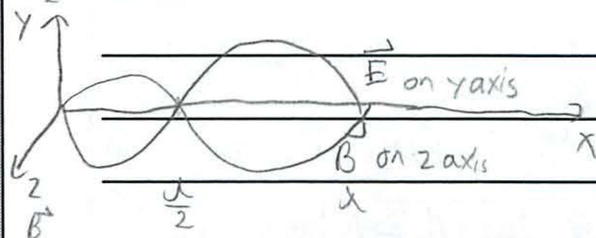
2) Why is this a "plane wave?"

$$\vec{E}(x,y,z,t) = E_{y,0}\sin\left(\frac{2\pi}{\lambda}(x-ct)\right)\hat{j}$$

$$\vec{B}(x,y,z,t) = \frac{1}{c}E_{y,0}\sin\left(\frac{2\pi}{\lambda}(x-ct)\right)\hat{k}$$

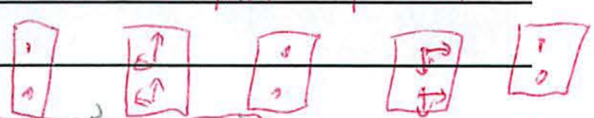
P30-20

depends where you start (phase shift, right)



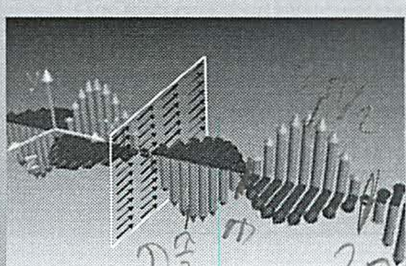
not really to scale

~~Solution on PDF printout~~



5 sheets at different distances
10 points – draw direction + magnitude $\vec{E}$ and $\vec{B}$ – just a different way to display same thing

Electromagnetic Radiation: Plane Waves



P30-21

There is one plane where both are 0

$2\pi = \lambda$
$\pi = \lambda/2$
$\pi/2 = \lambda/4$

Electromagnetic Waves

Both E & B travel like waves:

$$\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \qquad \frac{\partial^2 B_z}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B_z}{\partial t^2}$$

But there are strict relations between them:

$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \qquad \frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

Here, E_y and B_z are "the same," traveling along x axis

P30-22

Strict relationship b/w

Amplitudes of E & B

Let $E_y = E_0 f(x - vt)$; $B_z = B_0 f(x - vt)$

$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \Rightarrow -vB_0 f'(x-vt) = -E_0 f'(x-vt)$$

$$\boxed{\Rightarrow vB_0 = E_0}$$

E_y and B_z are "the same," just different amplitudes

P30-23

Strict phase + amp relationship

PRS EM Wave

The E field of a plane wave is:

$$\vec{\mathbf{E}}(z,t) = \hat{\mathbf{j}} E_0 \sin(kz + \omega t)$$

traveling in -z dir to keep argument same

The magnetic field of this wave is given by:

0%	1. $\vec{\mathbf{B}}(z,t) = \hat{\mathbf{i}} B_0 \sin(kz + \omega t)$
0%	2. $\vec{\mathbf{B}}(z,t) = -\hat{\mathbf{i}} B_0 \sin(kz + \omega t)$
0%	3. $\vec{\mathbf{B}}(z,t) = \hat{\mathbf{k}} B_0 \sin(kz + \omega t)$
0%	4. $\vec{\mathbf{B}}(z,t) = -\hat{\mathbf{k}} B_0 \sin(kz + \omega t)$
0%	5. I don't know

:20

Want to stay at same plate

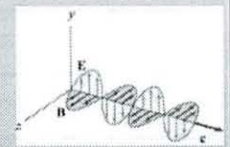
want $\vec{E} \times \vec{B}$ = -negitive z

B should be + $\hat{i}$

Properties of EM Waves

Travel (through vacuum) with speed of light

$$v = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3\times 10^8 \frac{m}{s}$$



At every point in the wave and any instant of time, E and B are in phase with one another, with

$$\frac{E}{B} = \frac{E_0}{B_0} = c$$

always in phase
magnitude related by this

E and B fields perpendicular to one another, and to the direction of propagation (they are **transverse**):

Direction of propagation = Direction of $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$

P30-25

poying vector

vector x y z
$k = \vec{E} \times \vec{B}$

Traveling E & B Waves

Wavelength: λ

Frequency : f

$$\vec{\mathbf{E}} = \hat{\mathbf{E}} E_0 \sin(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t)$$

Wave Number: $k = \frac{2\pi}{\lambda}$

Angular Freq.: $\omega = 2\pi f$

Period: $T = \frac{1}{f} = \frac{2\pi}{\omega}$

Speed: $v = \frac{\omega}{k} = \lambda f$

Direction: $+\hat{\mathbf{k}} = \hat{\mathbf{E}} \times \hat{\mathbf{B}}$

$$\frac{E}{B} = \frac{E_0}{B_0} = v$$

In vacuum...

$$= c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3\times 10^8 \frac{m}{s}$$

P30-26

Standing Waves

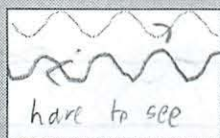
P30-27

Standing Waves

What happens if two waves headed in opposite directions are allowed to interfere?

$E_1 = E_0 \sin(kx - \omega t) \qquad E_2 = E_0 \sin(kx + \omega t)$

Superposition: $E = E_1 + E_2 = 2E_0 \sin(kx)\cos(\omega t)$



P30-28

Same wavelenght + Velocity

-still ossilating
-but not propagating

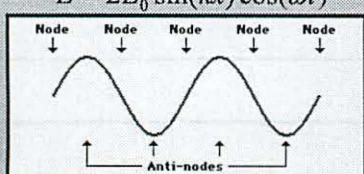
Standing Waves

Most commonly seen in resonating systems:
Musical Instruments, Microwave Ovens

ends fixed

fairly common

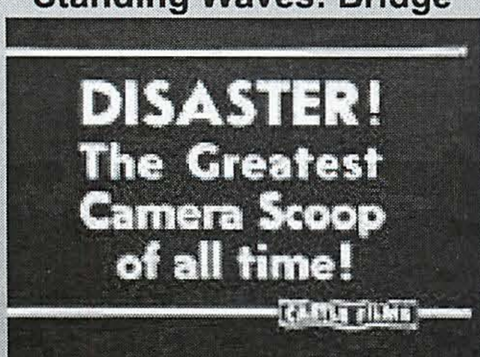
$E = 2E_0 \sin(kx)\cos(\omega t)$



P30-29

greatest ocilations

Standing Waves: Bridge



P30-30

galliping girdy bridge

Group Work: Standing Waves
Do Problem 2 In the Java Applet

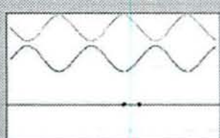
Want it to be – of F

same wavelenght, amp, velocity

Group Work: Standing Waves
Do Problem 2 In the Java Applet

$E_1 = E_0 \sin(kx - \omega t) \qquad E_2 = E_0 \sin(kx + \omega t)$

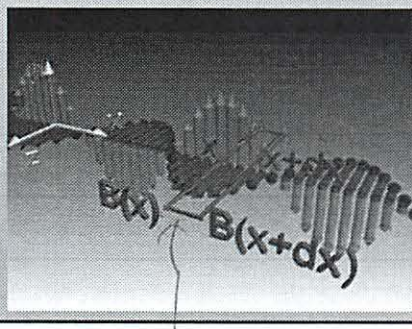
Superposition: $E = E_1 + E_2 = 2E_0 \sin(kx)\cos(\omega t)$



Appendix:
How Do Maxwell's Equations Lead to EM Waves?
(Optional)

Wave Equation

Start with Ampere-Maxwell Eq: $\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \varepsilon_0 \frac{d}{dt} \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$



Amperian loop

$\vec{E}$ = yellow – perp to loop

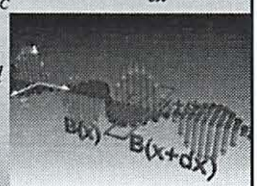
B = magnetic

Wave Equation

Start with Ampere-Maxwell Eq: $\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \varepsilon_0 \frac{d}{dt} \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$

Apply it to red rectangle:

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B_z(x,t)l - B_z(x+dx,t)l$$



$$\mu_0 \varepsilon_0 \frac{d}{dt} \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \mu_0 \varepsilon_0 \left(l\, dx \frac{\partial E_y}{\partial t} \right)$$

$$-\frac{B_z(x+dx,t) - B_z(x,t)}{dx} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

So in the limit that dx is very small:

$$-\frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

P30-35

Apply Maxwell's equations

derivative of magnetic field

← found relationship

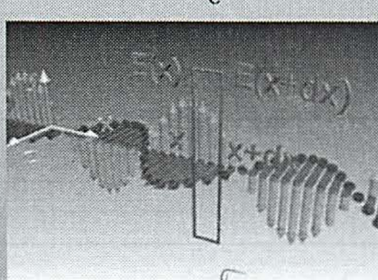
$\frac{d\Phi}{dt}$ electric field changing w/ respect to time

other side – bit further away

where field ⊥ does not count

Wave Equation

Now go to Faraday's Law $\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d}{dt} \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$



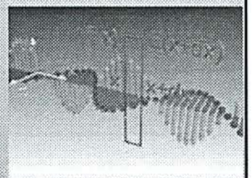
for other relationship

loop in other dir
– plane $\vec{E}$
– $\vec{B}$ penetrating

Wave Equation

Faraday's Law: $\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d}{dt}\int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$

Apply it to red rectangle:



similar

$$\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = E_y(x+dx,t)l - E_y(x,t)l$$

$$-\frac{d}{dt}\int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = -ldx\frac{\partial B_z}{\partial t}$$ Magnetic flux

$$\frac{E_y(x+dx,t) - E_y(x,t)}{dx} = -\frac{\partial B_z}{\partial t}$$

So in the limit that dx is very small: $\boxed{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}}$

P30-37

1D Wave Equation for E

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \qquad -\frac{\partial B_z}{\partial x} = \mu_0\varepsilon_0\frac{\partial E_y}{\partial t}$$

Take *x*-derivative of 1st and use the 2nd equation

$$\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial x}\right) = \frac{\partial^2 E_y}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{\partial B_z}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial x}\right) = \mu_0\varepsilon_0\frac{\partial^2 E_y}{\partial t^2}$$

$$\boxed{\frac{\partial^2 E_y}{\partial x^2} = \mu_0\varepsilon_0\frac{\partial^2 E_y}{\partial t^2}}$$

P30-38

Combine the 2 relationships

1D Wave Equation for E

$$\boxed{\frac{\partial^2 E_y}{\partial x^2} = \mu_0\varepsilon_0\frac{\partial^2 E_y}{\partial t^2}}$$

← when you see this form think wave equation

This is an equation for a wave. Let: $E_y = f(x - vt)$

$$\left.\begin{aligned}\frac{\partial^2 E_y}{\partial x^2} &= f''(x-vt)\\ \frac{\partial^2 E_y}{\partial t^2} &= v^2 f''(x-vt)\end{aligned}\right\} v^2 = \frac{1}{\mu_0\varepsilon_0}$$

P30-39

always related

remarkable observation - analize field w/ no charges & current but have a wave $\frac{1}{\mu_0\varepsilon_0} = v^2 = c^2$ Fixed velocity!

1D Wave Equation for B

$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \qquad \frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

Take x-derivative of 1st and use the 2nd equation

$$\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial t}\right) = \frac{\partial^2 B_z}{\partial t^2} = \frac{\partial}{\partial t}\left(-\frac{\partial E_y}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial t}\right) = \frac{1}{\mu_0 \varepsilon_0}\frac{\partial^2 B_z}{\partial x^2}$$

$$\boxed{\frac{\partial^2 B_z}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B_z}{\partial t^2}}$$

Similar for B field

Electromagnetic Radiation

Both E & B travel like waves:

$$\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \qquad \frac{\partial^2 B_z}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B_z}{\partial t^2}$$

But there are strict relations between them:

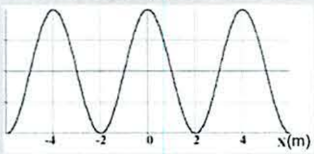
$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \qquad \frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

Here, E_y and B_z are "the same," traveling along x axis

in phase

This comes from Maxwell's Equations

PRS: Wave



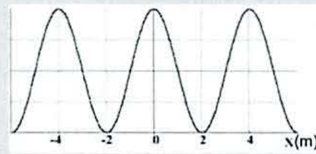
The graph shows a plot of the function y = cos(k x). The value of k is

- 0% 1. ½ m^{-1}
- 0% 2. ¼ m^{-1}
- 0% 3. π m^{-1}
- 0% 4. $\pi/2$ m^{-1}
- 0% 5. I don't know

:20

PRS Answer: Wave

Answer: 4. $k = \pi/2\ m^{-1}$



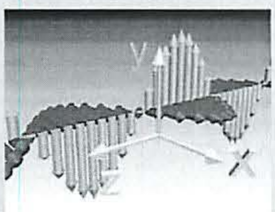
$\lambda = 4\ m \rightarrow k = 2\pi/\lambda = \pi/2\ m^{-1}$

$y = \cos(\pi x/2)$ is 1 at x = –4 m, 0 m, 4 m, etc.

P30- 2

PRS: Direction of Propagation

The figure shows the E (yellow) and B (blue) fields of a plane wave. This wave is propagating in the



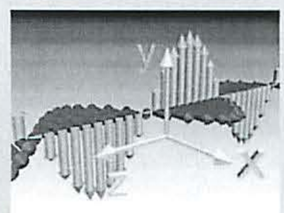
1. +x direction
2. –x direction
3. +z direction
4. –z direction
5. I don't know

20

PRS Answer: Propagation

Answer: 4. The wave is moving in the –z direction

The propagation direction is given by the direction of **E** x **B** (Yellow x Blue)



P30- 4

PRS: EM Wave

The E field of a plane wave is:

$$\vec{\mathbf{E}}(z,t) = \hat{\mathbf{j}} E_0 \sin(kz + \omega t)$$

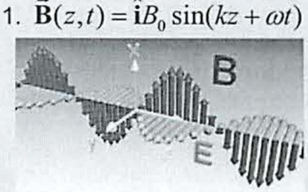
The magnetic field of this wave is given by:

1. $\vec{\mathbf{B}}(z,t) = \hat{\mathbf{i}} B_0 \sin(kz + \omega t)$
2. $\vec{\mathbf{B}}(z,t) = -\hat{\mathbf{i}} B_0 \sin(kz + \omega t)$
3. $\vec{\mathbf{B}}(z,t) = \hat{\mathbf{k}} B_0 \sin(kz + \omega t)$
4. $\vec{\mathbf{B}}(z,t) = -\hat{\mathbf{k}} B_0 \sin(kz + \omega t)$
5. I don't know

:20

PRS Answer: EM Wave

Answer: 1. $\vec{\mathbf{B}}(z,t) = \hat{\mathbf{i}} B_0 \sin(kz + \omega t)$



From the argument of the $\sin(kz + \omega t)$, we know the wave propagates in the -z direction.

So we have $\hat{\mathbf{E}} \times \hat{\mathbf{B}} = \hat{\mathbf{j}} \times ? = -\hat{\mathbf{k}}$

$$\Rightarrow \hat{\mathbf{B}} = \hat{\mathbf{i}}$$

P30- 6

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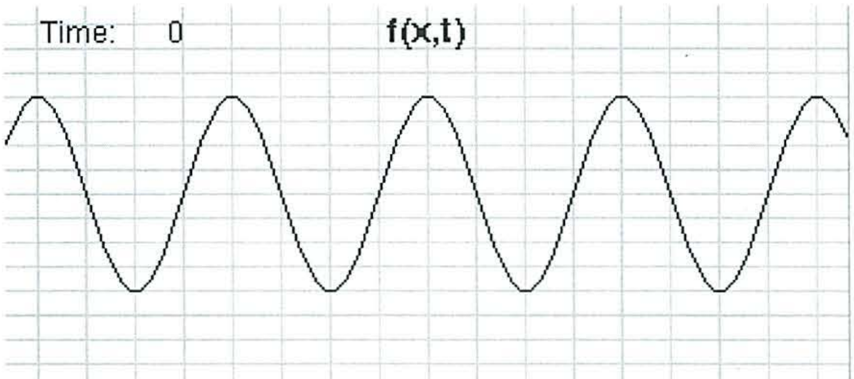
In Class W13D1_1 Solutions: Waves Java Applet

Problem: Find the wavelength, frequency, and speed of the wave in the top panel, f(x,t).



Solution:
The wavelength can be measured as the peak to peak distance: $\lambda = 4$

Thus the wavevector is given by $k = 2\pi/\lambda = \pi/2$

The speed can be measured by watching the wave travel. It moves one wavelength in 2 time units so the speed is v = 2.

So we can calculate both the frequency and angular frequency:
$\lambda f = v \Rightarrow f = v/\lambda = 1/2; \omega = 2\pi f = \pi$

The amplitude is F = 4.

So the equation can be obtained as:
$g(x,t) = F \cdot \cos(kx - \omega t) = 4 \cdot \cos(\pi x/2 - \pi t)$

The minus sign is because the wave is travelling to the right (+x direction)

Note that this applet doesn't use units (one unfortunate downside). Make sure you don't omit units when they are there.

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Department of Physics: 8.02

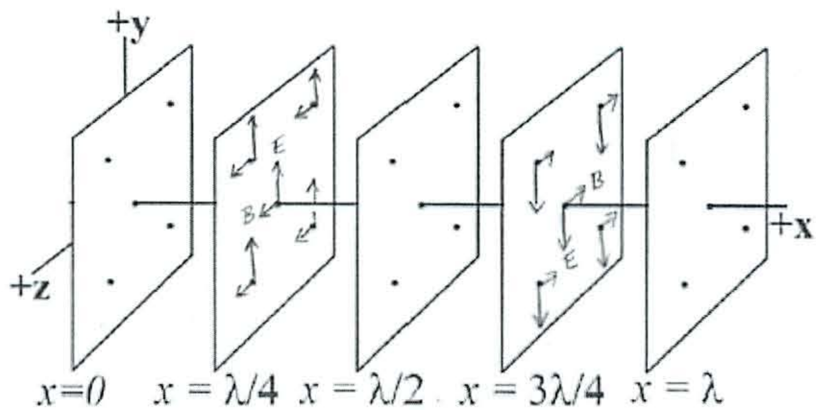
In Class W13D1_2 Solutions: Plane Waves

Problem: For the EM Wave:

$$\vec{\mathbf{E}}(x,y,z,t) = E_{y,0}\sin\left(\frac{2\pi}{\lambda}(x-ct)\right)\hat{\mathbf{j}}; \qquad \vec{\mathbf{B}}(x,y,z,t) = \frac{1}{c}E_{y,0}\sin\left(\frac{2\pi}{\lambda}(x-ct)\right)\hat{\mathbf{k}}$$

1) Plot E, B at each of the ten points pictured for t=0
2) Why is this a "plane wave?"

Solution:



This is a "plane wave" because for any plane (parallel to the yz plane) the electric and magnetic fields are doing the same thing at all points. For example, at x = 0 we find E & B are zero everywhere. So when the wave propagates it is as if entire planes were simply sliding along the propagation axis (in this case the x-axis)

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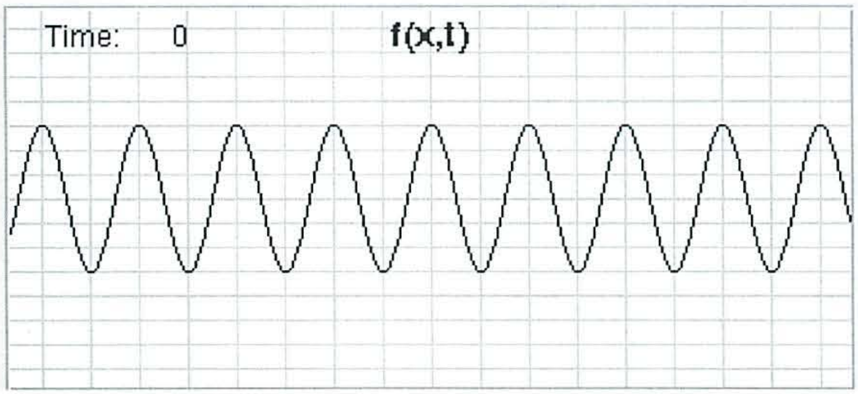
Department of Physics: 8.02

In Class W13D1_3 Solutions: Standing Waves Java Applet

Problem: (Problem 2 of the Java Applet) Find a wave function, g(x,t), that will produce a standing wave with a node at x=0, (i.e., at the center.)



Solution:
We first need to find the parameters of wave f(x,t)
The wavelength can be measured as the peak to peak distance: $\lambda = 2$

Thus the wavevector is given by $k = 2\pi/\lambda = \pi$

The speed can be measured by watching the wave travel. It moves one wavelength in 2 time units so the speed is v = 1.

So we can calculate both the frequency and angular frequency:
$\lambda f = v \Rightarrow f = v/\lambda = 1/2; \omega = 2\pi f = \pi$

The amplitude is F = 3.

So the equation can be obtained as:
$f(x,t) = F \cdot \cos(kx - \omega t) = 3 \cdot \cos(\pi x - \pi t)$

The minus sign is because the wave is travelling to the right (+x direction)
To make a standing wave we send an equal wave in the opposite direction. We want a node at zero, which means we also need to change the sign:

$g(x,t) = -F \cdot \cos(kx + \omega t) = -3 \cdot \cos(\pi x + \pi t)$

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Department of Physics

8.02 **Spring 2010**

Problem Set 11

Due: Tuesday, April 27 at 9 pm.

Hand in your problem set in your section slot in the boxes outside the door of 32-082. Make sure you clearly write your name and section on your problem set.

Text: Liao, Dourmashkin, Belcher; Introduction to E & M MIT 8.02 Course Notes.

Week Twelve Self Inductance & Magnetic Energy

M Apr 19-T Apr 20 Patriot's Day Holiday

Class 28 W12D2 W/R Apr 21/22	Expt. 9: Driven RLC Circuits, Maxwell's Equations and Displacement Current; Poynting Vector & Energy Flow.
Reading:	Course Notes: Sections 13.1-13.3, 13.12.3-13.12.4
Experiment:	Expt. 9: Driven RLC Circuits

Drop Date Thurs Apr 22

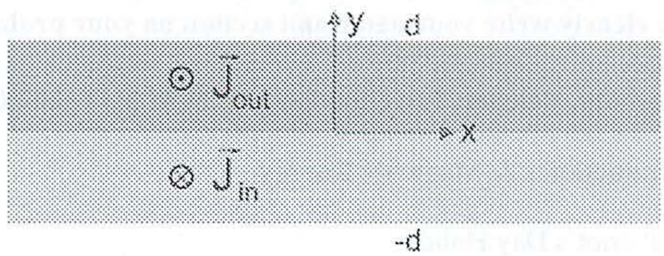
Class 29 W12D3 F Apr 23	PS09: Poynting Vector and Energy Flow in a Capacitor
Reading:	Course Notes: Sections 13.1-13.3, 13.12.3-13.12.4

Week Thirteen Electromagnetic Waves

Class 30 W13D1 M/T Apr 26/27	Maxwell's Equations, EM Waves
Reading:	Course Notes: Sections 13.3-13.4, 13.11, 13.12.1-13.12.2
Class 31 W13D2 W/R Apr 28/29	Exam 3 Review
Exam 3 Thursday April 29	**7:30 pm –9:30 pm**
W13D3 F Apr 30	No class day after exam

Problem 1: Current Slabs

The figure below shows two slabs of current. Both slabs of current are infinite in the x and z directions, and have thickness d in the y-direction. The top slab of current is located in the region $0 < y < d$ and has a constant current density $\vec{\mathbf{J}}_{out} = J\hat{\mathbf{z}}$ out of the page. The bottom slab of current is located in the region $-d < y < 0$ and has a constant current density $\vec{\mathbf{J}}_{in} = -J\hat{\mathbf{z}}$ into the page.



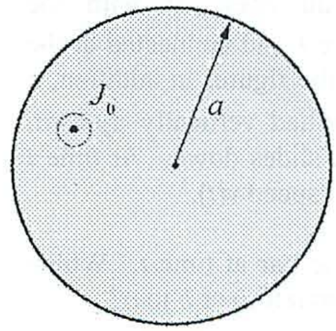
(a) What is the magnetic field for $|y| > d$? Justify your answer.

(b) Use Ampere's Law to find the magnetic field at $y = 0$. Show the Amperian Loop that you use and give the magnitude and direction of the magnetic field.

(c) Use Ampere's Law to find the magnetic field for $0 < y < d$. Show the Amperian Loop that you use and give the magnitude and direction of the magnetic field.

(d) Plot the x-component of the magnetic field as a function of the distance y on the graph below. Label your vertical axis.

Problem 2: An infinitely long wire of radius a carries a current density J_0 which is uniform and constant. The current points "out of" the page, as shown in the figure.

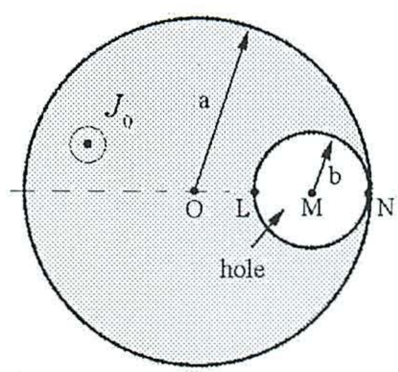


(a) Calculate the magnitude of the magnetic field $B(r)$ for (i) $r < a$ and (ii) $r > a$. For both cases show your Amperian loop and indicate (with arrows) the direction of the magnetic field.

(b) What happens to the answers above if the direction of the current is reversed so that it flows "into" the page ?

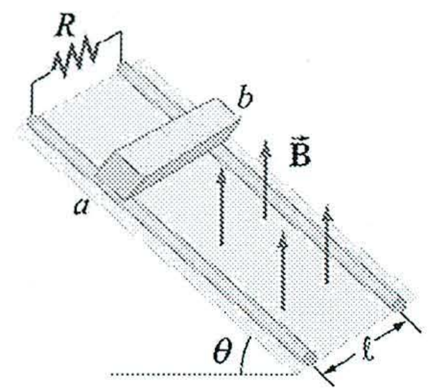
(c) Consider now the same wire but with a hole bored throughout. The hole has radius b (with $2b < a$) and is shown in the figure. We have also indicated four special points: O, L, M, and N. The point O is at the center of the original wire and the point M is at the center of the hole. In this new wire, the current density exists and remains equal to J_0 over the remainder of the cross section of the wire. Calculate the magnitude of the magnetic field at (i) the point M, (ii) at the point L, and (iii) at the point N. Show your work.

Hint: Try to represent the configuration as the "superposition" of two types of wires.



Problem 3: Sliding Bar on Wedges

A conducting bar of mass m slides down two frictionless conducting rails which make an angle θ with the horizontal, separated by a distance ℓ and connected at the top by a resistor R, as shown in the figure. In addition, a uniform magnetic field $\vec{\mathbf{B}}$ is applied vertically upward. The bar is released from rest and slides down. At time t the bar is moving along the rails at speed $v(t)$.



(a) Find the induced current in the bar at time t. Which way does the current flow, from a to b or b to a?

(b) Find the terminal speed v_T of the bar.

After the terminal speed has been reached,

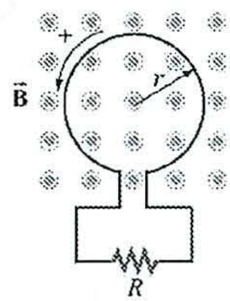
(c) What is the induced current in the bar?

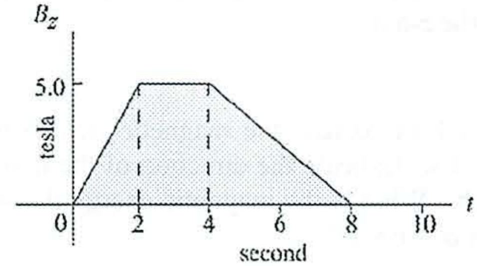
(d) What is the rate at which electrical energy is being dissipated through the resistor?

(e) What is the rate of work done by gravity on the bar? The rate at which work is done is $\vec{\mathbf{F}}\cdot\vec{\mathbf{v}}$. How does this compare to your answer in (d)? Why?

Problem 4 EMF Due to a Time-Varying Magnetic Field

A uniform magnetic field $\vec{\mathbf{B}}$ is perpendicular to a one-turn circular loop of wire of negligible resistance, as shown in the figure below. The field changes with time as shown (the z direction is out of the page). The loop is of radius $r = 50\text{ cm}$ and is connected in series with a resistor of resistance $R = 20\,\Omega$. The "+" direction around the circuit is indicated in the figure.





(a) What is the expression for EMF in this circuit in terms of $B_z(t)$ for this arrangement?

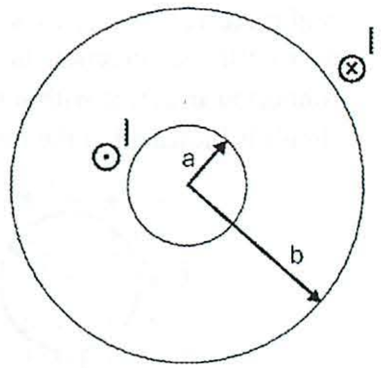
(b) Plot the EMF in the circuit as a function of time. Label the axes quantitatively (numbers and units). Watch the signs. Note that we have labeled the positive direction of the emf in the left sketch consistent with the assumption that positive $\vec{\mathbf{B}}$ is out of the paper.

(c) Plot the current I through the resistor R. Label the axes quantitatively (numbers and units). Indicate with arrows on the sketch the *direction* of the current through R during each time interval.

(d) Plot the power dissipated in the resistor as a function of time.

Problem 5: Inductor

An inductor consists of two very thin conducting cylindrical shells, one of radius a and one of radius b, both of length h. Assume that the inner shell carries current I out of the page, and that the outer shell carries current I into the page, distributed uniformly around the circumference in both cases. The z-axis is out of the page along the common axis of the cylinders and the r-axis is the radial cylindrical axis perpendicular to the z-axis.



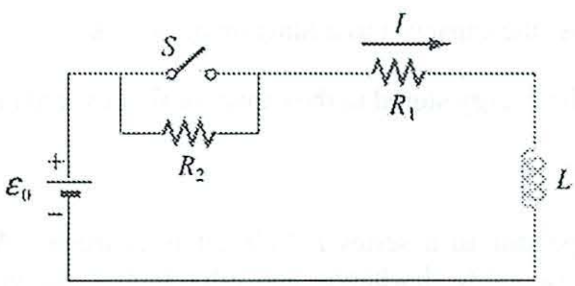
a) Use Ampere's Law to find the magnetic field between the cylindrical shells. Indicate the direction of the magnetic field on the sketch. What is the magnetic energy density as a function of r for $a < r < b$?

b). Calculate the inductance of this long inductor recalling that $U_B = \frac{1}{2} LI^2$ and using your results for the magnetic energy density in (a).

c) Calculate the inductance of this long inductor by using the formula $\Phi = LI = \int_{open\ surface} \vec{\mathbf{B}} \cdot \mathbf{d}\vec{\mathbf{A}}$ and your results for the magnetic field in (a). To do this you must choose an appropriate open surface over which to evaluate the magnetic flux. Does your result calculated in this way agree with your result in (b)?

Problem 6: Trying to open the switch on an *RL* Circuit

The *LR* circuit shown in the figure contains a resistor R_1 and an inductance L in series with a battery of emf ε_0. The switch S is initially closed. At $t = 0$, the switch S is opened, so that an additional very large resistance R_2 (with $R_2 \gg R_1$) is now in series with the other elements.



(a) If the switch has been *closed* for a long time before $t = 0$, what is the steady current I_0 in the circuit?

(b) While this current I_0 is flowing, at time $t = 0$, the switch S is opened. Write the differential equation for $I(t)$ that describes the behavior of the circuit at times $t \geq 0$. Solve this equation (by integration) for $I(t)$ under the approximation that $\varepsilon_0 = 0$. (Assume that the battery emf is negligible compared to the total emf around the circuit for times just after the switch is opened.) Express your answer in terms of the initial current I_0, and R_1, R_2, and L.

(c) Using your results from (b), find the value of the total emf around the circuit (which from Faraday's law is $-L\,dI/dt$) just after the switch is opened. Is your assumption in (b) that ε_0 could be ignored for times just after the switch is opened OK?

(d) What is the magnitude of the potential drop across the resistor R_2 at times $t > 0$, just after the switch is opened? Express your answers in terms of ε_0, R_1, and R_2. How does the potential drop across R_2 just after $t = 0$ compare to the battery emf ε_0, if $R_2 = 100R_1$?

Problem 7: *LC* Circuit

An inductor having inductance L and a capacitor having capacitance C are connected in series. The current in the circuit increase linearly in time as described by $I = Kt$. The capacitor initially has no charge. Determine

(a) the voltage across the inductor as a function of time,

(b) the voltage across the capacitor as a function of time, and

(c) the time when the energy stored in the capacitor first exceeds that in the inductor.

Problem 8: *LC* Circuit

(a) Initially, the capacitor in a series LC circuit is charged. A switch is closed, allowing the capacitor to discharge, and after time T the energy stored in the capacitor is one-fourth its initial value. Determine L if C and T are known.

(b) A capacitor in a series LC circuit has an initial charge Q_0 and is being discharged. The inductor is a solenoid with N turns. Find, in terms of L and C, the flux through each of the N turns in the coil at time t, when the charge on the capacitor is $Q(t)$. State any assumptions that you make.

(c) An LC circuit consists of a 20.0-mH inductor and a 0.500-μF capacitor. If the maximum instantaneous current is 0.100 A, what is the greatest potential difference across the capacitor?

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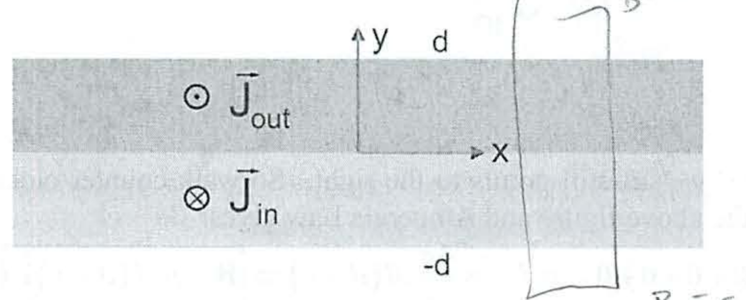
Problem Set 11 Solutions

Problem 1: Current Slabs

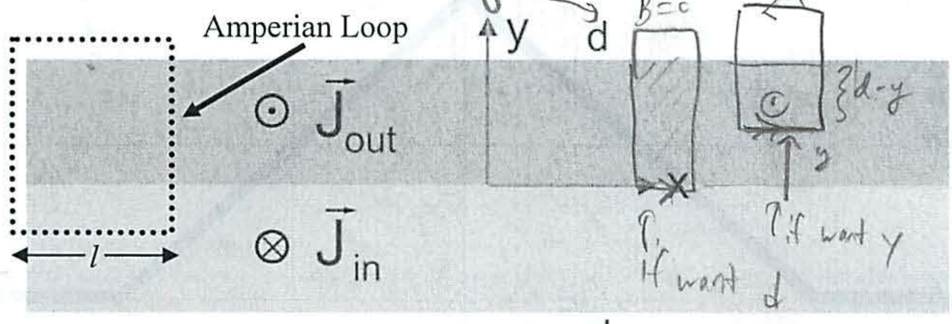
The figure below shows two slabs of current. Both slabs of current are infinite in the x and z directions, and have thickness d in the y-direction. The top slab of current is located in the region $0 < y < d$ and has a constant current density $\vec{\mathbf{J}}_{out} = J\hat{\mathbf{z}}$ out of the page. The bottom slab of current is located in the region $-d < y < 0$ and has a constant current density $\vec{\mathbf{J}}_{in} = -J\hat{\mathbf{z}}$ into the page.



(a) What is the magnetic field for $|y| > d$? Justify your answer.

Zero. The two parts of the slab create equal and opposite fields for $|y| > d$.

b) Use Ampere's Law to find the magnetic field at $y = 0$. Show the Amperian Loop that you use and give the magnitude and direction of the magnetic field.



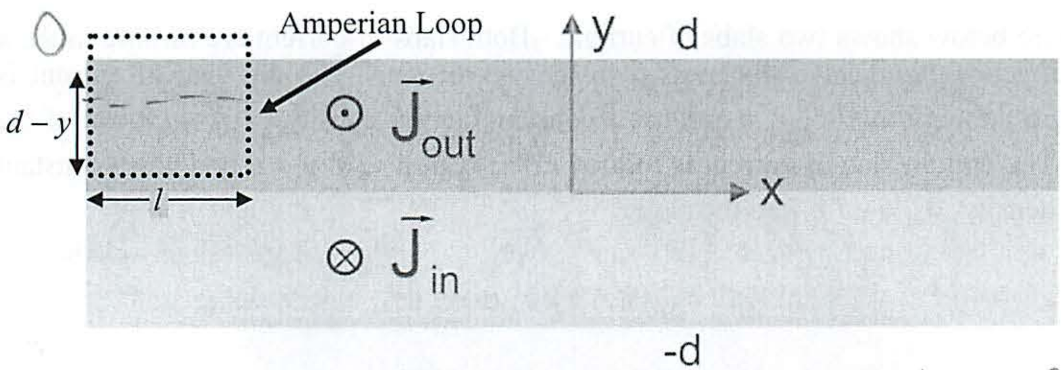
The field at $y = 0$ points to the right (both slabs make it point that way). So walk counter clockwise around the loop shown in the above figure and Ampere's Law gives:

$$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = Bl + 0 + 0 + 0 = \frac{4\pi}{c} I_{enc} = \mu_0 (Jld) \Rightarrow \boxed{\vec{\mathbf{B}} = \mu_0 Jd\,\hat{\mathbf{i}} \text{ (to the right)}}$$

think got this – but maybe for wrong problem

c) Use Ampere's Law to find the magnetic field for $0 < y < d$. Show the Amperian Loop that you use and give the magnitude and direction of the magnetic field.

measuring from here (0) → 0



~~why ccw?~~

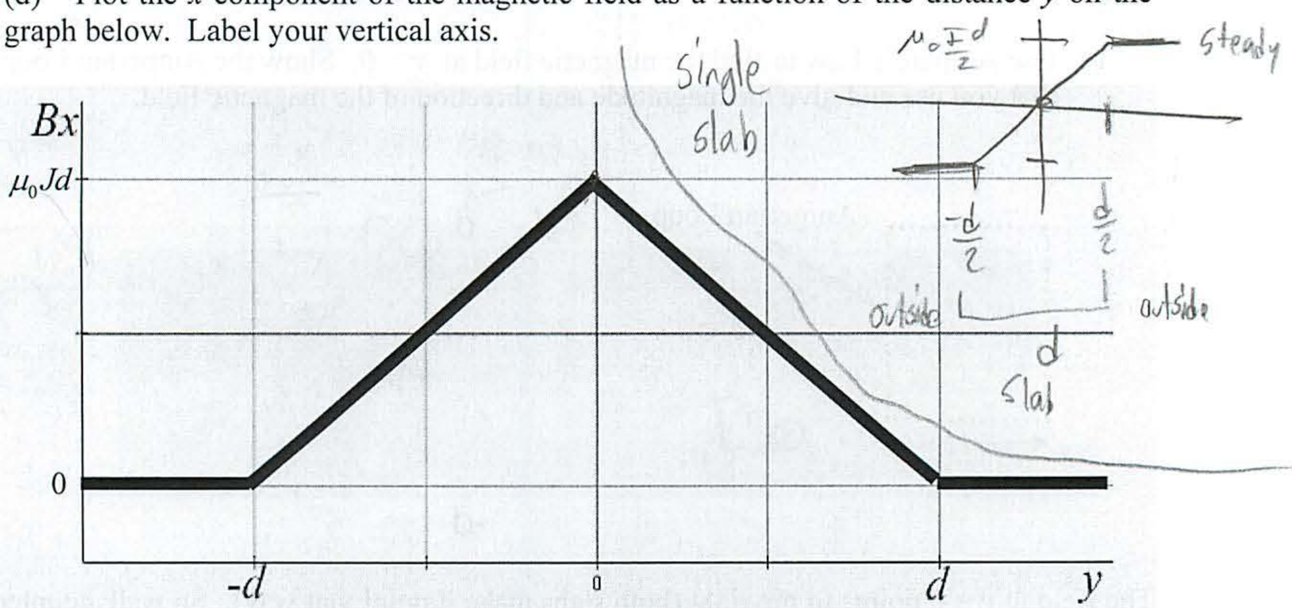
The field for $0 < y < d$ still points to the right. So walk counter clockwise around the loop shown in the above figure and Ampere's Law gives:

$$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = Bl + 0 + 0 + 0 = \mu_o I_{enc} = \frac{4\pi}{c} Jl(d-y) \Rightarrow \boxed{\vec{\mathbf{B}} = \mu_0 J(d-y)\,\hat{\mathbf{i}} \text{ (to the right)}}$$

why $d-y$

OH – they measure outside in

(d) Plot the x-component of the magnetic field as a function of the distance y on the graph below. Label your vertical axis.

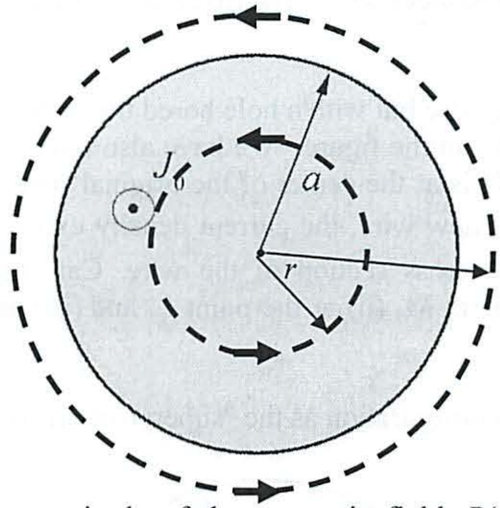


Completely got this wrong

Need to get the signs correct

Problem 2:

An infinitely long wire of radius a carries a current density J_0 which is uniform and constant. The current points "out of" the page, as shown in the figure.



a) Calculate the magnitude of the magnetic field $B(r)$ for (i) $r < a$ and (ii) $r > a$. For both cases show your Amperian loop and indicate (with arrows) the direction of the magnetic field.

The dashed lines above are the Amperian loops I will use for (i) and (ii). They both have a radius of r, and in both cases the paths are counterclockwise, as is the B field, due to a current out of the page (right hand rule).

(i) $r < a$.

From Ampere's Law:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = 2\pi r B = \mu_0 I_{penetrate} = \mu_0 J_0 \pi r^2 \Rightarrow \boxed{B = \frac{\mu_0 J_0 \pi r^2}{2\pi r} = \frac{\mu_0 J_0 r}{2} \text{ counterclockwise}}$$

(ii) $r > a$.

Now we just contain all of the current:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = 2\pi r B = \mu_0 I_{penetrate} = \mu_0 J_0 \pi a^2 \Rightarrow \boxed{B = \frac{\mu_0 J_0 \pi a^2}{2\pi r} = \frac{\mu_0 J_0 a^2}{2r} \text{ counterclockwise}}$$

this is fairly standard

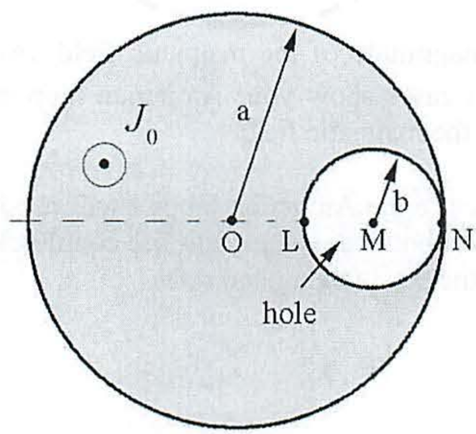
copied from notes - be able to do!

(b) What happens to the answers above if the direction of the current is reversed so that it flows "into" the page ?

If the direction of current flips then so does the direction of the magnetic field, so it is clockwise rather than counterclockwise. The magnitude of the field remains the same.

c) Consider now the same wire but with a hole bored throughout. The hole has radius b (with $2b < a$) and is shown in the figure. We have also indicated four special points: O, L, M, and N. The point O is at the center of the original wire and the point M is at the center of the hole. In this new wire, the current density exists and remains equal to J_0 over the remainder of the cross section of the wire. Calculate the magnitude of the magnetic field at (i) the point M, (ii) at the point L, and (iii) at the point N. Show your work.

Hint: Try to represent the configuration as the "superposition" of two types of wires.



The point here is that we have two wires superimposed on top of each other. The large (radius a) wire carries current out of the page while the smaller (radius b) wire carries current into the page (with the same current density). At all point L, M and N we are inside the large wire and on the right, so the counterclockwise B field is pointing up the page. What is happening from the small wire changes from place to place

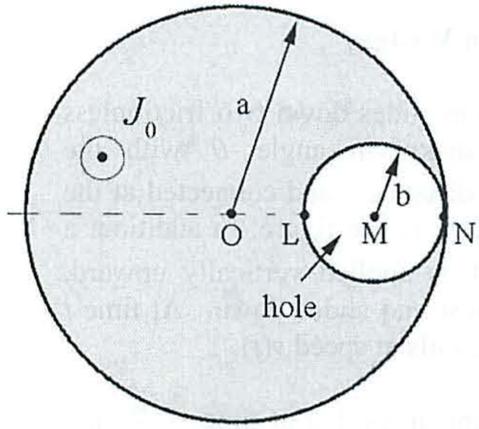
(i) the point M:

Here we are at the center of the small wire, so it contributes nothing. We are at a radius r = a - b inside the big wire, so from part (a.i) of this problem we have:

$$\boxed{B = \frac{\mu_0 J_0 (a-b)}{2} \text{ up}}$$

Dormashkin

very hard - perhaps too hard



(ii) at the point L:

Here we are to the left of the small wire (at a radius r = b), so the clockwise field (as we said in part b) is pointing up, just like the CCW field from the big wire We are at a radius r = a - 2b inside the big wire, so:

$$B = \frac{\mu_0 J_0 (a-2b)}{2} + \frac{\mu_0 J_0 b}{2} \text{up} = \frac{\mu_0 J_0 (a-b)}{2} \text{up}$$

(iii) at the point N:
Here we are to the right of the small wire (at a radius r = b), so the clockwise field is pointing down, opposite the CCW field from the big wire so they subtract rather than add We are at a radius r = a inside the big wire, so:

$$B = \frac{\mu_0 J_0 a}{2} - \frac{\mu_0 J_0 b}{2} \text{up} = \frac{\mu_0 J_0 (a-b)}{2} \text{up}$$

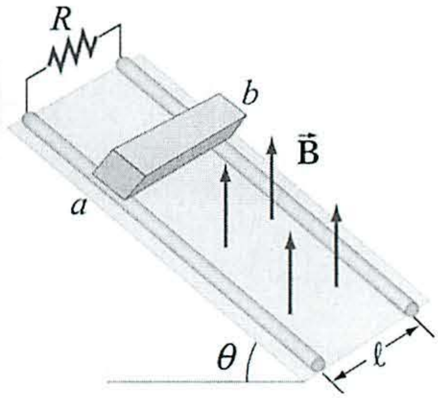
Same result everywhere - think that is what I had

A comment about people's work on this problem: I was stunned at how many people tried to do Ampere's law on the wire with a hole in it. Since the hole breaks the cylindrical symmetry of the problem you just can't do this. That is, since B is no longer constant around an Amperian centered on O, $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \neq 2\pi r B$. B isn't constant, so you can't just pull it out!

- did I do this
well superimposed hole over wire

Problem 3: Sliding Bar on Wedges

A conducting bar of mass m slides down two frictionless conducting rails which make an angle θ with the horizontal, separated by a distance ℓ and connected at the top by a resistor R, as shown in the figure. In addition, a uniform magnetic field $\vec{\mathbf{B}}$ is applied vertically upward. The bar is released from rest and slides down. At time t the bar is moving along the rails at speed $v(t)$.



(a) Find the induced current in the bar at time t. Which way does the current flow, from a to b or b to a?

The flux between the resistor and bar is given by

$$\Phi_B = B\ell x(t)\cos\theta$$

where $x(t)$ is the distance of the bar from the top of the rails.

Then,

$$\varepsilon = -\frac{d}{dt}\Phi_B = -\frac{d}{dt}B\ell x(t)\cos\theta = -B\ell v(t)\cos\theta$$

Because the resistance of the circuit is R, the magnitude of the induced current is

$$I = \frac{|\varepsilon|}{R} = \frac{B\ell v(t)\cos\theta}{R}$$

By Lenz's law, the induced current produces magnetic fields which tend to oppose the change in magnetic flux. Therefore, the current flows clockwise, from b to a across the bar.

(b) Find the terminal speed v_T of the bar.

At terminal velocity, the net force along the rail is zero, that is gravity is balanced by the magnetic force:

$$mg\sin\theta = IB\ell\cos\theta = \left(\frac{B\ell v_t(t)\cos\theta}{R}\right)B\ell\cos\theta$$

or

$$v_t(t) = \frac{Rmg\sin\theta}{(B\ell\cos\theta)^2}$$

After the terminal speed has been reached,

(c) What is the induced current in the bar?

$$I = \frac{B\ell v_t(t)\cos\theta}{R} = \frac{B\ell\cos\theta}{R}\left(\frac{Rmg\sin\theta}{(B\ell\cos\theta)^2}\right) = \frac{mg\sin\theta}{B\ell\cos\theta} = \frac{mg}{B\ell}\tan\theta$$

(d) What is the rate at which electrical energy is being dissipated through the resistor?

The power dissipated in the resistor is

$$P = I^2 R = \left(\frac{mg}{B\ell}\tan\theta\right)^2 R$$

(e) What is the rate of work done by gravity on the bar? The rate at which work is done is $\vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$. How does this compare to your answer in (d)? Why?

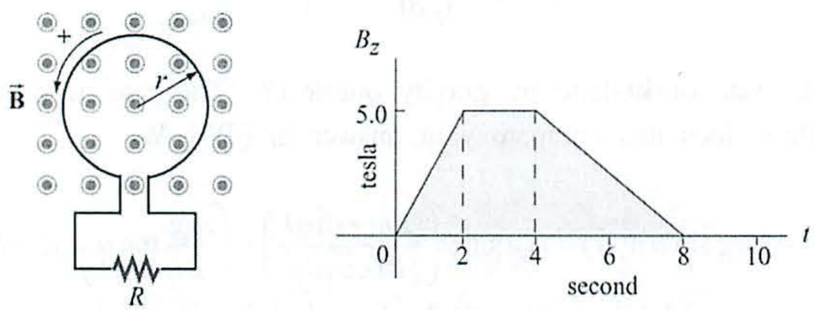
$$\vec{F} \cdot \vec{V} = (mg\sin\theta)v_t(t) = mg\sin\theta\left(\frac{Rmg\sin\theta}{(B\ell\cos\theta)^2}\right) = \left(\frac{mg}{B\ell}\tan\theta\right)^2 R = P$$

That is, they are equal. All of the work done by gravity is dissipated in the resistor, which is why the rod isn't accelerating past its terminal velocity.

this is hard w/o my work
I should have scanned it in

Problem 4 EMF Due to a Time-Varying Magnetic Field

A uniform magnetic field $\vec{\mathbf{B}}$ is perpendicular to a one-turn circular loop of wire of negligible resistance, as shown in the figure below. The field changes with time as shown (the z direction is out of the page). The loop is of radius $r = 50\ \text{cm}$ and is connected in series with a resistor of resistance $R = 20\ \Omega$. The "+" direction around the circuit is indicated in the figure. **In order to obtain credit you must show your work; partial answers without work will not be accepted.**



(a) What is the expression for EMF in this circuit in terms of B_z, t for this arrangement?

Solution: When we choose a "+" direction around the circuit shown in the figure above then we are also specifying that magnetic flux out of the page is positive. (The unit vector $\hat{\mathbf{n}} = +\hat{\mathbf{k}}$ points out of the page.) Thus the dot product becomes

? wtf ?

$$\vec{\mathbf{B}} \cdot \hat{\mathbf{n}} = \vec{\mathbf{B}} \cdot \hat{\mathbf{k}} = B_z . \qquad (0.1)$$

From the graph, the z-component of the magnetic field B_z is given by

$$B_z = \begin{cases} (2.5\ \text{T}\cdot\text{s}^{-1})t\ ;\ 0 < t < 2\ \text{s} \\ 5.0\ \text{T}\ ;\ 2\ \text{s} < t < 4\ \text{s} \\ 10\ \text{T} - (1.25\ \text{T}\cdot\text{s}^{-1})t\ ;\ 4\ \text{s} < t < 8\ \text{s} \\ 0\ ;\ t > 8\ \text{s} \end{cases} . \qquad (0.2)$$

The derivative of the magnetic field is then

$$\frac{dB_z}{dt} = \begin{cases} 2.5\ \text{T}\cdot\text{s}^{-1}\ ;\ 0 < t < 2\ \text{s} \\ 0\ ;\ 2\ \text{s} < t < 4\ \text{s} \\ -1.25\ \text{T}\cdot\text{s}^{-1}\ ;\ 4\ \text{s} < t < 8\ \text{s} \\ 0\ ;\ t > 8\ \text{s} \end{cases} . \qquad (0.3)$$

The magnetic flux is therefore

$$\Phi_{magnetic} = \iint \vec{\mathbf{B}} \cdot \hat{\mathbf{n}}\, d\vec{\mathbf{A}} = \iint B_z dA = B_z \pi r^2 . (0.4)$$

The electromotive force is

$$\mathcal{E} = -\frac{d}{dt}\Phi_{magnetic} = -\frac{dB_z}{dt}\pi r^2 . \qquad (0.5)$$

So we calculate the electromotive force by substituting Eq. (0.3) into Eq. (0.5) yielding

$$\mathcal{E} = \begin{cases} -(2.5\,\mathrm{T\cdot s^{-1}})\pi r^2 \,;\, 0 < t < 2\,\mathrm{s} \\ 0\,;\, 2\,\mathrm{s} < t < 4\,\mathrm{s} \\ (1.25\,\mathrm{T\cdot s^{-1}})\pi r^2 \,;\, 4\,\mathrm{s} < t < 8\,\mathrm{s} \\ 0\,;\, t > 8\,\mathrm{s} \end{cases} . \qquad (0.6)$$

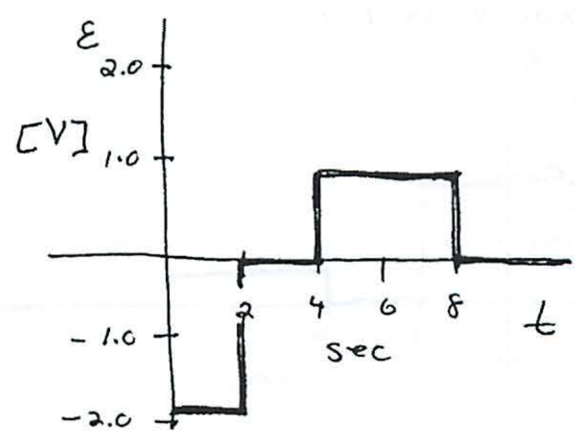
Using $r = 0.5\ \mathrm{m}$, the electromotive force is then

$$\mathcal{E} = \begin{cases} -1.96\,\mathrm{V}\,;\, 0 < t < 2\,\mathrm{s} \\ 0\,;\, 2\,\mathrm{s} < t < 4\,\mathrm{s} \\ 0.98\,\mathrm{V}\,;\, 4\,\mathrm{s} < t < 8\,\mathrm{s} \\ 0\,;\, t > 8\,\mathrm{s} \end{cases} \qquad (0.7)$$

Solution:

(b) Plot the EMF in the circuit as a function of time. Label the axes quantitatively (numbers and units). Watch the signs. Note that we have labeled the positive direction of the emf in the left sketch consistent with the assumption that positive $\vec{\mathbf{B}}$ is out of the paper.

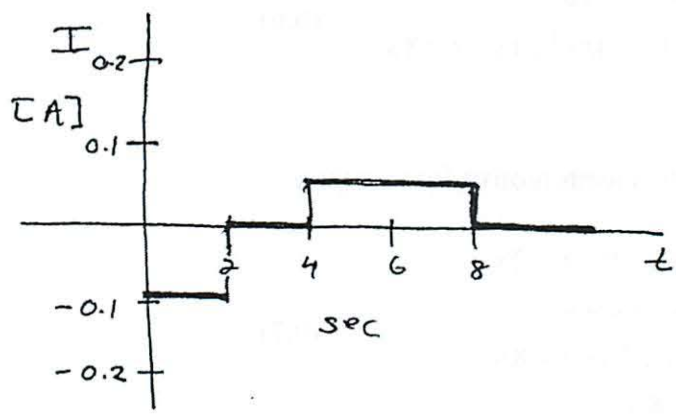
Solution:



(c) Plot the current I through the resistor R. Label the axes quantitatively (numbers and units). Indicate with arrows on the sketch the *direction* of the current through R during each time interval.

Solution: The current through the resistor ($R = 20\,\Omega$) is given by

$$I = \frac{\mathcal{E}}{R} = \begin{cases} -9.8\times10^{-2}\text{ A};\, 0 < t < 2\text{ s} \\ 0;\, 2\text{ s} < t < 4\text{ s} \\ 4.9\times10^{-2}\text{ A};\, 4\text{ s} < t < 8\text{ s} \\ 0;\, t > 8\text{ s} \end{cases}. \qquad (0.8)$$



(d) Plot the power dissipated in the resistor as a function of time.

Solution: The power dissipated in the resistor is given by

$$P = I^2 R = \begin{cases} 1.9\times10^{-1}\text{ W};\, 0 < t < 2\text{ s} \\ 0;\, 2\text{ s} < t < 4\text{ s} \\ 4.8\times10^{-2}\text{ W};\, 4\text{ s} < t < 8\text{ s} \\ 0;\, t > 8\text{ s} \end{cases}. \qquad (0.9)$$